Fields and Constants in the Theory of Gravitation

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Contents

Preface

This book ”Fields and Constants in the Theory of Gravitation”, CBPF-MO-002/02, Rio de Janeiro, 2002, is a translation from Russian of first 5 chapters from the book [3] by K.P. Stauyukovich and V.N. Melnikov “Hydrodynamics, Fields and Constants in the Theory of Gravitation”, Moscow, Energoatomizdat, 1983, which is the part of the latter, written by V.N. Melnikov on the basis of his papers with colleagues and related to fields and constants. Only first chapter, devoted to problems of fundamental physical constants is updated with account of the present observational data. Results in chapters 2-5, devoted to exact solutions remained in the original form. The only change is an addition of some sections with results, obtained after 1983.

Among the main results on exact solutions in GR with scalar, electromagnetic fields and their interactions one may find here the first black hole solution with the conformal scalar field (1970), first nonsingular particle-like solution with interacting scalar, electromagnetic and gravitational fields (1979), dilaton type interaction solutions for scalar and electromagnetic fields in GR (1979), first quantum cosmological solutions with massive and massless conformal scalar fields (1971) and with the cosmological constant (1972). The latter was the first demonstration of the creation of the Universe from nothing due to appearance of the potential barrier by the cosmological constant. All these results became even more important due to recent developments in string-branes theories and cosmological discoveries. Besides, here is the first nonsingular cosmological solution due to quantum gravitational effect of spontaneous breaking of the conformal scalar field gauge symmetry in the gravitational field (1978). As a whole [3] was the first book devoted to thorough study of fields in GR, especially of the scalar field in classical and quantum cases.

Similar program of obtaining exact solutions (integrable models) of multidimensional Einstein equations in arbitrary dimensions was realized later from 1987 and results are published in books:
1. V.N. Melnikov, ”Multidimensional Classical and Quantum Cosmology and Gravitation. Exact Solutions and Variations of Constants.” CBPF-NF-051/93, Rio de Janeiro, 1993, 93 pp.; also in:

”Cosmology and Gravitation”, ed. M. Novello, Edition Frontieres, Singapore, 1994, p. 147.
2. V.N. Melnikov, ”Multidimensional Cosmology and Gravitation”, CBPF-MO-002/95, Rio de Janeiro, 1995, 210 pp.; also in:

Cosmology and Gravitation. II, ed. M. Novello, Edition Frontieres, Singapore, 1996, p. 465.
3. V.N.Melnikov. ”Exact Solutions in Multidimensional Gravity and Cosmology III.” CBPF-MO-03/02, Rio de Janeiro, 2002, 297 pp.

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Chapter 1

Problem of Fundamental Constants Variations

1.1 Introduction: gravitation as a missing link of unified models[60]

The second half of the 20th century in the field of gravitation was devoted mainly to theoretical study and experimental varification of general relativity and alternative theories of gravitation with a strong stress on relations between macro and microworld fenomena or, in other words, between classical gravitation and quantum physics. Very intensive investigations in these fields were done in Russia by M.A.Markov, K.P.Staniukovich, Y.B.Zeldovich, D.D. Ivanenko, A.D.Sakharov and their colleagues starting from mid 60's. As a motivation there were: singularities in cosmology and black hole physics, role of gravity at large and very small (planckian) scales, attempts to create a quantum theory of gravity as for other physical fields, problem of possible variations of fundamental physical constants etc. A lot of work was done along such topics as [3]:

- particle-like solutions with a gravitational field,
- quantum theory of fields in a classical gravitational background,
- quantum cosmology with fields like a scalar one,
- self-consistent treatment of quantum effects in cosmology,
- development of alternative theories of gravitation: scalar-tensor, gauge, with torsion, bimetric etc.

As all attempts to quantize general relativity in a usual manner failed and it was proved that it is not renormalizable, it became clear that the promising trend is along the lines of unification of all physical interactions which started in the 70's. About this time the experimental investigation of gravity in strong fields and gravitational waves started giving a powerful speed up in theoretical studies of such objects as pulsars, black holes, QSO's, AGN's, Early Universe etc., which continues now.

But nowadays, when we think about the most important lines of future developments in physics, we may forsee that gravity will be essential not only by itself, but as a missing cardinal link of some theory, unifying all existing physical interactions: week, strong and electromagnetic ones [60]. Even in experimental activities some crucial next generation experiments verifiing predictions of unified schemes will be important. Among them are: STEP - testing the GR corner stone-Equivalence Principle, SEE - testing the inverse square law (or new nonnewtonian interactions), EP, possible variations of the newtonian constant G with time, absolute value of G with unprecedented accuracy

[39, 57]. Of course, gravitational waves problem, verification of torsional, rotational (GPB), 2nd order and strong field effects remain important also.

We may predict as well that thorough study of gravity itself and within the unified models will give in the next century and millenium even more applications for our everyday life as electromagnetic theory gave us in the 20th century after very abstract fundamental investigations of Faraday, Maxwell, Poincare, Einstein and others, which never dreamed about such enormous applications of their works.

Other very important feature, which may be envisaged, is an increasing role of fundamental physics studies, gravitation, cosmology and astrophysics in particular, in space experiments. Unique microgravity environments and modern technology outbreak give nearly ideal place for gravitational experiments which suffer a lot on Earth from its relatively strong gravitational field and gravitational fields of nearby objects due to the fact that there is no ways of screening gravity.

In the developement of relativistic gravitation and dynamical cosmology after A. Einstein and A. Friedmann, we may notice three distinct stages: first, investigation of models with matter sources in the form of a perfect fluid, as was originally done by Einstein and Friedmann. Second, studies of models with sources as different physical fields, starting from electromagnetic and scalar ones, both in classical and quantum cases (see [3]). And third, which is really topical now, application of ideas and results of unified models for treating fundamental problems of cosmology and black hole physics, especially in high energy regimes. Multidimensional gravitational models play an essential role in the latter approach.

The necessity of studying multidimensional models of gravitation and cosmology [1, 54, 58, 2] is motivated by several reasons. First, the main trend of modern physics is the unification of all known fundamental physical interactions: electromagnetic, weak, strong and gravitational ones. During the recent decades there has been a significant progress in unifying weak and electromagnetic interactions, some more modest achievements in GUT, supersymmetric, string and superstring theories.

Now, theories with membranes, p-branes and more vague M- and F-theories are being created and studied. Having no definite successful theory of unification now, it is desirable to study the common features of these theories and their applications to solving basic problems of modern gravity and cosmology. Moreover, if we really believe in unified theories, the early stages of the Universe evolution and black hole physics, as unique superhigh energy regions, are the most proper and natural arena for them.

Second, multidimensional gravitational models, as well as scalar-tensor theories of gravity, are theoretical frameworks for describing possible temporal and range variations of fundamental physical constants [3, 4, 5, 6]. These ideas have originated from the earlier papers of E. Milne (1935) and P. Dirac (1937) on relations between the phenomena of micro- and macro-worlds, and up till now they are under thorough study both theoretically and experimentally.

Lastly, applying multidimensional gravitational models to basic problems of modern cosmology and black hole physics, we hope to find answers to such long-standing problems as singular or nonsingular initial states, creation of the Universe, creation of matter and its entropy, acceleration, cosmological constant, origin of inflation and specific scalar fields which may be necessary for its realization, isotropization and graceful exit problems, stability and nature of fundamental constants [4], possible number of extra dimensions, their stable compactification etc.

Bearing in mind that multidimensional gravitational models are certain generalizations of general relativity which is tested reliably for weak fields up to 0.001 and partially in strong fields (binary pulsars), it is quite natural to inquire about their possible observational or experimental windows. From what we already know, among these windows are:

– possible deviations from the Newton and Coulomb laws, or new interactions,

– possible variations of the effective gravitational constant with a time rate smaller than the Hubble one,

– possible existence of monopole modes in gravitational waves,

– different behaviour of strong field objects, such as multidimensional black holes, wormholes and p-branes,

– standard cosmological tests etc.

Since modern cosmology has already become a unique laboratory for testing standard unified models of physical interactions at energies that are far beyond the level of the existing and future man-made accelerators and other installations on Earth, there exists a possibility of using cosmological and astrophysical data for discriminating between future unified schemes.

As no accepted unified model exists, in our approach we adopt simple, but general from the point of view of number of dimensions, models based on multidimensional Einstein equations with or without sources of different nature:

– cosmological constant,

– perfect and viscous fluids,

– scalar and electromagnetic fields,

– their possible interactions,

– dilaton and moduli fields,

– fields of antisymmetric forms (related to p-branes) etc.

Our program's main objective was and is to obtain exact self-consistent solutions (integrable models) for these models and then to analyze them in cosmological, spherically and axially symmetric cases. In our view this is a natural and most reliable way to study highly nonlinear systems. It is done mainly within Riemannian geometry. Some simple models in integrable Weyl geometry and with torsion were studied as well.

Here we dwell mainly upon some problems of fundamental physical constants, the gravitational constant in particular, upon the SEE project shortly and exact solutions in the spherically symmetric case, black hole and PPN parameters for these solutions in particular, within a multidimensional gravity, which is a natural continuation of works presented in other sections of the book.

1.2 Fundamental physical constants and their possible variations [60]

1.2.1. In any physical theory we meet constants which characterize the stability properties of different types of matter: of objects, processes, classes of processes and so on. These constants are important because they arise independently in different situations and have the same value, at any rate within accuracies we have gained nowadays. That is why they are called fundamental physical constants (FPC) [3, 4]. It is impossible to define strictly this notion. It is because the constants, mainly dimensional, are present in definite physical theories. In the process of scientific progress some theories are replaced by more general ones with their own constants, some relations between old and new constants arise. So, we may talk not about an absolute choice of FPC, but only about a choice corresponding to the present state of the physical sciences.

Really, before the creation of the electroweak interaction theory and some Grand Unification

Models, it was considered that this *choice* is as follows:

$$c,\ \hbar,\ \alpha,\ G_F,\ g_s,\ m_p\ (\text{or}\ m_e),\ G,\ H,\ \rho,\ \Lambda,\ k,\ I, \tag{1.1}$$

where α, G_F, g_s and G are constants of electromagnetic, weak, strong and gravitational interactions, H, ρ and Λ are cosmological parameters (the Hubble constant, mean density of the Universe and cosmological constant), k and I are the Boltzmann constant and the mechanical equivalent of heat which play the role of conversion factors between temperature on the one hand, energy and mechanical units on the other. After adoption in 1983 of a new definition of the meter ($\lambda = ct$ or $\ell = ct$) this role is partially played also by the speed of light c. It is now also a conversion factor between units of time (frequency) and length, it is defined with the absolute (null) accuracy.

Now, when the theory of electroweak interactions has a firm experimental basis and we have some good models of strong interactions, a more prefarable choice is as follows:

$$\hbar,\ (c),\ e,\ m_e,\ \theta_w,\ G_F,\ \theta_c,\ \Lambda_{QCD},\ G,\ H,\ \rho,\ \Lambda,\ k,\ I \tag{1.2}$$

and, possibly, three angles of Kobayashi-Maskawa — θ_2, θ_3 and δ. Here θ_w is the Weinberg angle, θ_c is the Cabibbo angle and Λ_{QCD} is a cut-off parameter of quantum chromodynamics. Of course, if a theory of four known now interactions will be created (M-, F-or other), then we will probably have another choice. As we see, the macro constants remain the same, though in some unified models, i.e. in multidimensional ones, they may be related in some manner (see below). From the point of view of these unified models the above mentioned ones are low energy constants.

All these constants are known with different *uncertainties*. The most precisely defined constant was and remain the speed of light c: its accuracy was 10^{-10} and now it is defined with the null accuracy. Atomic constants, e, $\hbar$, m and others are determined with errors $10^{-6} \div 10^{-8}$, G up to 10^{-4} or even worse, θ_w — up to 2%; the accuracy of H is also about 2%. the situation with other cosmological parameters (FPC): mean density estimations vary within 1-2%; for Λ we have now data that its corresponding density exceeds the matter density (0.7 of the total mass).

As to the *nature* of the FPC, we may mention several approaches. One of the first hypotheses belongs to J.A. Wheeler: in each cycle of the Universe evolution the FPC arise anew along with physical laws which govern this evolution. Thus, the nature of the FPC and physical laws are connected with the origin and evolution of our Universe.

A less global approach to the nature of dimensional constants suggests that they are needed to make physical relations dimensionless or they are measures of asymptotic states. Really, the speed of light appears in relativistic theories in factors like v/c, at the same time velocities of usual bodies are smaller than c, so it plays also the role of an asymptotic limit. The same sense have some other FPC: $\hbar$ is the minimal quantum of action, e is the minimal observable charge (if we do not take into account quarks which are not observable in a free state) etc.

Finally, FPC or their combinations may be considered as natural scales determining the basic units. While the earlier basic units were chosen more or less arbitrarily, i.e., the second, meter and kilogram, now the first two are based on stable (quantum) phenomena. Their stability is believed to be ensured by the physical laws which include FPC.

Another interesting problem, which is under discussion, is why the FPC have values in a very narrow range necessary for supporting life (stability of atoms, stars lifetime etc.). There exist several possible but far from being convincing explanations [40]. First, that it is a good luck, no matter how improbable is the set of FPC. Second, that life may exist in other forms and for another FPC set, of which we do not know. Third, that all possibilities for FPC sets exist in some universe. And the last but not the least: that there is some cosmic fine tuning of FPC: some unknown physical processes bringing FPC to their present values in a long-time evolution, cycles etc.

An exact knowledge of FPC and precision measurements are necessary for testing main physical theories, extention of our knowledge of nature and, in the long run, for practical applications of fundamental theories. Within this, such theoretical problems arise:

1) development of models for confrontation of theory with experiment in critical situations (i.e. for verification of GR, QED, QCD, GUT or other unified models);

2) setting limits for spacial and temporal variations of FPC.

As to a *classification* of FPC, we may set them now into four groups according to their generality:

1) Universal constants such as $\hbar$, which divides all phenomena into quantum and nonquantum ones (micro- and macro-worlds) and to a certain extent c, which divides all motions into relativistic and non-relativistic ones;

2) constants of interactions like α, θ_w, Λ_{QCD} and G;

3) constants of elementary constituencies of matter like m_e, m_w, m_x, etc., and

4) transformation multipliers such as k, I and partially c.

Of course, this division into classes is not absolute. Many constants move from one class to another. For example, e was a charge of a particular object – electron, class 3, then it became a characteristic of class 2 (electromagnetic interaction, $\alpha = \frac{e^2}{\hbar c}$ in combination with $\hbar$ and c); the speed of light c has been in nearly all classes: from 3 it moved into 1, then also into 4. Some of the constants ceased to be fundamental (i.e. densities, magnetic moments, etc.) as they are calculated via other FPC.

As to the *number* of FPC, there are two opposite tendencies: the number of "old" FPC is usually diminishing when a new, more general theory is created, but at the same time new fields of science arise, new processes are discovered in which new constants appear. So, in the long run we may come to some minimal choice which is characterized by one or several FPC, maybe connected with the so-called Planck parameters — combinations of c, $\hbar$ and G:

$$\begin{aligned} &L = \left(\frac{\hbar G}{c^3}\right)^{1/2} \sim 10^{-33}\ \text{cm}, \\ &m_L = (c\hbar/2G)^{1/2} \sim 10^{-5}\ \text{g}, \\ &\tau_L = L/c \sim 10^{-43}\ \text{s}. \end{aligned} \tag{1.3}$$

The role of these parameters is important since m_L characterizes the energy of unification of four known fundamental interactions: strong, weak, electromagnetic and gravitational ones, and L is a scale where the classical notions of space-time loose their meaning. We do not discuss some new ideas about low effective Planck lengths trying to explain the hyerarchy problem.

1.2.2. The problem of the gravitational constant G measurement and its stability is a part of a rapidly developing field, called gravitational-relativistic metrology (GRM). It has appeared due to the growth of measurement technology precision, spread of measurements over large scales and a tendency to the unification of fundamental physical interaction [6], where main problems arise and are concentrated on the gravitational interaction.

The main subjects of GRM are:

- general relativistic models for different astronomical scales: Earth, Solar System, galaxes, cluster of galaxies, cosmology - for time transfer, VLBI, space dynamics, relativistic astrometry etc.(pioneering works were done in Russia by Arifov and Kadyev, Brumberg in 60's);

- development of generalized gravitational theories and unified models for testing their effects in experiments;

- fundamental physical constants, G in particular, and their stability in space and time;

- fundamental cosmological parameters as fundamental constants: cosmological models studies, measurements and observations;

- gravitational waves (detectors, sources...);

- basic standards (clocks) and other modern precision devices (atomic and neutron interferometry, atomic force spectroscopy etc.) in fundamental gravitational experiments, especially in space...

There are *three problems related to G*, which origin lies mainly in unified models predictions:

1) absolute G measurements,

2) possible time variations of G,

3) possible range variations of G – non-Newtonian, or new interactions.

Absolute measurements of G. There are many laboratory determinations of G with errors of the order 10^{-3} and only 4 on the level of 10^{-4}. They are (in 10^{-11} m^3 kg^{-1} s^{-2}):

1. Facy and Pontikis, France	1972 —	6,6714 ± 0.0006
2. Sagitov et al., Russia	1979 —	6,6745 ± 0.0008
3. Luther and Towler, USA	1982 —	6,6726 ± 0.0005
4. Karagioz, Russia	1988 —	6,6731 ± 0.0004

From this data it is evident that the first three experiments contradict each other (the results do not overlap within their accuracies). And only the fourth experiment is in accord with the third one.

The official CODATA value of 1986

$$G = (6,67259 \pm 0.00085) \cdot 10^{-11} \cdot m^3 \cdot kg^{-1} \cdot s^{-2} \qquad (1.4)$$

is based on the Luther and Towler determination. But after very precise later measurements of G the situation became more vague.

As it may be seen from the Cavendish conference data [75], the results of 7 groups may agree with each other only on the level 10^{-3}. Some recent and precise G-measurement [56] diverge also from CODATA values [77]:

The most recent (less 50ppm) measurements of G are (in units $10^{-11} m^3 kg^{-1} s^{-2}$):

Gundlach and Merkowitz, 2000 (USA): $G = (6.674215 \pm 0.000092)$;

Armstrong, 2002 (New Zealand): $G = (6,673870 \pm 0,000270)$;

O.B. Karagioz (Moscow, Russia, 2003): $G = (6.6729 \pm 0.0005)$;

Luo Zhun (China, Wuhan, 2009) : $G = 6.67349 \pm 0.00018$;

T. Quinn et al., 2001 (BIPM): $G = (6.6693 \pm 0,0009)$ (1), $G = (6.6689 \pm 0.0014)(2)$;

Schlamminger et.al., 2006 (Switzerland): $G = (6.674252 \pm 0.000102)$

But, from 2006 CODATA still gives: $G = 6.67428(67)$ x $10^{-11} m^3 kg^{-1} s^{-2}$.

This means that either the limit of terrestrial accuracies has been reached or we have some new physics entering the measurement procedure [6]. The first means that, maybe we should turn to space experiments to measure G [39], and and second means that a more thorough study of theories generalizing Einstein's general relativity or unified theories is necessary.

There exist also some satellite determinations of G (namely $G \cdot M_{\text{Earth}}$) on the level of 10^{-9} and several less precise geophysical determinations in mines. Should we know G better, the mass of the Earth will be defined better accordingly.

The precise knowledge of G is necessary, first of all, as it is a FPC; next, for the evaluation of mass of the Earth, planets, their mean density and, finally, for construction of Earth models; for transition from mechanical to electromagnetic units and back; for evaluation of other constants through relations between them given by unified theories; for finding new possible types of interactions and geophysical effects; for some practical applications like increasing of modern gradiometers precision, as they demand a calibration by a gravitational field of a standard body depending on G: high accuracy of their calibration (10^{-5} - 10^{-6}) requires the same accuracy of G.

The knowledge of constants values has not only a fundamental meaning but also a metrological one. The modern system of standards is based mainly on stable physical phenomena. So, the stability of constants plays a crucial role. As all physical laws were established and tested during the last 2-3 centuries in experiments on the Earth and in the near space, i.e. at a rather short space and time intervals in comparison with the radius and age of the Universe, the possibility of slow *variations* of constants (i.e. with the rate of the evolution of the Universe or slower) cannot be excluded a priori.

So, the assumption of absolute stability of constants is an extrapolation and each time we must test it.

1.2.3. *Time Variations of G.* The problem of variations of FPC arose with the attempts to explain the relations between micro- and macro-world phenomena. Dirac was the first to introduce (1937) the so-called “Large Numbers Hypothesis” which relates some known very big (or very small) numbers with the dimensionless age of the Universe $T \sim 10^{40}$ (age of the Universe in seconds 10^{17}, divided by the characteristic elementary particle time 10^{-23} seconds). He suggested (after Milne in 1935) that the ratio of the gravitational to strong interaction strengths, $Gm_p^2/\hbar c \sim 10^{-40}$, is inversely proportional to the age of the Universe: $Gm_p^2/\hbar c \sim T^{-1}$. Then, as the age varies, some constants or their combinations must vary as well. Atomic constants seemed to Dirac to be more stable, so he chose the variation of G as T^{-1}.

After the original *Dirac hypothesis* some new ones appeared and also some generalized *theories* of gravitation admitting the variations of an effective gravitational coupling. We may single out three stages in the development of this field:

1. Study of theories and hypotheses with variations of FPC, their predictions and confrontation with experiments (1937-1977).

2. Creation of theories admitting variations of an effective gravitational constant in a particular system of units, analyses of experimental and observational data within these theories [1, 3] (1977-present).

3. Analyses of FPC variations within unified models [6, 4, 1] (present).

Within the development of the first stage from the analysis of the whole set of existed astronomical, astrophysical, geophysical and laboratory data, a conclusion was made [1, 2] that variations of atomic constants are excluded, but variations of the effective gravitational constant in the atomic system of units do not contradict the available experimental data on the level 10^{-12} y^{-1} and less. Moreover, in [30, 31] the conception was worked out that variations of constants are not absolute but depend on the system of measurements (choice of standards, units and devices using this or that fundamental interaction). Each fundamental interaction through dynamics, described by the corresponding theory, defines the system of units and the system of basic standards.

Earlier reviews of some hypotheses on variations of FPC and experimental tests can be found in [3, 4].

Following Dyson (1972), we can introduce dimensionless combinations of micro- and macro-constants:

$$\alpha = e^2/\hbar c = 7,3 \cdot 10^{-3}, \qquad \gamma = Gm^2/\hbar c = 5 \cdot 10^{-39},$$
$$\beta = G_F\ m^2 c/\hbar^3 = 9 \cdot 10^6, \qquad \delta = H\hbar/mc^2 = 10^{-42},$$
$$\varepsilon = \rho G/H^2 = 2 \cdot 10^{-3}, \qquad t = T/(e^2/mc^3) \approx 10^{40}$$

We see that α, β and ε are of order 1 and γ and δ are of the order 10^{-40}. Nearly all existing hypotheses on variations of FPC may be represented as follows:

Hyposesis 1 (standard):

$\alpha,\ \beta,\ \gamma$ are constant, $\delta \sim t^{-1},\ \varepsilon \sim t$.

Here we have no variations of G while δ and ε are determined by cosmological solutions.

Hyposesis 2 (Dirac):

$\alpha,\ \beta,\ \varepsilon$ are constant, $\gamma \sim t^{-1},\ \delta \sim t^{-1}$.

Then $\dot{G}/G = 5 \cdot 10^{-11}\text{year}^{-1}$ if the age of the Universe is taken to be $T = 2. \cdot 10$ years.

Hyposesis 3 (Gamow):

$\gamma/\alpha = Gm^2/e^2 \sim 10^{-37}$, so e^2 or α are varied, but not $G,\ \beta,\ \gamma;\ \varepsilon = \text{const},\ \alpha \sim t^{-1},\ \delta \sim t^{-1}$.

Then $\dot{\alpha}/\alpha = 10^{-10}\ \text{year}^{-1}$.

Hyposesis 4 (Teller):

trying to account also for deviations of α from 1, he suggested $\alpha^{-1} = \ln \gamma^{-1}$.

Then $\beta,\ \varepsilon$ are constants, $\gamma \sim t^{-1},\ \alpha \sim (\ln t)^{-1},\ \delta \sim t^{-1}$

$$\dot{\alpha}/\alpha = 5 \cdot 10^{-13}\text{year}^{-1}. \tag{1.5}$$

The same relation for α and γ was used also by Landau, DeWitt, Staniukovich, Terasawa and others, but in approaches other than Teller's.

Some other variants may be also possible, e.g. the Brans-Dicke theory with $G \sim t^{-r},\ \rho \sim t^{r-2}$, $r = [2 + 3\omega/2]^{-1}$, a combination of Gamow's and Brans-Dicke etc. [3].

1.2.4. There are different astronomical, geophysical and laboratory *data* on possible variations of FPC.

Astrophysical data:

a) from comparison of fine structure ($\sim \alpha^2$) and relativistic fine structure ($\sim \alpha^4$) shifts in spectra of radio galaxies, Bahcall and Schmidt (1967) obtained

$$|\dot{\alpha}/\alpha| \leq 2 \cdot 10^{-12}\ \text{year}^{-1}; \tag{1.6}$$

b) comparing lines in optical ($\sim Ry = me^4/\hbar^2$) and radio bands of the same sources in galaxies Baum and Florentin-Nielsen (1976) got the estimate

$$|\dot{\alpha}/\alpha| \leq 10^{-13}\ \text{year}^{-1}, \tag{1.7}$$

and for extragalactic objects

$$|\dot{\alpha}/\alpha| \leq 10^{-14}\ \text{year}^{-1}; \tag{1.8}$$

c) from observations of superfine structure in H-absorption lines of the distant radiosource Wolf et al. (1976) obtained that

$$|\alpha^2(m_e/m_p)g_p| < 2 \cdot 10^{-14} \; ; \tag{1.9}$$

from these data it is seen that Hyposeses 3 and 4 are excluded. Recent data only strengthen this conclusion. Comparing the data from absorption lines of atomic and molecular transition spectra in high redshifts QSO's, Varshalovich and Potekhin, Russia, [41] obtained for $z = 2.8 - 3.1$:

$$|\dot{\alpha}/\alpha| \leq 1.6 \cdot 10^{-14} \text{ year}^{-1} \tag{1.10}$$

and Drinkwater et al. [43]:

$$|\dot{\alpha}/\alpha| \leq 10^{-15} \text{ year}^{-1} \text{ for } z = 0.25 \tag{1.11}$$

and

$$|\dot{\alpha}/\alpha| \leq 5 \cdot 10^{-16} \text{ year}^{-1} \text{ for } z = 0.68 \tag{1.12}$$

for a model with zero deceleration parameter and $H = 75\text{km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$.

The same conclusion is made on the bases of *geophysical data*. Indeed,

a) α-decay of $U_{238} \to Pb_{208}$. Knowing abundancies of U_{238} and P_{238} in rocks and independently the age of these rocks, one obtains the limit

$$|\dot{\alpha}/\alpha| \leq 2 \cdot 10^{-13} \text{ year}^{-1}; \tag{1.13}$$

b) from spontaneous fission of U_{238} such an estimation was made:

$$|\dot{\alpha}/\alpha| \leq 2,3 \cdot 10^{-13} \text{ year}^{-1}. \tag{1.14}$$

c) finally, β-decay of Re_{187} to Os_{187} gave:

$$|\dot{\alpha}/\alpha| \leq 5 \cdot 10^{-15} \text{ year}^{-1} \tag{1.15}$$

We must point out that all astronomical and geophysical estimations are strongly model-dependent. So, of course, it is always desirable to have *laboratory tests* of variations of FPC.

a) Such a test was first done by the Russian group in the Committee for Standards (Kolosnitsyn, 1975). Comparing rates of two different types of clocks, one based on a Cs standard and another on a beam molecular generator, they found that

$$|\dot{\alpha}/\alpha| \leq 10^{-10} \text{ year}^{-1}. \tag{1.16}$$

b) From a similar comparison of a Cs standard and SCCG (Super Conducting Cavity Generator) clocks rates Turneaure et al. (1976) obtained the limit

$$|\dot{\alpha}/\alpha| \leq 4.1 \cdot 10^{-12} \text{ year}^{-1}. \tag{1.17}$$

c) Later, data were obtained by J. Prestage et al. [44] by comparing mercury and H-maser clocks. Their result is

$$|\dot{\alpha}/\alpha| \leq 3.7 \cdot 10^{-14} \text{ year}^{-1}. \tag{1.18}$$

More recent great progress is observed in atomic clocks. Measurement of optical clocks frequency ratios with single Al^+ and Hg^+ ions, 2008, gave relative variation of α at the level (see [78])

$$|\dot{\alpha}/\alpha| = (-1.6 \pm 2.3) \cdot 10^{-17} \text{ year}^{-1} \tag{1.19}$$

(the same order result as from Oklo data at z, corresponding to the epoch 10^9 years ago). These data at different z cannot be compared directly. For that one needs the full universe evolution.

All these limits were placed on the fine structure constant variations. From the analysis of decay rates of K_{40} and Re_{187}, a limit on possible variations of the *weak interaction constant* was obtained (see approach for variations of β, e.g. in [33])

$$|\dot{\beta}/\beta| \leq 10^{-10} \text{ year}^{-1}. \tag{1.20}$$

But the most strict data were obtained by A. Schlyachter in 1976 (Russia) from an analysis of the ancient natural nuclear reactor data in Gabon, Oklo, because the event took place $2 \cdot 10^9$ years ago. They are the following:

$$\begin{aligned} &|\dot{G}_s/G_s| < 5 \cdot 10^{-19} \text{ year}^{-1}, \\ &|\dot{\alpha}/\alpha| < 10^{-17} \text{ year}^{-1}, \\ &|\dot{G}_F/G_F| < 2 \cdot 10^{-12} \text{ year}^{-1}. \end{aligned} \tag{1.21}$$

Quite recently Damour and Dyson [42] repeated this analysis in more detail and gave more cautious results:

$$|\dot{\alpha}/\alpha| \leq 5 \cdot 10^{-17} \text{ year}{-1} \tag{1.22}$$

and

$$|\dot{G}_F/G_F| < 10^{-11} \text{ year}^{-1}. \tag{1.23}$$

So, we really see that all existing hyposeses with variations of atomic constants are excluded. We do not discuss here some new results on possible variation of α [76] as they are not confirmed and critisized.

1.2.5. Now we still have no unified theory of all four interactions. So it is possible to construct systems of measurements based on any of these four interactions. But practically it is done now on the basis of the mostly worked out theory — on electrodynamics (more precisely on QED). Of course, it may be done also on the basis of the gravitational interaction (as it was partially earlier). Then, different units of basic physical quantities arise based on dynamics of the given interaction, i.e. the atomic (electromagnetic) second, defined via frequency of atomic transitions or the gravitational second defined by the mean Earth motion around the Sun (ephemeris time).

It does not follow from anything that these two seconds are always synchronized in time and space. So, in principal they may evolve relative to each other, for example at the rate of the evolution of the Universe or at some slower rate.

That is why, in general, variations of the gravitational constant are possible in the atomic system of units (c, $\hbar$, m are constant) and masses of all particles — in the gravitational system of units (G, $\hbar$, c are constant by definition). Practically we can test only the first variant since the modern basic standards are defined in the atomic system of measurements. Possible variations of FPC must be tested experimentally but for this it is necessary to have the corresponding theories admitting such variations and their certain effects.

Mathematically these systems of measurement may be realized as conformally related metric forms. Arbitrary conformal transformations give us a transition to an arbitrary system of measurements.

We know that scalar-tensor and multidimensional theories are corresponding frameworks for these variations. So, one of the ways to describe variable gravitational coupling is the introduction of a *scalar field* as an additional variable of the gravitational interaction. It may be done by different means (e.g. Jordan, Brans-Dicke, Canuto [32] and others). We have suggested a variant of gravitational theory with a conformal scalar field (Higgs-type field [3, 31]) where Einstein's general relativity may be considered as a result of spontaneous symmetry breaking of conformal symmetry (Domokos, 1976) [3]. In our variant spontaneous symmetry breaking of the global gauge invariance leads to a nonsingular cosmology [35]. Besides, we may get variations of the effective gravitational constant in the atomic system of units when m, c, $\hbar$ are constant and variations of all masses in the gravitational system of units (G, c, $\hbar$ are constant). It is done on the basis of approximate [36] and exact cosmological solutions with local inhomogenity [37].

The effective gravitational constant is calculated using the equations of motions. Post-Newtonian expansion is also used in order to confront the theory with existing experimental data. Among the post-Newtonian parameters the parameter f describing variations of G is included. It is defined as

$$\frac{1}{GM}\frac{d(GM)}{dt} = fH. \tag{1.24}$$

According to Hellings' data [38] from the Viking mission,

$$\tilde{\gamma} - 1 = (-1.2 \pm 1.6) \cdot 10^3, \qquad f = (4 \pm 8) \cdot 10^{-2}. \tag{1.25}$$

In the theory with a conformal Higgs field [36, 37] we obtained the following relation between f and $\tilde{\gamma}$:

$$f = 4(\tilde{\gamma} - 1). \tag{1.26}$$

Using Hellings' data for $\tilde{\gamma}$, we can calculate in our variant f and compare it with f from [18]. Then we get $f = (-9,6 \pm 12,8) \cdot 10^{-3}$ which agrees with (1.24) within its accuracy.

We used here only Hellings' data on variations of G. But the situation with experiment and observations is not so simple. Along with [38], there are some other data [3, 4]:

1. From the growth of corals, pulsar spin down, etc. on the level

$$|\dot{G}/G| < 10^{-10} \div 10^{-11}\ \text{year}^{-1}. \tag{1.27}$$

2. Van Flandern's positive data (though not confirmed and critisized) from the analysis of lunar mean motion around the Earth and ancient eclipses data (1976, 1981):

$$|\dot{G}/G| = (6 \pm 2)10^{-11} \text{ year}^{-1}. \quad (1.28)$$

3. Reasenberg's estimates (1987) of the same Viking mission as in [38]:

$$|\dot{G}/G| < (0 \pm 2) \cdot 10^{-11} \text{ year}^{-1} \quad (1.29)$$

4. Hellings' result in the same form is

$$|\dot{G}/G| < (2 \pm 4) \cdot 10^{-12} \text{ year}^{-1}. \quad (1.30)$$

5. A result from nucleosythesis (Acceta et al., 1992):

$$|\dot{G}/G| < (\pm 0.9) \cdot 10^{-12} year^{-1}. \quad (1.31)$$

6. E.V.Pitjeva's result [55], Russia (1997), based on satellites and planets motion:

$$|\dot{G}/G| < (0 \pm 2) \cdot 10^{-12} year^{-1} \quad (1.32)$$

and

$$|\dot{G}/G| < 5 \cdot 10^{-13} year^{-1}, 2007. \quad (1.33)$$

The most reliable ones are based on lunar laser ranging (Muller et al, 1993; Williams et al, 1996; Nordtvedt, Erice, 2003). They are not better than 10^{-12} or $5 \cdot 10^{-13}$ per year (Pitjeva). Here, once more we see that there is a need for corresponding theoretical and experinmental studies. Probably, future space missions like Earth SEE-satellite [39], STEP or missions to other planets like PHOBOS-GRUNT-2009 (Russia-China, Russian Zenit rocket with two satellites: Russian lander and Chinese orbiter, 800-80000 km) and lunar laser ranging will be decisive steps in solving the problem of temporal variations of G and determining the fates of different theories which predict them, since the greater is the time interval between successive masurements and, of course, the more precise they are, the more stringent results will be obtained.

As we saw, different theoretical schemes lead to temporal variations of the effective gravitational constant:

1. Empirical models and theories of Dirac type, where G is replaced by $G(t)$.

2. Numerous scalar-tensor theories of Jordan-Brans-Dicke type where G depending on the scalar field $\sigma(t)$ appears.

3. Gravitational theories with a conformal scalar field arising in different approaches [3, 31].

4. Multidimensional unified theories in which there are dilaton fields and effective scalar fields appearing in our 4-dimensional spacetime from additional dimensions [19, 1]. They may help also in solving the problem of a variable cosmological constant from Planckian to present values.

As was shown by Marciano and in [4, 8, 43, 1] temporal variations of FPC are connected with each other in *multidimensional models* of unification of interactions. So, experimental tests on $\dot{\alpha}/\alpha$ may at the same time be used for estimation of $\dot{G}/G$ and vice versa. Moreover, variations of G are related also to the cosmological parameters ρ, Ω and q [8] which gives opportunities of raising the precision of their determination.

As variations of FPC are closely connected with the behaviour of internal scale factors, it is a direct probe of properties of extra dimensions and the corresponding unified theories [7, 8, 1], as well as gravitation and cosmological models [79].

1.2.6. *Non-Newtonian interactions, or range variations of G.* Nearly all modified theories of gravity and unified theories predict also some deviations from the Newton law (inverse square law, ISL) or composition-dependent violations of the Equivalence Principle (EP) due to appearance of new possible massive particles (partners) [4]. Experimental data exclude (at some level) the existence of these particles at nearly all ranges except less than *millimeter* and also at *meters and hundreds of meters* ranges. Of course, these results need to be verified in other independent experiments, probably in space ones [39].

In the Einstein theory G is a true constant. But, if we think that G may vary with time, then, from a relativistic point of view, it may vary with distance as well. In GR massless gravitons are mediators of the gravitational interaction, they obey second-order differential equations and interact with matter with a constant strength G. If any of these requirements is violated, we come in general to deviations from the Newton law with range (or to generalization of GR).

In [5] we analyzed several classes of such theories:

1. Theories with massive gravitons like bimetric ones or theories with a Λ-term.
2. Theories with an effective gravitational constant like the general scalar-tensor ones.
3. Theories with torsion.
4. Theories with higher derivatives (4th-order equations etc.), where massive modes appear leading to short-range additional forces.
5. More elaborated theories with other mediators besides gravitons (partners), like supergravity, superstrings, M-theory etc.
6. Theories with nonlinearities induced by any known physical interactions (Born-Infeld etc.)
7. Phenomenological models where the detailed mechanism of deviation is not known (fifth or other force).

In all these theories some effective or real masses appear leading to Yukawa-type deviation from the Newton law, characterized by strength and range.

There exist some model-dependant estimations of these forces. The most well-known one belongs to Scherk (1979) from supergravity where the graviton is accompanied by a spin-1 partner (graviphoton) leading to an additional repulsion. Other models were suggested by Moody and Wilczek (1984) – introduction of a pseudo-scalar particle – leading to an additional attraction between macro-bodies with the range $2 \cdot 10^{-4}$ cm $< \lambda <$ 20 cm and strength α from 1 to 10^{-10} in this range. Another supersymmetric model was elaborated by Fayet (1986, 1990), where a spin-1 partner of a massive graviton gives an additional repulsion in the range of the order 10^3 km and α of the order 10^{-13}.

A scalar field to adjust Λ was introduced also by S. Weinberg in 1989, with a mass smaller than 10^{-3}eV$/c^2$, or a range greater than 0.1 mm. One more variant was suggested by Peccei, Sola and Wetterich (1987) leading to additional attraction with a range smaller than 10 km. Some p-brane models also predict non-Newtonian additional interactions in the mm range, what is intensively discussed nowadays. About PPN parameters for multidimensional models with p-branes see below.

1.2.7. SEE - Project

We saw that there are three problems connected with G. There is a promising new multi-purpose space experiment SEE - Satellite Energy Exchange [39], which addresses all these problems and may be more effective in solving them than other laboratory or space experiments.

This experiment is based on a limited 3-body problem of celestial mechanics: small and large masses in a drag-free satellite and the Earth. Unique horse-shoe orbits, which are effectively one-dimensional, are used in it.

The aims of the SEE-project are to measure: Inverse Square Law (ISL) and Equivalence Principle (EP) at ranges of meters and the Earth radius, G-dot and the absolute value of G with unprecedented accuracies.

We studied some aspects of the SEE-project [57] :

1. Wide range of trajectories with the aim of finding optimal ones:
- circular in spherical field;
- the same plus Earth quadrupole modes;
- elliptic with eccentricity less than 0.05.
2. Estimations of other celestial bodies influence.
3. Estimation of relative influence of trajectories to changes in G and α.
4. Modelling measurement procedures for G and α by different methods, for different ranges and for different satellite altitudes: optimal - 1500 km, ISS free flying platform - 500 km and also for 3000 km.
5. Estimations of some sources of errors:
- radial oscillations of the shepherd's surface;
- longitudal oscillations of the capsule;
- transversal oscillations of the capsule;
- shepherd's nonsphericity;
- limits on the quadrupole moment of the shepherd;
- limits on addmissible charges and time scales of charging by high energy particles etc.
6. Error budgets for G, G-dot and $G(r)$.

The general conclusion is that the SEE-project may really improve our knowledge of G and $G(r)$ values by 3-4 orders and G-dot about one order better than we have nowadays.

1.3 Multidimensional Models [1,2,54,58,73,M4]

The history of the multidimensional approach begins with the well-known papers of T.K. Kaluza and O. Klein on 5-dimensional theories which opened an interest to investigations in multidimensional gravity. These ideas were continued by P. Jordan who suggested to consider the more general case $g_{55} \neq$ const leading to a theory with an additional scalar field. They were in some sense a source of inspiration for C. Brans and R.H. Dicke in their well-known work on a scalar-tensor gravitational theory. After their work a lot of investigations have been performed using material or fundamental scalar fields, both conformal and non-conformal (see details in [3] and other sections of this book).

A revival of ideas of many dimensions started in the 70's and continues now. It is completely due to the development of unified theories. In the 70's an interest to multidimensional gravitational models was stimulated mainly by (i) the ideas of gauge theories leading to a non-Abelian generalization of the Kaluza-Klein approach and (ii) by supergravitational theories. In the 80's the supergravitational theories were "replaced" by superstring models. Now it is heated by expectations connected with the overall M-theory. In all these theories, 4-dimensional gravitational models with extra fields were obtained from some multidimensional model by dimensional reduction based on the decomposition of the manifold

$$M = M^4 \times M_{\rm int}, \tag{1.34}$$

where M^4 is a 4-dimensional manifold and $M_{\rm int}$ is some internal manifold (mostly considered to be compact earlier, but now variants with large space dimensions are rather popular).

The earlier papers on multidimensional gravity and cosmology dealt with multidimensional Einstein equations and with a block-diagonal cosmological or spherically symmetric metric defined on the manifold $M = R \times M_0 \times \cdots \times M_n$ of the form

$$g = -dt \otimes dt + \sum_{r=0}^{n} a_r^2(t) g^r \tag{1.35}$$

where (M_r, g^r) are Einstein spaces, $r = 0, \ldots, n$. In some of them a cosmological constant and simple scalar fields were also used [15].

Such models are usually reduced to pseudo-Euclidean Toda-like systems with the Lagrangian

$$L = \frac{1}{2} G_{ij} \dot{x}^i \dot{x}^j - \sum_{k=1}^{m} A_k \mathrm{e}^{u_i^k x^i} \tag{1.36}$$

and the zero-energy constraint $E = 0$.

It should be noted that pseudo-Euclidean Toda-like systems are not well-studied yet. There exists a special class of equations of state that gives rise to Euclidean Toda models [9].

Cosmological solutions are closely related to solutions with spherical symmetry [16]. Moreover, the scheme of obtaining the latter is very similar to the cosmological approach [1]. The first multidimensional generalization of such type was considered by D. Kramer and rediscovered by A.I. Legkii, D.J. Gross and M.J. Perry (and also by Davidson and Owen). In [52] the Schwarzschild solution was generalized to the case of n internal Ricci-flat spaces and it was shown that a black hole configuration takes place when the scale factors of internal spaces are constants. It was shown there also that a minimally coupled scalar field is incompatible with the existence of black holes. In [10] an analogous generalization of the Tangherlini solution was obtained, and an investigation of singularities was performed in [26]. These solutions were also generalized to the electrovacuum case with and without a scalar field [11, 13, 12]. Here, it was also proved that BHs exist only when a scalar field is switched off. Deviations from the Newton and Coulomb laws were obtained depending on mass, charge and number of dimensions. In [12] spherically symmetric solutions were obtained for a system of scalar and electromagnetic fields with a dilaton-type interaction and also deviations from the Coulomb law were calculated depending on charge, mass, number of dimensions and dilaton coupling. Multidimensional dilatonic black holes were singled out. A theorem was proved in [12] that "cuts" all non-black-hole configurations as being unstable under even monopole perturbations. In [14] the extremely charged dilatonic black hole solution was generalized to a multicenter (Majumdar-Papapetrou) case when the cosmological constant is non-zero.

We note that for $D = 4$ the pioneering Majumdar-Papapetrou solutions with a conformal scalar field and an electromagnetic field were considered in [24].

At present there exists a special interest to the so-called M- and F-theories etc. These theories are “supermembrane” analogues of the superstring models in $D = 11, 12$ etc. The low-energy limit of these theories leads to models governed by the Lagrangian

$$\mathcal{L} = R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a \in \Delta} \frac{\theta_a}{n_a!} \exp[2\lambda_a(\varphi)](F^a)^2, \tag{1.37}$$

where g is a metric, $F^a = dA^a$ are forms of rank $F^a = n_a$, and φ^α are scalar fields.

In [19] it was shown that, after dimensional reduction on the manifold $M_0 \times M_1 \times \cdots \times M_n$ and when the composite p-brane ansatz is considered, the problem is reduced to the gravitating self-interacting σ-model with certain constraints. For electric p-branes see also [17, 18, 20] (in [20] the composite electric case was considered). This representation may be considered as a powerful tool for obtaining different solutions with intersecting p-branes (analogs of membranes). In [46, 29] Majumdar-Papapetrou type solutions were obtained (for the non-composite electric case see [17, 18] and for the composite electric case see [20]). These solutions correspond to Ricci-flat (M_i, g^i), $i = 1, \dots, n$ and were generalized to the case of Einstein internal spaces [19]. The obtained solutions take place when certain *orthogonality relations* (on couplings parameters, dimensions of “branes”, total dimension) are imposed. In this situation a class of cosmological and spherically symmetric solutions was obtained [27]. Special cases were also considered in [22]. Solutions with a horizon were considered in detail in [21, 27]. In [21, 28] some propositions related to (i) interconnection between the Hawking temperature and the singularity behaviour, and (ii) to multitemporal configurations were proved.

It should be noted that multidimensional and multitemporal generalizations of the Schwarzschild and Tangherlini solutions were considered in [13, 25], where the generalized Newton formulas in a multitemporal case were obtained.

We note also that there exists a large variety of Toda solutions (open or closed) when certain intersection rules are satisfied [27].

We continued our investigations of p-brane solutions based on the sigma-model approach in [46, 18, 20]. (For the pure gravitational sector see [17, 45, 46].)

We found a family of solutions depending on one variable describing the (cosmological or spherically symmetric) “evolution” of $(n + 1)$ Einstein spaces in the theory with several scalar fields and forms. When an electro-magnetic composite p-brane ansatz is adopted, the field equations are reduced to the equations for a Toda-like system.

In the case when n “internal” spaces are Ricci-flat, one space M_0 has a non-zero curvature, and all p-branes do not “live” in M_0, we found a family of solutions to the equations of motion (equivalent to equations for Toda-like Lagrangian with zero-energy constraint [27]) if certain *block-orthogonality relations* on p-brane vectors U^s are imposed. These solutions generalize the solutions from [27] with an orthogonal set of vectors U^s. A special class of “block-orthogonal” solutions (with coinciding parameters ν_s inside blocks) was considered earlier in [28].

We considered a subclass of spherically symmetric solutions. This subclass contains non-extremal p-brane black holes for zero values of “Kasner-like” parameters. A relation for the Hawking temperature was presented (in the black hole case).

We also calculated the Post-Newtonian Parameters β and γ (Eddington parameters) for general spherically symmetric solutions and black holes in particular [54]. These parameters depending on

p-brane charges, their worldvolume dimensions, dilaton couplings and number of dimensions may be useful for possible physical applications.

Some specific models in classical and quantum multidimensional cases with p-branes were analysed in [47]. Exact solutions for the system of scalar fields and fields of forms with a dilatinic type interactions for *generalized intersection rules* were studied in [48], where the PPN parameters were also calculated.

Finally, a *stability* analysis for solutions with p-branes was carried out [49]. It was shown there that for some simple p-brane systems multidimensional black branes are stable under monopole perturbations while other (non-BH) sperically symmetric solutions turned out to be unstable. Other problems of multidimensional cosmology and gravitation with different matter sources see in [73].

Now we return to our earlier investigations for systems of fields in 4 dimensions (exact solutions and their studies).

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1.4 Theory of Gravitation with Conformal Higgs Scalar Field [67]

The model. The theory of gravitation with the fundamental conformal invariant scalar field without a self-interaction was suggested and studied in [67, 68]. Its relation to the LNH was analyzed in [69]. Later as its generalization the variant with conformal Higgs field was suggested [35] which reduces to [67, 68] at sufficiently large characteristic sizes of systems under consideration. Following [35] let us take the Lagrangian in the form

$$L = L_g + L_\varphi + L_m \tag{1.38}$$

where $L_g = (R - 2\Lambda)/2\varkappa$ is the Einstein-Hilbert gravitational field Lagrangian with the cosmological constant Λ and

$$L_\varphi = g^{\alpha\beta}\nabla_\alpha\varphi^*\nabla_\beta\varphi - (m^2 + R/6)\varphi^*\varphi - (\lambda/6)(\varphi^*\varphi)^2 \tag{1.39}$$

is the scalar field Lagrangian conformally invariant for $m = 0,\ c = \hbar = 1$. L_m is the Lagrangian of other fields and macroscopic matter including also interaction Lagrangians. $\varkappa$ is the Einstein gravitational constant. In particular, $L_\varphi + L_m$ may be the Weinberg-Salam or GUT or other unified theories Lagrangians written in a covariant form. The role of quantum vacuum effects in such a theory will be investigated in Chapter 5. In particular we shall show there that the Einstein-Hilbert term may not be introduced explicitly: it appears due to the spontaneous symmetry breaking of the conformal symmetry. It will be shown also there that the spontaneous symmetry breaking of the gauge symmetry leads to the absence of a singular state in the homogeneous and isotropic cosmology. Here we dwell upon another feature of this theory, namely on a possibility of explanation of a possible variation of the effective gravitational constant with time in the atomic system of units.

Conformal Transformations and Variations of Constants [31, 69]. It was stressed by many authors that the LNH couldn't be extrapolated to early stages of the Universe Evolution. At sufficiently large radii of the Universe $a \gg m^{-1}$ in (1.37) it is possible to ignore the self-interaction of a scalar field. Then, (1.38) will belong to the usual class of scalar-tensor theories of the type:

$$L = [(R/\varkappa)\ h(\varphi) + (1/2)\ l(\varphi)\ \varphi_{,\mu}\ \varphi^{,\mu} + \Lambda(\varphi) + L_m]\sqrt{-g} \tag{1.40}$$

where $h(\varphi)$ and $l(\varphi)$ are arbitrary functions of φ, Λ is a variable cosmological term. The particular choice of functions $h(\varphi)$ and $l(\varphi)$ fixes the definite variant of a scale-tensor theory of gravitation. For example, in theories with a broken conformal invariance

$$h(\varphi) = 1 \pm (\varkappa/6)\ \varphi^2,\ l(\varphi) =\ \pm 1\ ,$$

where (+) sign corresponds to a negative energy of a scalar field and (-) sign corresponds to the case of the conformally invariant (m=0) scalar field with a positive energy contribution [67]. Experimental consequences of such theories describied by (1.40) were studied in detail in [72]. In particular in [68] and [72] exact spherically symmetric solutions corresponding to (1.40) were found and post-newtonian parameters were calculated in [72]. Classical tests of GR define values of three parameters for different variants of scalar-tensor theories. Namely, the effect of a light ray or a radio wave deviation in the gravitational field of an isolated body (i.e. of the Sun) depends crucially on a value of φ_0 which is the cosmological value of the φ-field.

Theories of gravitation described by the Lagrangian (1.40) correspond to variable gravitational constant. It is the so called atomic or Jordan-Brans-Dicke (JBD) frame. Let us perform a conformal transformation

$$g_{\mu\nu} = h^{-1}(\varphi)\ \overline{g}_{\mu\nu} \tag{1.41}$$

Then, for $g^{\mu\nu}$, $\sqrt{-g}$ and R we shall have

$$\begin{aligned} g^{\mu\nu} &= h(\varphi)\ \overline{g}^{\mu\nu};\ \sqrt{-g} = h^{-2}(\varphi)\ \sqrt{-\overline{g}}; \\ R &= h(\varphi)\ [\overline{R} + 3\ \overline{\Box}\ \ln\ h(\varphi) - (3/2)\ \{\ln\ h(\varphi)\}_{,\mu} \\ &\quad \times \{(\ln\ h(\varphi)\}^{,\mu}]\ , \end{aligned}$$

where $\overline{R}$ and $\overline{\Box}$ are calculated in the metrie $\overline{g}_{\mu\nu}$. If we suppose that $\Lambda(\varphi) = 0$, then (1.40) is transformed to the so called quasi-Einstein form (or Einstein, or gravitational frame):

$$\overline{L} = [\overline{R}/2\varkappa + (1/2)\ \Phi(\varphi)_{,\mu}\ \varphi_{,\nu}\ \overline{g}^{\mu\nu} + h^{-2}(\varphi)\ L_m]\ \sqrt{-\overline{g}}\ , \tag{1.42}$$

where

$$\Phi(\varphi) = \left[\frac{l(\varphi)}{R(\varphi)} - \frac{3}{2\varkappa}\frac{h'^2(\varphi)}{h^2(\varphi)}\right];\ h'(\varphi) \equiv dh/d\varphi.$$

The following field equations correspond to the Lagrangian (1.42):

$$\overline{G}_{\mu\nu} \equiv \overline{R}_{\mu\nu} - (1/2)\ \overline{g}_{\mu\nu}\ \overline{R} = \varkappa\overline{T}_{\mu\nu} - \overline{\Phi}(\varphi)\overline{\tau}_{\mu\nu}; \tag{1.43}$$

$$\overline{\Box}\varphi + (1/2)\ \varphi_{,\mu}\ \varphi^{,\mu}\frac{d\ \ln\ \Phi(\varphi)}{d\varphi} = \frac{1}{\overline{\Phi}(\varphi)}\frac{\partial\sqrt{-\overline{g}}\ \overline{L}_m}{\sqrt{-\overline{g}}\ \partial\varphi}, \tag{1.44}$$

where $\overline{\tau}_{\mu\nu}$ is a usual energy-momentum tensor of a minimally coupled scalar field and $\overline{\Phi}(\varphi) = \varkappa\ \Phi(\varphi);\ \overline{L}_m = h^{-2}(\varphi)\ L_m$.

It is known [74] that under the scalar field transformation

$$\psi = \psi(\varphi);\ \left|\frac{d\psi}{d\varphi}\right|^2 = \left|\frac{l(\varphi)}{h(\varphi)} - \frac{3}{2\varkappa}\ \frac{h'^2(\varphi)}{h^2(\varphi)}\right| \tag{1.45}$$

the Lagrangian (1.42) is reduced to

$$\mathcal{L} = [\overline{R}/2\varkappa + (\alpha/2)\ \psi_{,\mu}\psi^{,\mu} + h^{-2}L_m]\sqrt{-\overline{g}}, \tag{1.46}$$

with

$$\alpha = \text{sgn}\left[\frac{l(\varphi)}{h(\varphi)} - \frac{3}{2\varkappa}\frac{h'^2(\varphi)}{h^2(\varphi)}\right], \tag{1.47}$$

i.e. to the usual Einstein form with the exception that matter is interacting now also directly with the scalar field, but not only via the metric.

Let us consider particular cases of scalar-tensor theories of gravitation with broken conformal invariance, due to R-term. In respect to conformally invariant theories they have an advantage that the scalar field is a truly dynamical one, so the relation between different systems of measurements (frames) is quite definite. Then, for $h^+(\varphi)$ and $l^+(\varphi)$, transforming scalar field according to

$$\psi(\varphi) = \sqrt{6/\varkappa}\ \text{arctg}(\sqrt{\varkappa/6}\ \varphi) \tag{1.48}$$

we shall have:

$$L = (\overline{R} + \varkappa\psi_{,\mu}\psi^{,\mu} + 3\varkappa\overline{L}_m)\sqrt{-\overline{g}} \tag{1.49}$$

with

$$\overline{L}_m = \cos^4(\sqrt{\varkappa/6}\ \psi)L_m. \tag{1.50}$$

The following system of field equations corresponds to the Lagrangian (1.49) after variation with respect to the scalar field ψ and metric $\overline{g}^{\mu\nu}$:

$$\left.\begin{aligned} \overline{G}_{\mu\nu} \equiv\ & \overline{R}_{\mu\nu} - (1/2)\ \overline{g}_{\mu\nu}\overline{R} = \varkappa\overline{T}_{\mu\nu} - \overline{\tau}_{\mu\nu}; \\ \Box\psi =\ & -f(\psi)\overline{T}\ , \end{aligned}\right\} \tag{1.51}$$

where

$$\overline{\tau}_{\mu\nu} = \varkappa\ (\psi_{,\mu}\psi_{,\nu} - (1/2)\ \overline{g}_{\mu\nu}\ \psi_{,\alpha}\psi^{,\alpha}); \tag{1.52}$$

$$f(\psi) = \sqrt{\frac{\varkappa}{6}}\ \text{tg}\left(\sqrt{\frac{\varkappa}{6}}\psi\right);$$

$$\overline{T} = (2\overline{g}_{\mu\nu}/\sqrt{-\overline{g}})\frac{\partial(\sqrt{-\overline{g}}\ \overline{L}_m)}{\partial\overline{g}_{\mu\nu}}. \tag{1.53}$$

For $h^-(\varphi)$ and $l^-(\varphi)$ the transformation

$$\psi(\varphi) = \sqrt{6/\varkappa}\ \text{arcth}(\sqrt{\varkappa/6}\ \varphi) \tag{1.54}$$

leads to similar equations (1.51) but with difference in that the sign of the scalar field energy-momentum tensor is changed to the opposite one and the function $f(\psi)$ is defined now as:

$$f(\psi) = -\sqrt{\varkappa/6}\ \text{th}(\sqrt{\varkappa/6}\ \psi)$$

It is seen that conformal transformations of metric with the corresponding transformation of a scalar field are used as some mathematical tool leading to Einstein type equations. But, there is also a physical meaning of these transformations related to a transformation of the system of units (measurements). Really, if a metric is transformed according to (1.41), then the unit of length is transformed as $\bar{l} = h^{1/2}(\varphi)\, l$. The same is valid for the unit of time if the speed of light is constant. As to a transformation of masses the demand of the Planck constant invariance leads to the relation $\overline{m} = h^{-1/2} m$. As we see, masses of macro objects transform in the same manner. For the problem of transformation of micro objects Lagrangian see papers of Damour.

We want to point out that the above mentioned conformal transformations lead to another variant of the gravitation theory (1.51) - to the theory with variable masses depending on the scalar field and constant $\varkappa$, in which basic units (constant scales by definition) are G, c and $\hbar$, or equivalently, the Planck mass, length and time:

$$m_L = (\hbar c/G)^{1/2};\ L = \hbar m_L^{-1} c^{-1};\ t_L = Lc^{-1},$$

which do not depend on particle properties.

It is known that the Poincare-invariance garanties the constancy of rest masses of particles but when gravity is present it doesn't take place. So, theoretical arguments do not forbid the hypothesis of variable masses. Using reverse transformation (+) we obtain the variant (1.40) with variable $\varkappa$ and constant c, $\hbar$ and m.

Let us note that in the variant of the theory with variable masses (1.51) equations of motion of test bodies are changed with respect to GR. One may obtain the generalized formulas for calculation of effects of a geodesic motion similar to Eddington-Robertson-Schiff formulas.

Now, let us study a homogeneous and isotropic model within the approach (1.51) and for both signs $(\pm)$.

The metric is taken in the usual form:

$$-ds^2 = dt^2 - a^2(t)[d\chi^2 + A^2(d\theta^2 + \sin^2\theta d\Phi^2)]$$

where $A = \sin\chi,\ \chi,\ \mathrm{sh}\chi$ for elliptic, flat and hyperbolic spaces correspondingly.

Not null matter energy-momentum tensor components in a comoving frame are:

$$T_0^0 = -\varepsilon;\ T_k^l = \delta_k^l p,$$

where ε is an energy density and p is a pressure.

Let us take as an equation of state the following: $\varepsilon - 3p = 0$ (radiation). Then, the scalar field equation in frame (1.51) takes the form:

$$(d/dt)\ (a^3\dot{\psi}) = 0,\ \text{where}\ \dot{\psi} = d\psi/dt.$$

Its integration leads to

$$\dot{\psi} a^3 = b, \tag{1.55}$$

where b is a constant.

Using this result we transform the system (1.51) to the following:

$$\left.\begin{aligned} -(3/a^2)(\dot{a}^2 + \beta_1) &= -\varkappa\varepsilon + (\sigma/2)\varkappa b^2/a^6; \\ -(1/a^2)(\beta_1 + \dot{a}^2 + 2a\ddot{a}) &= \varkappa p - (\sigma/2)\varkappa b^2/a^6, \end{aligned}\right\} \tag{1.56}$$

where $\beta_1 = 1, 0, -1$ for three types of Friedman universe (closed, flat and open); $\sigma = 1$ corresponds to negative scalar field energy and $\sigma = -1$ to a positive one.

¿From (1.56) we obtain the equation

$$(6/a^2)(\beta_1 + \dot{a}^2 + a\ddot{a}) = \sigma\varkappa b^2/a^6,$$

and its integral is

$$\dot{a}^2 = -m/a^4 - \beta_1 + C_1/a^2, \tag{1.57}$$

with $C_1 = \text{const},\ m = \sigma\varkappa b^2/6$.

Substituting (1.57) into the first equation of the system (1.56) we find

$$\varepsilon(a) = 3C_1/\varkappa a^4,$$

which leads to $C_1 > 0$. Otherwise the energy density of usual matter could be negative.

For the chosen equation of state $p = \varepsilon/3$; $\varepsilon a^4 = \varepsilon_0 a_0^4$, so

$$E_0 = \frac{2\pi^2\beta_2}{a_0}\left(\frac{3C_1}{\varkappa}\right),$$

where $V = 2\pi^2\beta_2 a^3$ is a space volume; $\beta_2 = 1, 2/3\pi$ and $(1/\pi)[-\chi_0 + (1/2)\mathrm{sh}2\chi_0]$ for elliptic, flat and hyperbolic spaces. Now we may define the constant C_1 via the total energy of matter:

$$C_1 = \varkappa a_0 E_0/6\pi^2\beta_2.$$

We may calculate also the scalar factor $a(t)$ behavior integrating equation (1.57). In a general case ($\beta_1 = 1, -1$) solutions may be expressed via elliptic integrals of the first and second kind. Let us restrict ourselves to the flat space model. For $\beta_1 = 0$ (1.57) leads to

$$\frac{1}{2}\ \frac{a\sqrt{C_1a^2 - m}}{C_1} + \frac{1}{2}\ \frac{m}{C_1^{3/2}}\ \ln|a\sqrt{C_1a^2 - m}| = t + C_2, \tag{–}$$

with C_2=const defined from initial conditions, $\sigma = 1$ $(m > 0)$ in this case and the scalar field is described by

$$\psi = (b/\sqrt{m})\ \arccos(\sqrt{m}/\sqrt{C_1}a) + \text{const}.$$

¿From (–) we see that the initial singularity is absent, $a \geq a_0$, and this is due to the negative contribution of the scalar field energy.

For the second case ($\sigma = -1$) the initial singularity remains as the conformally invariant scalar field is the additional attraction field. The scalar field $a(t)$ for $\beta_1 = 0$ is defined by

$$\frac{1}{2}\ \frac{a\sqrt{C_1a^2 - m}}{C_1} - \frac{|m|}{2C_1^{3/2}}\ \ln|\sqrt{C_1}a + \sqrt{C_1a^2 + |m|}| = t + C_3.$$

Integration of the scalar field equation leads to

$$\psi = \frac{-b}{\sqrt{|m|}}\ \ln\left|\frac{\sqrt{|m|} + \sqrt{C_1a^2 + |m|}}{\sqrt{C_1}a}\right| + \psi_0.$$

For large t the scale factor in both cases varies like $a^2 \sim 2\sqrt{C_1}t$. As $\psi a^3 = b =$ const the asymptotic behaviour of the ψ-field is

$$\psi \sim \psi_0 - b/\sqrt{2}C_1^{3/4}t^{1/2}, \tag{1.58}$$

where ψ_0 is the asympotic value of ψ. As we pointed out earlier, relativistic gravitational effects are defined by ψ_0. For example, in $\sigma = 1$ case the deviation of light and radio waves in the gravitational field of the Sun is defined by the formula

$$\chi = 4(m/d)\ (1 + \overline{\gamma})/2$$

where m is the mass of the Sun, d is an impact parameter and;

$$\overline{\gamma} = \frac{\overline{A} - (\sqrt{\varkappa}/6)k\psi_0}{\overline{A} + \sqrt{\varkappa}k\psi_0/\sqrt{6}};$$

$\overline{A}$ and k are constants. It is seen that for $\psi_0 = 0$ we come to the usual GR value for χ.

Using the solution obtained for the scalar field we may write down the evolution of masses in gravitational (Planck) units:

$$\overline{m} = \begin{cases} (1 - b^2\varkappa/4C_1^{3/2}6t)\ m, & \sigma = 1 \\ (1 + b^2\varkappa/4C_1^{3/2}6t)\ m, & \sigma = -1. \end{cases}$$

One may also pass to the system of units where masses are constant but the effective gravitational coupling is variable (i.e. the function of a scalar field).

As $\varkappa_{eff} \equiv \varkappa\cos^2(\sqrt{\varkappa/6}\psi)$ for $\sigma = 1$ and $\varkappa_{eff} \equiv \varkappa\mathrm{ch}^2\sqrt{\varkappa/6}\psi$ for $\sigma = -1$ the asymptotic behaviour of $\varkappa_{eff}$ is the following (see (1.58)):

$$\varkappa_{eff} = \begin{cases} \varkappa\ (1 - A/t), & \sigma = 1 \\ \varkappa(1 + A/t), & \sigma = -1. \end{cases} \tag{1.59}$$

where $A = b^2\varkappa/12C_1^{3/2}$ is constant.

As it is seen, in the case of the material conformal field ($\sigma = 1$) $\varkappa_{eff}$ is diminishing with time as it is in the original Dirac's hypothesis. We may evaluate it roughly if we suppose that the energy densities of matter and of the scalar field are of the same order. Then,

$$(\dot{G}/G)_0 \approx -\, 4{\cdot}10^{-11}\ \mathrm{year}^{-1}$$

This result for $\sigma = -1$ may be even less if the contribution of the scalar field to the total mass is less than the usual matter.

Expressions for $\varkappa_{eff} = \varkappa(t)$ may be found directly without using transformations from one system of units to another. For that we shall treat (1.40) for the particular case of $h(\varphi)$ and $l(\varphi)$ as (+). The system of field equation then takes the form:

$$G_{\mu\nu} = \varkappa T_{\mu\nu}/\tilde{A} + (1/\tilde{A})\{\tilde{A}_{;\mu;\nu} - g_{\mu\nu}\ \tilde{A}^{;\alpha}_{;\alpha} - \sigma\varkappa[\varphi_{,\mu}\ \varphi_{,\nu} - (1/2)\ g_{\mu\nu}\ \varphi_{,\alpha}\ \varphi^{,\alpha}]\};$$
$$\varphi^{;\alpha}_{;\alpha} - (1/6)R\ \varphi = 0,$$

where $\tilde{A} = 1 + \omega\ (\varkappa/6)\ \varphi^2$, $\sigma = \pm1$, $\omega = \pm1$ ($\sigma = -1$, $\omega = -1$ correspond as earlier to the material scalar field).

In the case of a homogeneous and isotropic model let us restrict ourselves to the variant $\sigma = -1$, $\omega = -1$ and the equation of state for matter as $\varepsilon - 3p = 0$. Then, for the scale factor we have:

$$a = \sqrt{2a_0 t - \beta_1 t^2}, \tag{1.60}$$

with $\beta_1 = 1,\ 0,\ -1$ for closed, flat and open models correspond ingly. Integrating the scalar field equations we get:

$$\varphi = \left\{ \begin{array}{l} -\dfrac{C}{aa_0},\ \beta_1 = 0; \\ \\ -\dfrac{C\sqrt{a_0^2 - a^2}}{a_0^2 a},\ \beta_1 = 1; \\ \\ -C(a_0^2 + a^2)^{1/2}/aa_0^2,\ \beta_1 = -1, \end{array} \right\} \tag{1.61}$$

where a_0, C are integration constants, the value of φ at infinity is taken as zero. The energy density is $\varepsilon = 3a_0^2/\varkappa a^4$.

¿From (1.60) and (1.61) it follows that $\varkappa = \varkappa(t)$ of (1.59) type is realized for the flat model and is diminishing with time if $\sigma = -1$.

So, the results obtained asymptotically in two systems of units (c, $\hbar$ and G are constant and m, c, $\hbar$ are constant) are coinsiding and correspond roughly to the Large Number Hypothesis.

1.5. Possible time variations of G in general ST theories

1.5.1 Introduction

Dirac's Large Numbers Hypothesis (LNH) is the origin of many theoretical explorations of time-varying G. According to LNH, the value of $\dot{G}/G$ should be approximately the Hubble rate. Although it has become clear in recent decades that the Hubble rate is too high to be compatible with experiment, the enduring legacy of Dirac's bold stroke is the acceptance by modern theories of non-zero values of $\dot{G}/G$ as being potentially consistent with physical reality.

A striking feature of the present status of theoretical physics is that there is no satisfactory theory unifying all four known interactions; most modern unification theories do not admit unique and universal constant values of physical constants and of the Newtonian gravitational coupling constant G in particular. In this section we discuss various bounds that may be suggested by scalar-tensor theories. Although the bounds on $\dot{G}$ and $G(r)$ are in some classes of theories rather wide on purely theoretical grounds since any theoretical model contains a number of adjustable parameters, we note that observational data concerning other phenomena, in particular, cosmological data, may place limits on the possible ranges of these adjustable parameters.

Here we restrict ourselves to the problem of $\dot{G}$ (for $G(r)$ see [3, 4, 5, 60])). We show that ST-theories predict the value of $\dot{G}/G$ to be 10^{-12}/yr or less. The significance of this fact for experimental and observational determinations of the value of or upper bound on $\dot{G}$ is the following: any determination with error bounds significantly below 10^{-12}/yr (combined with experimental bounds on other parameters) will typically be compatible with only a small portion of existing theoretical models and will therefore cast serious doubt on the viability of all other models. In short, a tight bound on $\dot{G}$, in conjunction with other astrophysical observations, will be a very effective "theory killer" and/or significantly reduce the class of viable theories. Any step forward in this direction will be of utmost significance.

Some estimations for $\dot{G}$ were done long ago in the frames of general scalar tensor theories using the values of cosmological parameters (Ω, H, q etc) known at that time [3, 31]. It is easy to show that for modern values they predict $\dot{G}/G$ at the level of 10^{-12}/yr and less (see also recent estimations of A. Miyazaki [61], predicting time variations of G at the level of $10^{-13}\mathrm{yr}^{-1}$ for a Machian-type cosmological solution in the Brans-Dicke theory).

The most reliable experimental bounds on $\dot{G}/G$ (radar ranging of spacecraft dynamics [38]) and laser lunar ranging [62] give the limit of 10^{-12}/yr), so any results at this level or less will be very important for solving the fundamental problem of variations of constants and for discriminating between viable unified theories. So, realization of such multipurpose new generation type space experiments like Satellite Energy Exchange (SEE) for measuring $\dot{G}$ and also absolute value of G and Yukawa type forces at meters and Earth radius ranges [39, 57, 64] become extremely topical.

So, in what follows, we shall discuss predictions for $\dot{G}$ from generalized scalar-tensor theories [66].

1.5.2 Scalar-tensor cosmology and variations of G

The purpose of this section is to estimate the order of magnitude of variations of the gravitational constant G due to cosmological expansion in the framework of scalar-tensor theories (STT) of gravity, using modern data on cosmological parameters. Some estimations within the theory of gravity with conformal scalar field were done in 80's Ref. [31, 69, 3].

Consider the general (Bermann-Wagoner-Nordtvedt) class of STT where gravity is characterized by the metric $g_{\mu\nu}$ and the scalar field φ; the action is

$$S = \int d^4x\sqrt{g}\{f(\varphi)\mathcal{R}[g] + h(\varphi)g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - 2U(\varphi) + L_m\}. \tag{1.62}$$

Here $\mathcal{R}[g]$ is the scalar curvature, $g = |\det(g_{\mu\nu})|$; f, h and U are certain functions of φ, varying from theory to theory, L_m is the matter Lagrangian.

This formulation of the theory corresponds to the Jordan conformal frame, in which matter particles move along geodesics and hence the weak equivalence principle is valid, and non-gravitational fundamental constants do not change. In other words, this is the frame well describing the existing laboratory, geophysical and cosmological observations.

Among the three functions of φ entering into (1.62) only two are independent since there is a freedom of transformations $\varphi = \varphi(\varphi_{\mathrm{new}})$. We use this arbitrariness, choosing $h(\varphi) \equiv 1$, as is done, e.g., in Ref. [63]. Another standard parametrization is to put $f(\varphi) = \varphi$ and $h(\varphi) = \omega(\varphi)/\varphi$ (the Brans-Dicke parametrization of the general theory (1.62)). In our parametrization $h \equiv 1$, the Brans-Dicke parameter $\omega(\varphi)$ is $\omega(\varphi) = f(f_\varphi)^{-2}$; here and henceforth, the subscript φ denotes a derivative with respect to φ. The Brans-Dicke STT is the particular case $\omega = \mathrm{const}$, so that in (1.62)

$$f(\varphi) = \varphi^2/(4\omega), \qquad h \equiv 1. \tag{1.63}$$

The field equations that follow from (1.62) read

$$\Box\varphi - \tfrac{1}{2}\mathcal{R}\,f_\varphi + U_\varphi = 0, \tag{1.64}$$

$$f(\varphi)\Big(\mathcal{R}_\mu^\nu - \tfrac{1}{2}\delta_\mu^\nu\mathcal{R}\Big) = -\varphi_{,\mu}\varphi^{,\nu} + \tfrac{1}{2}\delta_\mu^\nu\varphi^{,\alpha}\varphi_{,\alpha} - \delta_\mu^\nu U(\varphi) + (\nabla_\mu\nabla^\nu - \delta_\mu^\nu\Box)f - T^\nu_{\mu\,(\mathrm{m})}, \tag{1.65}$$

where $\Box$ is the D'Alembert operator, and the last term in (1.65) is the energy-momentum tensor of matter.

Consider now isotropic cosmological models with the standard FRW metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \tag{1.66}$$

where $a(t)$ is the scale factor of the Universe, and $k = 1,\ 0,\ -1$ for closed, spatially flat and hyperbolic models, respectively. Accordingly, we assume $\varphi = \varphi(t)$ and the energy-momentum tensor of matter in the perfect fluid form $T^{\nu}_{\mu\,(\mathrm{m})} = \mathrm{diag}(\rho, -p, -p, -p)$ (ρ is the density and p is the presuure).

The field equations in this case can be written as follows:

$$\ddot\varphi + 3\frac{\dot a}{a}\dot\varphi - \frac{3}{a^2}(a\ddot a + \dot a^2 + k) + U_\varphi = 0, \tag{1.67}$$

$$\frac{3f}{a^2}(\dot a^2 + k) = \tfrac{1}{2}\dot\varphi^2 + U - 3\frac{\dot a}{a}\dot f + \rho, \tag{1.68}$$

$$\frac{f}{a^2}(2a\ddot a + \dot a^2 + k) = -\tfrac{1}{2}\dot\varphi^2 + U - \ddot f - 2\frac{\dot a}{a}\dot f - p. \tag{1.69}$$

To connect these equations with observations, let us fix the time t at the present epoch (i.e., consider the instantaneous values of all quantities) and introduce the standard observables:

$H = \dot a/a$ (the Hubble parameter),

$q = -a\ddot a/\dot a^2$ (the deceleration parameter),

$\Omega_m = \rho/\rho_{\mathrm{cr}}$ (the matter density parameter),

where ρ_{cr} is the critical density, or, in our model, the r.h.s. of Eq. (1.68) in case $k = 0$: $\rho_{\mathrm{cr}} = 3fH^2$. This is slightly different from the usual definition $\rho_{\mathrm{cr}} = 3H^2/8\pi G$ where G is the Newtonian gravitational constant. The point is that the locally measured Newtonian constant in STT differs from $1/(8\pi f)$; provided the derivatives $U_{\varphi\varphi}$ and $f_{\varphi\varphi}$ are sufficiently small, one has [63]

$$8\pi G_{\mathrm{eff}} = \frac{1}{f}\frac{2\omega + 4}{2\omega + 3}. \tag{1.70}$$

(First this fact was pointed out and investigated in [36, 37] on the basis of cosmological solutions with local inhomogeneities and equations of particle motion.)

Since, according to the solar-system experiments, $\omega \geq 2500$, for our order-of-magnitude reasoning we can safely put $8\pi G = 1/f$, and, in particular, our definition of ρ_{cr} now coincides with the standard one.

The time variation of G, to a good approxiamtion, is

$$\dot G/G \approx -\dot f/f = gH, \tag{1.71}$$

where, for convenience, we have introduced the coefficient g expressing $\dot G/G$ in terms of the Hubble parameter H.

Eqs. (1.67)–(1.69) contain too many arbitrary parameters for making a good estimate of g. Let us now introduce some restrictions according to the current state of observational cosmology:

(i) $k = 0$ (a spatially flat cosmological model, so that the total density of matter equals ρ_{cr});

(ii) $p = 0$ (the pressure of ordinary matter is negligible compared to the energy density);

(iii) $\rho = 0.3\,\rho_{\rm cr}$ (the ordinary matter, including its dark component, contributes to only 0.3 of the critical density; unusual matter, which is here represented by the scalar field, comprises the remaining 70 per cent).

Then Eqs. (1.68) and (1.69) can be rewritten in the form

$$\tfrac{1}{2}\dot\varphi^2 + U - 3H\dot f = 2.1 H^2 f, \qquad (1.72)$$

$$-\tfrac{1}{2}\dot\varphi^2 + U - 2H\dot f - \ddot f = (1-2q)H^2 f. \qquad (1.73)$$

Subtracting (1.73) from (1.72), we exclude the "cosmological constant" U, which can be quite large but whose precise value is hard to estimate. We obtain

$$\dot\varphi^2 - H\dot f + \ddot f = (1.1 + 2q)H^2 f. \qquad (1.74)$$

The first term in Eq. (1.74) can be represented in the form

$$\dot\varphi^2 = \dot f^2 (df/d\varphi)^{-2} = \dot f^2 \omega/f,$$

and $\dot f/f$ can be replaced with $-gH$. The term $\ddot f$ can be neglected for our estimation purposes. To see this, let us use as an example the Brans-Dicke theory, in which $f = \varphi^2/(4\omega)$. We then have

$$\ddot f = (\dot\varphi^2 + \varphi\ddot\varphi)/(2\omega);$$

here the first term is the same as the first term in Eq. (1.74), times the small parameter $1/(2\omega)$. Assuming that $\varphi\ddot\varphi$ is of the same order of magnitude as $\dot\varphi^2$ (or slightly greater), we see that, generically, $|\ddot f| \ll \dot\varphi^2$. Note that our consideration is not restricted to the Brans-Dicke theory and concerns the model (1.62) with an arbitrary function $f(\varphi)$ and an arbitrary potential $U(\varphi)$.

Neglecting $\ddot f$, we see that (1.74), divided by $H^2 f$, leads to a quadratic equation with respect to g:

$$\omega g^2 + g - q' = 0, \qquad (1.75)$$

where $q' = 1.1 + 2q$.

According to modern observations, the Universe is expanding with an acceleration, so that the parameter q is, roughly, -0.5 ± 0.2, hence we can take $|q'| \le 0.4$.

In case $q' = 0$ we simply obtain $g = -1/\omega$. Assuming

$$H = h_{100}\cdot 100\ {\rm km/(s.Mpc)} \approx h_{100}\cdot 10^{-10}\ {\rm yr}^{-1}$$

and $\omega \ge 2500$, we come to the estimate

$$|\dot G/G| \le 4{\cdot}10^{-14} h_{100}\ {\rm yr}^{-1}, \qquad (1.76)$$

where h_{100} is, by modern views, close to 0.7. So (1.76) becomes

$$|\dot G/G| \le 3{\cdot}10^{-14}\ {\rm yr}^{-1}. \qquad (1.77)$$

For nonzero values of q', solving the quadratic equation (1.75) and assuming $q'\omega \gg 1$, we arrive at the estimate $|g| \sim \sqrt{q'/\omega}$, so that, taking $q' = 0.4$ and again $\omega \ge 2500$, we have instead of (1.76)

$$|\dot G/G| \le 1.3{\cdot}10^{-12} h_{100}\ {\rm yr}^{-1} \approx 0.9{\cdot}10^{-12}{\rm yr}^{-1}, \qquad (1.78)$$

where we have again put $h_{100} = 0.7$.

We conclude that, in the framework of the general STT, modern cosmological observations, taking into account the solar-system data, restrict the possible variation of G to values less than 10^{-12}/yr. This estimate may be considerably tightened if the matter density parameter Ω_m and the (negative) deceleration parameter q will be determined more precisely. If we put new estimations for $\omega \geq 40000$, than possible $\dot{G}/G$ be less than 10^{-13}/yr.

1.5.3 Conclusion

Summarizing the above considerations, we can conclude:

Restrictions of possible nonzero values of $\dot{G}$ give no restriction on the possible class of generalized gravitation theories, but in the framework of some fixed theory any restriction on $\dot{G}$ restricts the possible class of models.

We note that similar estimations of G-dot may be done for different multidimensional models [65], giving the result on the level of 10^{-12}/yr and less.

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Chapter 2

Solutions with Fields in General Relativity

2.1. Solution for Massless Static Conformal Scalar Field in Spherical Symmetry [40]

Here we consider the spherically symmetric, static, self-consistent problem of the interaction of scalar and gravitational fields. For the scalar field we use, in contrast, e.g., [1, 2], the equation

$$(\Box + m^2 + \frac{R}{6})\varphi = 0, \tag{2.1}$$

where $\Box = \nabla^\alpha \nabla_\alpha$ is the generally covariant d'Alambert operator, and R is the scalar curvature. This equation is conformally invariant for $m = 0$ and leads to the correct semiclassical transition [3, 4, 5].

We restrict the analysis to a massless scalar field. The Lagrangian

$$\mathcal{L} = \frac{R}{2\kappa} + \nabla^\alpha \varphi^* \nabla_\alpha \varphi - \frac{R}{6}\varphi^* \varphi \tag{2.2}$$

corresponds to the field equations

$$\begin{aligned} R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R &= -\kappa T^\nu_\mu, \qquad (2.3)\\ (\Box + \frac{R}{6})\varphi &= 0\ , \qquad (2.4) \end{aligned}$$

where the energy-momentum tensor of the scalar field,

$$\begin{aligned} T^\nu_\mu = \nabla_\mu \varphi^* \nabla^\nu \varphi &+ \nabla^\nu \varphi^* \nabla_\mu \varphi - \delta^\nu_\mu (\nabla^\alpha \varphi^* \nabla_\alpha \varphi - \frac{R}{6}\varphi^* \varphi) - \\ &- \frac{1}{3}(R^\nu_\mu + \nabla^\nu \nabla_\mu - \delta^\nu_\mu \Box)\varphi^* \varphi \qquad (2.5) \end{aligned}$$

has the properties

$$T = T^{\alpha}_{\alpha}, \qquad T_{\nu\mu} = T_{\mu\nu}, \qquad \nabla_{\alpha} T^{\alpha}_{\nu} = 0. \tag{2.6}$$

The notation here is the same as in [4, 5]. From eqs. (2.6) and (2.3) we conclude that

$$R = 0. \tag{2.7}$$

For the static, spherically symmetric case we have

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi_1^2), \tag{2.8}$$
$$\varphi = \varphi(r) \tag{2.9}$$

and in the system (2.3) and (2.4) the only independent equations are the three equations which can be written in the form (the prime denotes d/dr):

$$\frac{1}{r^2} - \frac{\lambda'}{r} - \frac{e^{\lambda}}{r^2} = \frac{\kappa}{3}[\varphi^{*\prime}\varphi' + \frac{\nu'}{2}(\varphi^*\varphi)'](1 - \frac{\kappa}{3}\varphi^*\varphi)^{-1}, \tag{2.10}$$

$$\frac{1}{r^2} + \frac{\nu'}{r} - \frac{e^{\lambda}}{r^2} = \kappa[\varphi^{*'}\varphi^{'} + \frac{1}{6}(\frac{4}{r} + \nu^{'})(\varphi^*\varphi)^{'}](1 - \frac{\kappa}{3}\varphi^*\varphi)^{-1}, \tag{2.11}$$

$$\varphi^{''} + (\frac{2}{r} + \frac{\nu^{'} - \lambda^{'}}{2})\varphi^{'} = 0. \tag{2.12}$$

Equations (2.12) can be integrated once in a simple manner:

$$\varphi^{'} r^2 e^{(\nu-\lambda)/2} = c_1. \tag{2.13}$$

In particular, if we have $c_1 = 0$, we have $\varphi =$ const; with $|\varphi| = \sqrt{\frac{3}{\kappa}}$ we see from the system (2.10-2.12) that the unique condition $R = 0$ is imposed on the metric, while with $|\varphi| \neq \sqrt{\frac{3}{\kappa}}$ the solution is the Schwarzchild metric, for which we have

$$\nu^{'} + \lambda^{'} = 0. \tag{2.14}$$

We will attempt to find a solution which satisfies Eq. (2.14) in the case $c_1 \neq 0$. In this case Eq. (2.7) becomes

$$\lambda^{''} - \lambda'^2 + \frac{4\lambda'}{r} + \frac{2}{r^2}(e^{\lambda} - 1) = 0 \tag{2.15}$$

and can be integrated by means of the substitution $z = e^{-\lambda}$. Then, using the Euclidean boundary condition at infinity,

$$\lambda \to 0, \qquad \nu \to 0 \quad \text{and} \quad \varphi \to 0 \quad \text{for} \quad r \to \infty, \tag{2.16}$$

we find a solution of (2.10-2.12):

$$e^{\nu} = e^{-\lambda} = (1 - \frac{a}{r})^2, \quad \varphi = -\frac{c_1}{r - a}, \quad |c_1|^2 = \frac{3a^2}{\kappa}. \tag{2.17}$$

N.A.Zaitsev and others have analized the interaction of metric and scalar fields in various theories of gravitation and have found the integral of Eqs. (2.10-2.12) without the assumption (2.14):

$$e^{\bar{\lambda}} = 1 - cy - c_2y^2, \quad (2.18)$$
$$e^{\bar{\nu}} = \frac{1}{x^2}(y^{-2} - cy^{-1} - c_2), \quad (2.19)$$
$$x^2 = x_0^2|\frac{1 - cy - c_2y^2}{y^2}||\frac{2c_2y + c - 2D}{2c_2y + c + 2D}|^{c/2D}, \quad (2.20)$$

where y is an independent parameter, $c_2 = \kappa|c_1|^2 > 0, 2D = \sqrt{c^2 + 4c_2}, x = r\sqrt{A}$,

$$A = 1 - \frac{\kappa}{3}\varphi^*\varphi, \quad e^{\bar{\nu}} = e^{\nu}A, \quad e^{\bar{\lambda}} = e^{\lambda}[1 + \frac{\kappa}{6A}x\frac{d}{dx}(\varphi^*\varphi)]^2, \quad (2.21)$$

x_0 and c are integration constants.

Integrating (2.13), and using Eqs. (2.21) and boundary condition (2.16), we can write the solution in the convenient form

$$r(\xi) = \frac{D}{1-\xi}\xi^{1/2(1-\gamma-\beta)}(1 + \xi^{\beta}),$$
$$e^{\nu(\xi)} = \frac{1}{4}\xi^{\gamma-\beta}(1 + \xi^{\beta})^2,$$
$$e^{\lambda(\xi)} = 4\xi[1 - \gamma + (1 + \gamma)\xi - \beta(1 - \xi)\frac{1 - \xi^{\beta}}{1 + \xi^{\beta}}]^{-2},$$
$$\varphi(\xi) = \sqrt{\frac{3}{x}}\frac{1 - \xi^{\beta}}{1 + \xi^{\beta}}\frac{c_1}{|c_1|}, \quad (2.22)$$

where

$$\gamma = -\frac{c}{2D}, \quad \beta = \sqrt{\frac{1 - \gamma^2}{3}}, \quad \xi = |\frac{y_2}{y_1}|\frac{(y_1 - y)}{(y - y_2)}, \quad (2.23)$$

and $y_1 > 0$ and $y_2 < 0$ are the roots of the trinomial $1 - cy - c_2y^2$. The independent parameter ξ and the integration constants γ and D lie in the ranges

$$0 < \xi < 1, \qquad -1 < \gamma < 1, \qquad 0 < D < \infty, \quad (2.24)$$

where $r(1) = \infty$. With $\gamma < 1/2$ we have $r(0) = 0$;

at $\gamma = 1/2$ the solution takes the simple form (2.17) where $a = D/2 > 0$.

At $\gamma > 1/2$ the value of r is limited from below by the value $r^* > 0$; the corresponding value of ξ^* is found from the trascendental equation

$$1 - \gamma - \beta + (1 - \gamma + \beta)\xi^{\beta} + (1 + \gamma + \beta)\xi + (1 + \gamma - \beta)\xi^{\beta+1} = 0, \quad (2.25)$$

which has the unique solution on the interval $0 < \xi < 1$.

At large r solution (2.22) has the asymptotic form

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2\gamma D}{r}, \qquad \varphi = \sqrt{\frac{3}{\kappa}}\frac{\beta D}{r}, \quad (2.26)$$

i.e., behaves like the Schwarzschild solution with mass

$$M = \frac{8\pi\gamma D}{\kappa}. \tag{2.27}$$

We see that this solution can be physically meaningful only for $\gamma > 0$.

For $\gamma = 1/2$ the metric is equal to the Reissner-Nordstrom metric, whose mass is given by (2.27) and "charge" is $q = D/\sqrt{2\kappa}$.

With γ not equal to 1/2 there is an essential singularity at $\xi = 0$:

$|\varphi(0)| = \sqrt{\frac{3}{\kappa}}), e^{\lambda(0)} = 0, r(0) = 0$ and $e^{\nu(0)} = \infty$ for $\gamma < 1/2$;
$r(0) = \infty$ and $e^{\nu(0)} = 0$ for $\gamma > 1/2$.

True, in the case $\gamma > 1/2$ on the sphere $\xi = \xi^*$ we have $e^\lambda(\xi^*) = \infty$ for finite e^ν and φ , but this singularity is a pure coordinate singularity and can be easily removed by transforming from r to the radial coordinate ξ.

We write the proper energy of the scalar field outside the sphere $\xi = \xi_0$:

$$E = \int T_0^0 \sqrt{-g} d^3x = 4\pi \int_{\xi_0}^{1} T_0^0 e^{(\nu+\lambda)/2} r^2 \frac{dr}{d\xi} d\xi. \tag{2.28}$$

Integration yields

$$E(r_0) = \frac{4\pi a^2}{\kappa r_0} \qquad \text{for} \qquad \gamma \pm \frac{1}{2}.$$

$$E(\xi_0) = \frac{2\pi\gamma D}{\kappa} + \frac{\pi D}{\kappa}[(\beta - \gamma)\xi_0^{-\beta} - (\beta + \gamma)\xi_0^{\beta}] \quad \text{for} \quad |\gamma| \neq \frac{1}{2}.$$

We will treat separately the cases $\gamma \leq 1/2$ and $\gamma > 1/2$.

In the case $\gamma \leq 1/2$, the density T_0^0 is positive everywhere, and the integral (2.28) diverges at the center. However, the total energy of matter (scalar field) and gravitational field (2.27) is finite; we can interpret this result as a compensation of the energy of the scalar field by the negative gravitational energy.

In the absense of singularities the total energy is given by the Tolman equation

$$M = \int (2T_0^0 - T)\sqrt{-g} d^3x \tag{2.29}$$

Formally, using this equation, integrating over the region $r > r_0$, and using (2.6), we find that for any r_0 we have $M = 2E(r_0)$. This result is the same as (2.27) with $\xi_0^\beta = (\beta - \gamma)/(\beta + \gamma)$, where $\gamma < 1/2$ and with $r = a$ for $\gamma = 1/2$. It can be interpreted as a presence of an effective cut-off in the formula for the total energy, by analogy with the conclusions reached in [6] for the electromagnetic field. According to (2.29) a similar cut-off should be found for any massless matter $(T = 0)$, if we assume the total gravitational energy to be negative.

In the case $\gamma > 1/2$, on the other hand, we have $T_0^0 < 0$ when $\xi^{2beta} < (\gamma - \beta)/(\gamma + \beta)$, so we find an infinite negative value for the proper energy of the scalar field. On the other hand, mass (2.27) is finite and positive, so the gravitational energy is finite and positive, and the appearance of the "hole" in the solution can be interpretated as a result of the repulsive effect of the gravitational forces. The scalar and metric field interchange roles insofar as their effect on the solution is concerned.

2.2. Solutions with Static Scalar and Electromagnetic Fields. PPN Formalism [41]

2.2.1 Introduction

Modern gravitational experiments concern mainly weak fields which faintly differ from Newtonian ones. Thus, it is of interest to analyse various theories of gravitation and the ways of their comparison with experiment on post-Newtonian (PN) level.

In the papers by Thorne and Will [7, 8] the most general form of PN metric (parametrized PN, or PPN approximation) is proposed, containing nine arbitrary parameters varying from theory to theory. In the PPN formalism a theory of gravitation is supposed to be a metric one, i.e. the matter equations of motion are of the form

$$\nabla_a T^a_\mu = 0, \tag{2.30}$$

where ∇_a denotes a covariant derivative and T^ν_μ is the matter energy-momentum tensor including all non-gravitational fields.

On the other hand, there exists a method of finding exact static spherically-symmetric solutions for a broad class of scalar-tensor theories. Here a comparison of an exact vacuum solution for this class of theories with the PN metric is carried out. It will be shown that the PN approximation requires a special value for the source scalar charge-mass ratio, whereas in the exact solution the scalar charge C is an independent arbitrary constant. It is clear that specific scalar charge (in respect to mass) different from the standard one, may be discovered in nature. To describe this possibility in a consistent way one should introduce the scalar charge explicitly into the theory. We discuss some variants and consequences of this innovation.

Possible behaviour of scalar-tensor vacuum and electrovac spherically-symmetric solutions is discussed. To describe their properties in a compact way, we propose a classification of arbitrary spherically-symmetric metrics.

2.2.2. Field equations

Consider a class of scalar-tensor theories with the Lagrangian density

$$L = A(\varphi)R + B(\varphi)g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} - 2\Lambda(\varphi) + 2k_0 L_m(\Psi(\varphi)g^{\alpha\beta}, \dots), \tag{2.31}$$

where φ is the scalar field which we assume to be real (see 2.2.7); L_m is the matter Lagrangian; $k_0 = 8\pi c^{-4}G_0$ is an initial gravitational constant, generally speaking, a non-Einsteinian one; A, B, Λ and Ψ are arbitrary functions; $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature and $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ is the Ricci tensor. The Riemann tensor $R^\mu_{\nu\varrho\sigma}$, is defined by the formula

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)A_\sigma = R^\alpha_{\sigma\mu\nu}A_\alpha$$

with an arbitrary vector A_μ. The metric tensor $g_{\mu\nu}$ has the signature $(+---)$. Greek indices range from 0 to 3, Latin ones from 1 to 3; $\varphi_{,\alpha} \equiv \partial\varphi/\partial x^\alpha$.

Suppose that at spatial infinity the metric is flat and the φ field tends to some constant value φ_0. It is useful to divide Lagrangian (2.31) by the (nonzero) constant $A_0 = A(\underset{0}{\varphi})$. Denote

$$\bar{A}(\varphi) = AA_0^{-1}, \quad \bar{B}(\varphi) = BA_0^{-1}, \quad \bar{k} = k_0 A_0^{-1} = 8\pi\bar{G}c^{-4}. \tag{2.32}$$

Obviously $\bar{A}(\underset{0}{\varphi}) = 1$.

The transformation [9, 10]

$$g_{\mu\nu} = F(\psi)\tilde{g}_{\mu\nu}, \quad F(\psi) = [\bar{A}(\varphi)]^{-1}; \tag{2.33}$$

$$\frac{d\varphi}{d\psi} = \bar{A}\left|\bar{A}\bar{B} + \frac{3}{2}\bar{A}_\varphi^2\right|^{-4}, \quad \bar{A}_\varphi \equiv \frac{d\bar{A}}{d\varphi}; \quad \mathrm{sign}\left(AB + \frac{3}{2}A_\varphi^2\right) = n \tag{2.34}$$

reduces equation (2.31) to the form

$$\tilde{L} = \tilde{R} + n\tilde{g}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} - 2F^2(\psi)\Lambda + 2\bar{k}F^2(\psi)L_m(F^{-1}\Psi\tilde{g}^{\mu\nu}, \dots)\;, \tag{2.35}$$

where a tilde marks quantities obtained with the help of $\tilde{g}_{\mu\nu}$. Noting that formula (2.34) admits addition of an arbitrary constant to ψ, we put

$$\psi(\underset{0}{\varphi}) = \underset{0}{\psi} = 0. \tag{2.36}$$

Then according to (2.33)

$$F(0) = 1. \tag{2.37}$$

As the tensor $T_{\mu\nu}$, is yielded by the variation

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int L_m\sqrt{-g}d^4x, \quad d^4x \equiv dx_0dx^1dx^2dx^3, \tag{2.38}$$

in theory (2.31), the equations of motion have the form (2.30) only if $\Psi \equiv \mathrm{const}$. A conformal transformation of the type (2.33) allows us, however, to achieve this for any theory (2.31) [11]. Besides, here we are dealing with distances small on cosmological scale and so, it is reasonable to put $\Lambda \equiv 0$. Thus we consider theory (2.31) under assumptions

$$\Lambda(\varphi) \equiv 0; \quad \Psi(\varphi) \equiv l. \tag{2.39}$$

As in (2.31) a transformation of φ field is possible

$$\varphi \to \tilde{\varphi}, \quad \varphi = \varphi(\tilde{\varphi}), \tag{2.40}$$

where $\varphi(\tilde{\varphi})$ is an arbitrary function; for a given concrete choice of gravitation theory the form of the coefficients A and B is not unique. Under assumptions (2.39) such a choice should be set by means of one function, e.g. $\omega(\varphi)$ in Nordtvedt's notation [12]

$$A(\varphi) = \varphi, \quad B(\varphi) = \omega(\varphi)/\varphi. \tag{2.41}$$

It is often more convenient to set a theory by means of the function $F(\psi)$ which is invariant under transformation (2.40).

Taking into account (2.39), we get the field equations from the transformed Lagrangian (2.35):

$$\tilde{G}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = -\bar{k}F(\psi)T_{\mu\nu} - n\tilde{S}_{\mu\nu}(\psi); \tag{2.42}$$

$$2\Box\psi + nF\frac{dF}{d\psi}\bar{k}T = 0, \tag{2.43}$$

where $\Box = \tilde{g}_{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta$ and the tensor

$$\tilde{S}_{\mu\nu}(\psi) = \psi_{,\mu}\psi_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}. \tag{2.44}$$

The matter tensor $T_{\mu\nu}$ is here untransformed and $T = g^{\alpha\beta}T_{\alpha\beta}$.

2.2.3. The post-Newtonian approximation

The PN metric is known to be a solution of the field equations in the form of an expansion in powers of reciprocal velocity of light c. In this expansion the components g_{00}, g_{0i} and g_{ik} should be found with an accuracy of $O(c^{-4})$, $O(c^{-3})$ and $O(c^{-2})$ respectively.

Let us find such a solution of equations (2.42), (2.43). Following Chandrasekhar [13] and Will [14], assume that the tensor $T_{\mu\nu}$ describes a perfect fluid:

$$T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu - g_{\mu\nu}p, \tag{2.45}$$

where p is the pressure, u_μ is the unity-normalized four-velocity and ε is the energy density, from which the rest mass density ϱ is distinguished:

$$\varepsilon = \varrho(c^2 + \Pi). \tag{2.46}$$

Now the quantities in equations (2.42) and (2.43) are presented as series

$$\begin{aligned} &\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}; \quad h_{\mu\nu} = \underset{1}{h}_{\mu\nu} + \underset{2}{h}_{\mu\nu} + \ldots, \quad \psi = \underset{1}{\psi} + \underset{2}{\psi} + \ldots; \\ &F(\psi) = 1 + F_0'\psi + \frac{1}{2}F_0''\psi^2 + \ldots; \quad F_0' = \frac{dF}{d\psi}(0); \quad F_0'' = \frac{d^2F}{d\psi^2}(0); \quad \ldots \end{aligned} \tag{2.47}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric and the indices below denote orders in c^{-1}.

The metric to be found $g_{\mu\nu}$ will be determined with the required accuracy if one finds $\tilde{g}_{\mu\nu}$ with the same accuracy and also ψ to within $O(c^{-4})$.

As $\bar{k} = O(c^{-4})$ and

$$T_{00} = \underset{-2}{T}_{00} + \underset{0}{T}_{00} + \ldots; \quad T_{0i} = \underset{-1}{T}_{0i} + \ldots; \quad T_{ik} = \underset{0}{T}_{ik} + \ldots, \tag{2.48}$$

in equation (2.43) $\psi = \underset{2}{\psi} + \underset{4}{\psi} + \ldots$ Hence the tensor $\tilde{S}_{\mu\nu}(\psi)$ begins at the fourth order. Together with (2.48) this gives right to put

$$h_{00} = \underset{2}{h}_{00} + \underset{4}{h}_{00} + \ldots; \quad h_{0i} = \underset{3}{h}_{0i} + \ldots; \quad h_{ik} = -\delta_{ik}(1 - \underset{2}{h}_{00}) + \ldots. \tag{2.49}$$

The first two orders of equation (2.43) are of the form

$$\partial_{ii}\underset{2}{\psi} = 4\pi\bar{G}nF_0'\varrho c^{-2}; \tag{2.50}$$

$$\partial_{ii}\underset{4}{\psi} - \partial_{00}\underset{2}{\psi} + \underset{2}{h}_{00}\partial_{ii}\underset{2}{\psi} = 4\pi\bar{G}n\varrho[c^{-4}F_0'(\Pi - 3p/\varrho) + c^{-2}(F'^2_0 + F_0'')\underset{2}{\psi}], \tag{2.51}$$

where $\partial_\alpha = \partial/\partial x^\alpha$. (Note that $\partial_0 = \partial/c\partial t$ adds a unity to the order.) Equations (2.42) are solved similarly to paper [13]. Equation (2.50) and the $^{(}00)$ component of (2.42) give in the senior order Newton's law, *i.e.*

$$\begin{aligned} &g_{00} = F(\psi)\tilde{g}_{00} = l - 2c^{-2}U = l - 2c^{-2}GI, \\ &I(\vec{x}, t) = \int d^3x' \frac{\varrho(\vec{x}', t)}{|\vec{x} - \vec{x}'|}, \quad d^3x \equiv dx^1dx^2dx^3, \end{aligned} \tag{2.52}$$

where U is the Newtonian gravitational potential. The Newtonian constant of gravitation is

$$G = \bar{G}(l + \eta), \quad \eta = \frac{1}{2}nF'^2_0. \tag{2.53}$$

Solving then equation (2.51) and the $({}_{0i})$ and $({}_{00})$ components of (2.42) in the third and fourth order respectively and using the gauge condition determining the choice of coordinates in the third order

$$\partial_i h_{0i} = \frac{1}{2}(3-\eta)\partial_0 h_{00}, \tag{2.54}$$

we get finally

$$g_{00} = 1 - 2c^{-2}U + c^{-4}(2\beta_0 U^2 - 4G\Phi),$$
$$g_{0i} = Gc^{-3}(\frac{7}{2}\Delta_i V_i + \frac{1}{2}\Delta_2 W_i), \quad g_{ik} = -\delta_{ik}(+2c^{-2}\gamma_0 U), \tag{2.55}$$

where

$$V_i(\vec{x},t) = \int d^3x' \frac{\varrho(\vec{x}',t)v^i(\vec{x}',t)}{|\vec{x}-\vec{x}'|};$$
$$W_i(\vec{x},t) = \int d^3x' \frac{\varrho(\vec{x}',t)v^k(\vec{x}',t)(x^i-x^{i'})(x^k-x^{k'})}{|\vec{x}-\vec{x}'|^3};$$
$$\Phi(\vec{x},t) = \int d^3x' \frac{\varrho(\vec{x}',t)\chi(\vec{x}',t)}{|\vec{x}-\vec{x}'|}; \quad \chi = \beta_1 v^2 + \beta_2 U + \frac{1}{2}\beta_3\Pi + \frac{3}{2}\beta_4 p/\varrho;$$

$v^2 = v^i v^i$; v^i is the fluid three-velocity. The constants

$$\beta_0 = \frac{1+2\eta+\lambda}{(1+\eta)^2}; \quad \beta_2 = \frac{1-\eta-\eta^2-\lambda}{(1+\eta)_2}; \quad \beta_3 = \Delta_2 = 1;$$
$$\beta_1 = \frac{1}{1+\eta}; \quad \Delta_1 = \frac{7-\eta}{7(1+\eta)}; \quad \beta_4 = \gamma_0 = \frac{1-\eta}{1+\eta}, \tag{2.56}$$

with

$$\eta = \frac{1}{2}nF'^2_0; \quad \lambda = \frac{1}{2}n\eta F''_0 \tag{2.57}$$

are the concrete values of Will's PPN parameters [8] for the theories with Lagrangian (2.31) (with changes in notation: $\gamma \to \gamma_0$, $\beta \to \beta_0$). Two parameters from [8], Σ and ζ here equal zero due to gauge (2.54). Transformation formulae to another gauge are given in [8].

Note that for any values of η and λ parameters (2.56) satisfies all the constraints [14], characterizing an asymptotically Lorentz-invariant theory, containing PN conservation laws for energy, momentum and angular momentum. Thus, the theory (2.31) possesses all these properties.

The expressions for η and λ in terms of the initial functions $A(\varphi)$ and $B(\varphi)$ and their derivatives values at $\varphi = \underset{0}{\varphi}$ are:

$$\eta = A_\varphi^2(2AB + 3A_\varphi^2)^{-1};$$
$$\lambda = \eta^3 A_\varphi^{-4}[3A_\varphi^2(AB + A_\varphi^2) + A^2(A_\varphi B_\varphi - 2BA_{\varphi\varphi})]. \tag{2.58}$$

These formulae are insensitive to multiplying (2.31) by a constant and hence are valid as well without the assumption $A(\underset{0}{\varphi} y) = 1$.

In Nordtvedt's notation (2.41) formula (2.58) looks especially simple:

$$\eta = (2\omega+3)^{-1}; \quad \lambda = (2\omega+3)^{-3}(2\omega+3+\omega_\varphi \underset{0}{\varphi}), \tag{2.59}$$

Expressions (2.56) generalize the result of paper [12] where the PN metric for point masses is obtained.

2.2.4 An exact spherically-symmetric solution [41]

In the Lagrangian (2.31) we take a concrete L_m:

$$L_m = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} \tag{2.60}$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor. Again we make use of transformation (2.33), (2.34). Due to conformal invariance of (2.60) the transformed equations (13) and (14) take the form

$$\tilde{G}^{\nu}_{\mu} = -\bar{k}[-\tilde{F}^{\nu\alpha}F_{\mu\alpha} + \frac{1}{4}\delta^{\nu}_{\mu}\tilde{F}^{\alpha\beta}F_{\alpha\beta}] - n\tilde{S}^{\nu}_{\mu}(\psi), \tag{2.61}$$
$$\Box\psi = 0, \tag{2.62}$$

where $\tilde{F}^{\mu\nu} = \tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}F_{\alpha\beta}$. To these one should add the Maxwell equations

$$\tilde{\nabla}_{\alpha}\tilde{F}^{\alpha\mu} = 0. \tag{2.63}$$

The set of equations (2.61)-(2.63) is solved in Ref. [15] and more completely in [16, 17, 41] under the assumption that the field is spherically-symmetric and static,

$$d\tilde{s}^2 = \tilde{g}_{\alpha\beta}dx^{\alpha}dx^{\beta} = \mathrm{e}\tilde{y}(z)(dx^0)^2 - \mathrm{e}\tilde{\alpha}(z)dz^2 - \mathrm{e}\tilde{\beta}(z)d\Omega^2; \tag{2.64}$$

$F_{01} = F_{01}(z) = -F_{10}$; the rest $F_{\mu\nu} = 0$; $\psi = \psi(z)$, where $d\Omega^2 = (dx^2)^2 + \sin^2 x^2(dx^3)^2$. Let us use the simplest method [17]. So, we choose the coordinate z, so that

$$\tilde{\alpha}(z) = 2\tilde{\beta}(z) + \tilde{\gamma}(z). \tag{2.65}$$

Then, the solution of equations (2.62) and (2.63) has the form

$$\psi(z) = Cz; \quad \tilde{F}^{10} = q\,\mathrm{e}{-\tilde{\alpha}(z)}, \tag{2.66}$$

where we have put $z = 0$ at spatial infinity and taken into account the condition (2.36). The integration constant q has the meaning of electric charge and C may be similarly called "scalar charge".

Under the assumptions (2.64) among the equations (2.61) there are two independent ones. Using (2.66) they may be written in the form

$$\mathrm{e}\tilde{\alpha}(\tilde{G}^1_1 + \tilde{G}^2_2) = \frac{1}{2}(\tilde{\beta}'' + \tilde{\gamma}'') - \mathrm{e}\tilde{\beta} + \tilde{\gamma} = 0;$$
$$\mathrm{e}\tilde{\alpha}\tilde{G}^1_1 = -\mathrm{e}\tilde{\beta} + \tilde{\gamma} + \frac{1}{4}[(\tilde{\beta}' + \tilde{\gamma}')^2 - \tilde{\gamma}'^2] = \frac{n}{C}^2 - Q^2\,\mathrm{e}\tilde{\gamma}, \tag{2.67}$$

where a prime denotes d/dz and $Q^2 = \bar{k}q^2/2$.

Integrating (38), we finally get for the metric $g_{\mu\nu}$:

$$ds^2 = \mathrm{e}\gamma(z)(dx^0)^2 - \mathrm{e}\alpha(z)dz^2 - \mathrm{e}\beta(z)d\Omega^2 \tag{2.68}$$
$$= F(\psi)\left\{Q^{-2}\frac{(dx^0)^2}{s^2(h, z+z_1)} - Q^2\frac{s^2(h, z+z_1)}{s^2(k, z)}\left[\frac{dz^2}{s^2(k, z)} + d\Omega^2\right]\right\}, \tag{2.69}$$

where the function

$$S(a, x) = \begin{cases} a^{-1}\sinh ax & \text{for } a > 0, \\ x & \text{for } a = 0, \\ a^{-1}\sin ax & \text{for } a < 0; \end{cases} \tag{2.70}$$

and the integration constants h and k are connected by the relation

$$k^2 \operatorname{sign} k = h^2 \operatorname{sign} h + \frac{1}{2} n C^2. \tag{2.71}$$

For an arbitrary metric (2.68) the boundary condition that the space-time is flat at infinity is formulated as

$$\mathrm{e}\gamma \to 1; \quad \mathrm{e}\beta \to \infty; \quad \frac{1}{4}\beta'^2\,\mathrm{e}\beta - \alpha \to l \tag{2.72}$$

when z tends to some z_∞. For the metric (2.69) $z_\infty = 0$ and the condition (2.72) is fulfilled completely if

$$s^2(h, z_1) = Q^{-2} \tag{2.73}$$

(remember that $F(0) = 1$). With no loss of generality we may regard the coordinate z definition domain to be

$$0 < z < z_{\max} \tag{2.74}$$

where $z_{\max}$ depends on which of the functions $F(\psi)$, $s(h, z + z_1)$ or $s(k, z)$ will be the first to go to zero or infinity or, symbolically,

$$z_{\max} = \min\{\underset{a}{\infty};\ \underset{b}{\infty(F)};\ \underset{e}{\mathrm{zero}(F)};\ \underset{d}{\mathrm{zero}[(k,z)]};\ \underset{e}{\mathrm{zero}[s(h, z+z_1)]}\}. \tag{2.75}$$

The letters a, b, c, d, e mark the corresponding variants of the solution. Evidently for special values of the integration constants the quantities in the curly brackets in (2.75) may coincide. For these variants we use double or triple notations, *e. g.* cd or cde.

In the case $q = 0$ (vacuum) the equations (2.67) under the condition (2.72) yield

$$ds^2 = F(\psi)\left\{\mathrm{e}{-2hz}(dx^0)^2 - \frac{\mathrm{e}2hz}{s^2(k,z)}\left[\frac{dz^2}{s^2(k,z)} + d\Omega^2\right]\right\}. \tag{2.76}$$

The definition domain for z is again given by (2.74) and (2.75) but obviously in (46) the "e" possibility is absent.

Solution (2.76) contains three arbitrary constants h, C and $\underset{0}{\varphi}$. The latter is involved in $F(\psi)$ in an indirect form.

In order to have a convenient description of possible properties of the metrics (2.69) and (2.76), we introduce a classification of arbitrary spherically-symmetric metrics basing on the behaviour of the functions $\mathrm{e}\beta(z)$ and $\mathrm{e}\gamma(z)$; $\mathrm{e}\beta$ has an invariant meaning of the area of the sphere $z = \mathrm{const}$ (divided by 4π) and $\mathrm{e}\beta$ is an invariant (within the given physically preferable reference frame) time slowing-down factor in respect to distant points where the space-time may be treated as flat. The convergence or divergence of the integral

$$\int \mathrm{e}(\alpha - \gamma)/2dz \tag{2.77}$$

when the radial coordinate z tends to the end $\bar{z}$ of its definition domain, is also of interest. If the integral (2.77) converges, then (in the evident meaning) the points at $z = \bar{z}$ are observable.

We shall denote the behaviour of the metric (2.68) by means of two figures, the first one corresponding to the behaviour of $\mathrm{e}\gamma$ when $z \to \bar{z}$ and the second one to that of $\mathrm{e}\beta$. Namely we write 1, 2 or 3 if the function tends to zero, infinity or a finite value, respectively. Besides, we denote the convergence or divergence of integral (2.77) by a plus or minus sign. *E. g.* the Schwarzschild metric belongs to class 13 because when the spherical radius r tends to the gravitational radius r_g, $\mathrm{e}\gamma(z) \to 0$, $\mathrm{e}\beta(z) = r^2 \to \mathrm{const}$ and the sphere $r = r_g$, is invisible for an observer at rest.

Let us give a brief characteristic of the obtained classes.

11, 21, 31. The three-space includes the center. Moreover, in class 31 there are singularity-free metrics. This is so if the local euclidity conditions are fulfilled: for $z \to \bar{z}$

$$\mathrm{e}\gamma \to \mathrm{const}, \quad \gamma' \to 0, \quad (z-\bar{z})^2 \,\mathrm{e}\alpha - \beta \to 1. \tag{2.78}$$

12, 22, 32. The three-space has a "neck" that means that $\mathrm{e}\beta(z)$ has a minimum. In case 32 $z = \bar{z}$ corresponds to another spatial infinity (not always flat) rather than to a singularity. A two-dimensional analog of such geometry is a hyperboloid of one sheet.

13, 23. To study the geometry completely it is necessary to convert to another reference frame perhaps allowing to continue the metric further than $z = \bar{z}$.

33. The coordinate z should be changed to a more licky one ($z \to z'(z)$) allowing to penetrate further than $z = \bar{z}$ by analytic continuation. One will naturally find one of the rest eight classes.

The proposed classification allows us to describe the solutions (2.69) and (2.76) in a compact way, see Table I. The table shows that classes 13 and 23 containing Schwarzschild-type singularities may appear only for special values of the constants, whatever the function $F(\psi)$ is.

Of certain interest are the solutions of class 32 which occur for general values of the constants only for $n = -1$. It is easily assured that at $z \to z_{\max}$ the 32 metrics (2.69) and (2.76) become flat but the time has in general another rate than at $z = 0$. One may try to apply such metrics to describe topologies of the kind of Wheeler handles [18] but, unlike the familiar Schwarzschild and Reissner-Nordstrom metrics, these ones are singularity-free. It may be shown that under natural assumptions on the orders of the quantities involved in (2.69) and (2.76), the "necks" dimensions, *i.e.* $\mathrm{e}\beta_{\min}/2$, are of the order of gravitational radii $2GMc^{-2}$ for corresponding masses.

Note that for the solution (2.69) the complete electromagnetic field energy in the outer region of an arbitrary sphere $z = z^*$ may be found:

$$\int T_0^0 \sqrt{-g} d^3x = \frac{4\pi}{\bar{k}} \int_0^{z^*} \frac{dz}{s^2(h, z+z_1)} = \frac{4\pi}{\bar{k}} \left[\frac{s'}{s}(h, z_1) - \frac{s'}{s}(h, z^* + z_1) \right]. \tag{2.79}$$

The integral (2.79) taken over the whole space ($z^* = z_{\max}$ is finite for all variants of $z_{\max}$ given in (2.78) but those containing e.

TABLE I

Possible behaviour of the metrics (2.69) and (2.76)

Variants by (46)	Behaviour	Solution (2.69)				Solution (2.76)	
		$n=1$, $h\geq 0$	$n=1$, $h<0$	$n=-1$, $h\geq 0$	$n=-1$, $h<0$	$n=1$ or $n=-1$, $\lvert h\rvert \geq \lvert C\rvert/\sqrt{2}$	$n=-1$, $\lvert h\rvert < \lvert C\rvert/\sqrt{2}$
a	$(^*)$	+	−	+	−	+	−
b	22_+	+	+	+	+	+	+
c	11_+	+	+	+	+	+	+
d	32_-	−	−	−	+	−	+
e	21_+	+	+	+	+	−	−
bd	22_-	−	−	−	+	−	+
cd	$(^{**})_-$	−	−	−	+	−	+
be	$(^*_*)_-$	+	+	+	+	−	−
ce	$(^{***})_+$	+	+	+	+	−	−
de	23_+	−	−	−	+	−	−
bde	22_+	−	−	−	+	−	−
cde	$(^{***})_+$	−	−	−	+	−	−

Comments: In the six right-hand columns the "$+$" sign means that there exist $F(\psi)$ for which the metric behaves correspondingly. Otherwise the "$-$" sign is written.
$(^*)$ is any of the nine classes depending on the integration constants values and the form of $F(\psi)$. Classes 13, 23, 31, 32 may occur only for special choices of the constants and class 33, moreover, for special $F(\psi)$.
$(^{**})$ means classes 11, 12 or 13; $(^*_*)$ classes 21, 22 or 23; $(^{***})$ classes 11, 21 or 31, depending on the function $F(\psi)$.

2.2.5 The exact solution and the post-Newtonian metric

Static spherically-symmetric PN metric out of the gravitational field sources, as well as vacuum metric (2.76) in the asymptotic region, may be presented as a series in inverse powers of the so-called isotropic radius r:

$$ds^2 = \left(1 + \frac{\xi_1}{r} + \frac{\xi_2}{r^2} + \dots\right)(dx^0)^2 - \left(l - \frac{\eta_1}{r} + \dots\right)(dr^2 + r^2 d\Omega^2). \qquad (2.80)$$

Moreover, in the PN expansion the factors $\xi_1, \dots, \eta_1, \dots$ are in their turn power series in c^{-1}:

$$\xi_1 = -\frac{2GM_0}{c^2} + O(c^{-4}); \quad \xi_2 = 2\beta_0 \frac{G^2 M_0^2}{c^4} + O(c^{-6});$$
$$\eta_1 = 2\gamma_0 \frac{GM_0}{c^2} + O(c^{-4}) \ , \qquad (2.81)$$

where

$$M_0 = 4\pi \int \varrho(r') r'^2 dr'.$$

For exact solution (2.76) ξ_1, ξ_2 and η_1 are combinations of the integration constants:

$$\xi_1 = \lim_{z\to\infty}[-\varepsilon\gamma' \,\mathrm{e}(2\beta-\alpha)/2] = -2h + CF_0';$$
$$\xi_2 = \frac{1}{2}\lim_{z\to\infty} \mathrm{e}(3\beta-\alpha)/2[\varepsilon\gamma' + \mathrm{e}(\beta-\alpha)/2(\gamma'' + \gamma'^2 + \frac{1}{2}\beta'\gamma' - \frac{1}{2}\alpha'\gamma')]$$
$$= \frac{1}{2}[C^2(F_0'' - F'^2_0) + (2h - CF_0')^2];$$
$$\eta_1 = \lim_{z\to\infty} \mathrm{e}\beta/2[2 - |\beta'|\,\mathrm{e}(\beta-\alpha)/2] = 2h + CF_0', \tag{2.82}$$

where $\varepsilon = \operatorname{sign}\beta'$ at $z \to z_\infty$. After the first equality sign the expressions for ξ_1, ξ_2 and η_1 are given for arbitrary metric (2.68) satisfying the boundary condition (2.72).

Comparing (2.81) and (2.82), we obtain the constants h and C as expansions in c^{-1}:

$$h = \frac{GM_0}{c^2(1+\eta)} + O(c^{-4}); \quad C = -nF_0'\frac{GM_0}{c^2(1+\eta)} + O(c^{-4}). \tag{2.83}$$

In the PN approximation only the senior terms of these series are taken into consideration. (The third integration constant of the solution (2.76), $\underset{0}{\varphi}$ is the same as in the PN approximation.)

Thus, in the solution (2.76) the constant h is related to the active gravitational mass which in accordance with (2.83) is mainly determined by the rest mass M_0 if the PN expansion is applicable. As for the independent constant C, it is forcedly connected with the mass in the PN metric.

2.2.6 Discussion

The results of Section 2.2.5 may be interpreted as imputing a special value to the ratio of the scalar charge density and the rest mass density in theory (2.31). If this ratio differs from its standard value dictated by (2.83), it should affect observable phenomena. Thus, we conclude that it is desirable to introduce the scalar charge explicitly into the theory, writing in the Lagrangian (2.31) an additional term of the form, say,

$$S(\varphi)j(x), \tag{2.84}$$

where $S(\varphi)$ is a certain function and $j(x)$ is independent of φ and plays the role of charge density which naturally should be related to some matter parameters. Such a term describes direct scalar field – matter interaction and its appearance does not look unexpected.

However, if the term (2.84) is present, theory (2.31) is no more a metric one. On the right side of equation of motion (2.30) there emerges an expression proportional to $Sj_{,\mu}$. Nevertheless, if the quantity j is proportional to the energy – momentum tensor trace or to the matter density, then the senior order of c^{-1} expansion, similar to 2.2.3, gives Newton's law for the interaction of point particles. The PN metric and the PN equations of motion are got in a rather cumbersome manner in this case and we will not give them here. The whole situation is rather well illustrated by the following simple formal modification of the theory. In equation (2.43) for the transformed scalar field ψ in the second term (which plays the role of scalar charge density) write an indefinite factor σ, leaving equations (2.42) for the metric field unchanged. Then, carrying out the expansion in c^{-1}, it is easily assured that for $\sigma = \mathrm{const}$, in the senior order Newton's law with the constant

$$G = \bar{G}\left(1 + \frac{1}{2}\sigma nF'^2_0\right) \tag{2.85}$$

is valid and the PN metric again takes form (2.55) with PPN parameters (2.56), where instead of η and λ there stand

$$\eta^+ = \sigma\eta; \quad \lambda^+ = \sigma^2\lambda. \tag{2.86}$$

This metric accords with exact solution (2.76) with arbitrary C and h. (Note that inclusion of terms of the type (2.84) does not alter the form of the exterior vacuum solution.) Introduction of σ increases the uncertainty in interpreting the results of PN effects measurements. In fact, if such measurements are in agreement with (2.56), then they establish within this "σ formalism" only the values of η^+ and λ^+ with unknown σ. (For expressions for concrete effects using the PPN parameters see *e.g.* in paper [14].) More details may be obtained only by studying effects out of the PN frames, *e.g.* time variation of the gravitational constant due to the Universe expansion.

In the notation (2.31) the local constant of gravitation and its variation due to variation of φ are given in the σ formalism by the formulae, generalizing those given by Nordtvedt (Ref. [12], Appendix):

$$G = \frac{G_0}{\varphi}\frac{2\omega + 3 + \sigma}{2\omega + 3}; \quad \frac{G_{,\alpha}}{G} = \left[1 + \frac{2\sigma\omega_{\varphi}\varphi}{(2\omega+3)(2\omega+3+\sigma)}\right]\frac{\varphi_{,\alpha}}{\varphi}. \tag{2.87}$$

However, one should be careful in comparing this kind of formulae with observations using cosmological solutions for φ because perhaps scalar fields, local from the cosmological viewpoint (*e. g.* that of Galaxy) may mask possible changes of the cosmological background.

Note that, besides scalar-tensor theories, variable gravitational coupling was considered by Staniukovich [19].

Up to now, we have been using the assumption that in the senior order of the c^{-1} expansion, Newton's law should be fulfilled. However, it seems physically more plausible to introduce scalar charge as an elementary particle characteristic, *e. g.* as a function of the particle rest mass. Then, one obtains different values of the constant G for different substances.

2.2.7 Examples

Complex scalar field. There is no evident reason to consider the scalar field φ in scalar-tensor theories to be real *a priori*. The complexity of φ may play some role in cosmological (see [21]) or other problems where it yields an additional degree of freedom. However, we will convince ourselves that changing of $A(\varphi)$, $B(\varphi)$ and $\varphi_{,\alpha}\varphi_{,\beta}$ in the Lagrangian (2.31) for $A(|\varphi|)$, $B(|\varphi|)$ and $\overset{*}{\varphi}_{,\alpha}\varphi_{,\beta}$ respectively, does not alter the metrics obtained in 2.2.3 and 2.2.4.

Denote

$$x = |\varphi|; \quad y = \arg\varphi. \tag{2.88}$$

The transformation (2.33), (2.34) (where instead of φ and ψ we write x and $\tilde{x}$) brings the initial equations to the form

$$\tilde{G}_{\mu\nu} = -\bar{k}F(\tilde{x})T_{\mu\nu} - n\tilde{S}_{\mu\nu}(\tilde{x}) - \bar{B}Fx^2\tilde{S}_{\mu\nu}(y), \tag{2.89}$$

$$2n\Box\tilde{x} + F\frac{dF}{d\tilde{x}}\bar{k}T - \frac{d}{d\tilde{x}}(\bar{B}Fx^2)\tilde{g}^{\alpha\beta}y_{,\alpha}y_{,\beta} = 0, \tag{2.90}$$

$$\bar{B}Fx^2\Box y + \frac{d}{d\tilde{x}}(\bar{B}Fx^2)\tilde{g}^{\alpha\beta}\tilde{x}_{,\alpha}y_{,\beta} = 0 \tag{2.91}$$

with the notations (2.32) and (2.44) under the assumptions (2.39). From a PPN expansion for (2.91) it follows:

$$y = \underset{0}{y} = \text{const} \tag{2.92}$$

and hence $\tilde{S}_{\mu\nu}(y) \equiv 0$, and thus y does not enter the further calculations. Thus the formulae (2.56) for the PPN parameters remain unchanged but in (2.58) the index φ should be changed for x.

A static spherically-symmetric solution with the electric field may be found using the transformation

$$g_{\mu\nu} = \bar{A}^{-1}\tilde{g}_{\mu\nu}; \quad \frac{dx}{d\tilde{x}} = \frac{x}{\tilde{x}}\left(\frac{\bar{A}\bar{B}}{\bar{A}\bar{B} + (3/2)\bar{A}_x^2}\right)^{1/2} \tag{2.93}$$

which leads to the field equations

$$\tilde{G}_\mu^\nu = -\tilde{k}[-\tilde{F}^{\nu\alpha}F_{\mu\alpha} + \frac{1}{4}\delta_\mu^\nu\tilde{F}^{\alpha\beta}F_{\alpha\beta}] - nD(\overset{*}{\psi}\,\psi)[\tilde{g}^{\nu\alpha}\overset{*}{\psi}_{,\mu}\psi_{,\alpha} - \frac{1}{2}\delta_\mu^\nu\tilde{g}^{\alpha\beta}\overset{*}{\psi}_{,\alpha}\psi_{,\beta}], \tag{2.94}$$

$$\Box\psi + \frac{d}{d\psi}(\ln D)\tilde{g}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} = 0, \tag{2.95}$$

with the Maxwell equations (2.63) added, where

$$\psi = \tilde{x}\,\mathrm{e}iy; \quad nD(\overset{*}{\psi}\,\psi) = \bar{B}x^2/(\bar{A}\tilde{x}^2); \quad n = \mathrm{sign}\,B. \tag{2.96}$$

Under the assumptions (2.64), (2.65) the equation (2.95) is reduced to ,

$$\psi'' + \frac{d}{d\psi}(\ln D)\varphi'^2 = 0. \tag{2.97}$$

Dividing this equation by ψ' and adding its complex-conjugate, we obtain an exact differential equation which gives:

$$D(\overset{*}{\psi}\,\psi)\,\overset{*}{\psi}{}'\,\psi' = C^2 = \mathrm{const}, \tag{2.98}$$

and the further calculation repeats exactly that for the real field.

Some particular cases. We point out some particular cases of gravitation theories described by Lagrangian (2.31) under assumptions (2.39).

1. General relativity:

$$A(\varphi) \equiv 1; \quad B(\varphi) \equiv 0; \quad F(\varphi) \equiv 1; \quad \eta = \lambda = 0. \tag{2.99}$$

2. The Brans-Dicke theory [22]:

$$\begin{aligned} &A(\varphi) = \varphi; \quad B(\varphi) = \omega/\varphi; \quad \omega = \mathrm{const} \neq -3/2; \\ &F(\psi) = \exp[-\psi/\sqrt{|\omega + 3/2|}]; \quad \eta = (2\omega + 3)^{-1}; \quad \lambda = (2\omega + 3)^{-2}. \end{aligned} \tag{2.100}$$

The PPN parameters calculated from here using (2.56) coincide with those given in paper [8].

3. The Zaitsev-Kolesnikov theory with a conformally covariant scalar field [9, 16, 23]:

$$\begin{aligned} &A(\varphi) = 1 + \frac{1}{6}\varphi^2; \quad B(\varphi) = -1; \\ &F(\psi) = \frac{\cos^2[(\psi + \psi_0)/\sqrt{6}]}{\cos^2(\psi^0/\sqrt{6})}; \quad \psi_0/\sqrt{6} = \arctan(\varphi/\sqrt{6}); \\ &\eta = -\frac{1}{18}\overset{2}{\underset{0}{\varphi}}; \quad \lambda = -\frac{1}{108}(1 - \frac{1}{6}\overset{2}{\underset{0}{\varphi}}) = \frac{1}{6}\eta(1 + 3\eta). \end{aligned} \tag{2.101}$$

4. This scheme includes also the case of material conformally covariant scalar field in general relativity which has been studied in papers [4, 41, 17]:

$$A(\varphi) = 1 - \frac{1}{3}k_0\varphi^2; \quad B(\varphi) = 2k_0, \tag{2.102}$$

with Einstein value of the constant k_0;

$$F(\psi) = \frac{\cosh^2[(\psi+\psi_0)/\sqrt{6}]}{\cosh^2(\psi_0/\sqrt{6})}; \quad \psi_0/\sqrt{6} = \tanh^{-1}(\sqrt{k_0/3\varphi}). \tag{2.103}$$

For this $F(\psi)$ solutions (2.69) and (2.76) completely coincide with the corresponding solutions in [17] if one changes the notation in the following way:

$$\begin{aligned} &k \to k\cosh^2 z_0; \quad \frac{|C|z}{\sqrt{6}} \to z; \quad \frac{\psi_0}{\sqrt{6}} \to z_0; \\ &\frac{\sqrt{6}h}{|C|} \to h; \quad \frac{\sqrt{6}k}{|C|} \to k; \quad C^2 \to 2k|C|^2\cosh^4 z_0. \end{aligned} \tag{2.104}$$

The properties of metrics (2.69) and (2.76) for the concrete cases 2, 3 and 4 are given in Table 2. In case 4 at $h = |C|/\sqrt{6}$ these metrics belong to class 33_+ and their further study takes place in the transformed coordinates:

$$y = \coth(|C|z/\sqrt{6}) \text{ for } \psi_0 \neq 0; \quad r = (y+y_1)\sqrt{C^2/6} \text{ for } \psi_0 = 0, \tag{2.105}$$

in which the metric takes the form

$$\begin{aligned} &ds^2 = (y+y_0)^2\left\{\frac{(dx^0)^2}{(y+y_1)^2} - \frac{1}{6}C^2\frac{y+y_1}{y^4}(dy^2 + y^2 d\Omega^2)\right\} \text{ for } \psi_0 \neq 0, \\ &ds^2 = (1-r_0/r)^2(dx^0)^2 - (1-r_0/r)^{-2}dr^2 - r^2 d\Omega^2 \text{ for } \psi_0 = 0. \end{aligned} \tag{2.106}$$

where

$$\begin{aligned} &y_0 = \tanh(\psi_0/\sqrt{6}); \quad r_0 = y_1|C|/\sqrt{6}; \\ &y_1 = \begin{cases} \coth(|C|z_1/\sqrt{6}) & \text{for solution (2.69)}, \\ 1 & \text{for solution (2.76)}. \end{cases} \end{aligned} \tag{2.107}$$

TABLE II

Properties of the solutions (2.69) and (2.76) in some particular cases

Example 2. Brans-Dicke theory

Constants	$\omega > -3/2$		$\omega < -3/2$					
	$h < 0$ or $z_1 < 0$	$h > 0$ $z_1 > 0$	$h < \lvert C\rvert/\sqrt{2}$ or $z_1 < 0$			$h \geq \lvert C\rvert/\sqrt{2}$, $z_1 > 0$		
						$h < h_0$	$h = h_0$	$h > h_0$
Variants by (2.75)	e	a	d	e	de	a	a	a
Behaviour	21_+	11_+	32_-	21_+	23_+	12_-	13_-	11_-

Example 3. Zaitsev-Kolesnikov theory

Variation by (2.75)	c	d	e	cd	ce	de	cde
Bechaviour	11_+	32_-	21_+	13_-	31_+	23_+	31_+

Example 4. Conformally covariant scalar field in general relativity

Constant	$h > \lvert C\rvert/\sqrt{6}$, $z_1 > 0$	$h < 0$ or $z_1 < 0$	$0 \leq h < \lvert C\rvert/\sqrt{6}$ $z_1 > 0$	$h = \lvert C\rvert/\sqrt{6}$, $z_1 > 0$		
				$\psi_0 > 0$	$\psi_0 = 0$	$\psi_0 < 0$
Variations by (2.75)	a	e	a	a	a	a
Behaviour	12_+	21_+	21_+	32_-	13_-	11_+

$$h_0 = \frac{1}{4}|C|(|\omega + 3/2|^{-1/2} + 2|\omega + 3/2|^{1/2})$$

Comment: Variants of the behaviour of solution (2.69) are given in the table. To get variants for (2.76) one should reject those involving e.

2.3 Cosmological Solutions with Conformal Scalar Field [4]

2.3.1 In papers by M.A.Markov [24, 25] and K.P. Stanukovich [19] a hypothesis on the existence of particles with dimensions coinciding with their gravitational radius and mass about 10^{-5} g was suggested. They based their arguments on dimensional considerations. In this paper an attempt is made to obtain similar results by solving a set of scalar and gravitational fields equations.

It should be noted here that such an approach to the case of Dirac and Einstein equations was advanced first by Markov [25].

2.3.2 Consider a system with Lagrangian density

$$L = \frac{1}{2}\left\{g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \left(m^2 + \frac{R}{6}\right)\varphi^2\right\} + \frac{R - 2\lambda}{2æ}, \tag{2.108}$$

where φ is a scalar field, which is assumed to be real; æ is the gravitational constant, λ is the "cosmological constant", R is a scalar curvature and ∇_μ – covariant derivative.

We take a metric as $ds^2 = g_{\mu\nu}dx^\mu x^\nu$. The sign of curvature tensor is chosen so that

$$\nabla_\mu\nabla_\nu A_\sigma - \nabla_\nu\nabla_\mu A_\sigma = -R^\alpha_{\mu\nu\sigma}A_\alpha \ ,$$

where A_μ is an arbitrary vector. Greek indices range over 0, 1, 2, 3. Here and then we use the system of units with $c = \hbar = 1$.

Variating the action integral

$$S = \int z\sqrt{-g}d^4x, \quad d^4x = dx^0dx^1dx^2dx^3 \tag{2.109}$$

with respect to φ and $g^{\mu\nu}$, we get the equation of scalar field

$$\left(\Box + m^2 + \frac{R}{6}\right)\varphi = 0, \tag{2.110}$$

where $\Box = \nabla^\mu\nabla_\mu$ is the D'Alambert operator, and that of gravitational field

$$R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R + \delta^\nu_\mu\lambda = -æT^\nu_\mu \tag{2.111}$$

with energy-momentum tensor of matter (scalar field)

$$T^\nu_\mu = \nabla_\mu\varphi\nabla^\nu\varphi - \frac{1}{2}\delta^\nu_\mu\left[\nabla^\rho\varphi\nabla_\rho\varphi - \left(m^2 + \frac{R}{6}\right)\varphi^2\right] - \frac{1}{6}(R^\nu_\mu + \nabla_\mu\nabla^\nu - \delta^\nu_\mu\Box)\varphi^2 \tag{2.112}$$

respectively, Tensor (2.112) has the properties:

$$T_{\mu\nu} = T_{\nu\mu}; \quad \nabla_\mu T^\mu_\nu = 0; \quad T = T^\mu_\mu = m^2\varphi^2 \tag{2.113}$$

In the scalar field formalism we follow the paper by Chernikov and Tagirov [26], since equation (2.110) is conformal invariant for $m = 0$ (Penrose, [27]) and admits a proper quasiclassical transition [26, 28].

2.3.3 As a solution for the metric let us assume the Einstein static model

$$ds^2 = dt^2 - \rho^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\Phi^2)], \quad \rho = \text{const.} \tag{2.114}$$

and consider one-particle state of φ field. Since the object to be found is a closed micro-universe, then its total charge must be zero. It was the reason why the real φ field was chosen. One-particle normalisation condition for non-quantised φ field

$$i\int_\Sigma(\overset{*}{\varphi}\cdot\partial_\alpha\varphi - \partial_\alpha\overset{*}{\varphi}\cdot\varphi)d\Sigma^\alpha = 1, \tag{2.115}$$

where $d\Sigma^\alpha$ is a vector square element on an arbitrary spacelike hypersurface Σ, is inconsistent on the assumption that it is real. Therefore we assume the scalar field to be secondary-quantised.

According to [28], Heisenberg operator $\hat\varphi$, satisfying equation (2.110), is written in the form

$$\hat\varphi = \frac{1}{\sqrt{2}\rho^{3/2}}\sum_{s=0}^{\infty}\sum_{\sigma=1}^{(s+1)^2}P^{s\sigma}(x^1, x^2, x^3)\left\{\frac{\mathrm{e}ikt}{\sqrt{k}}c^+_{s\sigma} + \frac{\mathrm{e}-ikt}{\sqrt{k}}c^-_{s\sigma}\right\}, \tag{2.116}$$

where $P^{s\sigma}$ are orthonormalized harmonic polynomials on a unit three-sphere, $c^\pm_{s\sigma}$ are creation (+) and annihilation (−) operators for particles with a definite momentum s [26] (σ denoting other quantum numbers), obeying canonical commutation rules

$$[c^\pm_{s\sigma}, c^\pm_{s'\sigma'}] = 0; \quad [c^-_{s\sigma}, c^+_{s'\sigma'}] = \delta_{ss'}\delta_{\sigma\sigma'}; \tag{2.117}$$

$$k = k(s) = \sqrt{\frac{(s+1)^2}{\rho^2} + m^2}. \tag{2.118}$$

Write the assumed one-particle φ field state as

$$| 1 >=| s\sigma >= c^+_{s\sigma} | 0 >, \tag{2.119}$$

where $\mid 1>$ is an eigen-vector of the energy operator,

$$\hat{E} = \int \hat{T}_{00} d\Sigma = \frac{1}{2} \sum_{s=0}^{\infty} \sum_{\sigma=1}^{(s+1)^2} k(c^+_{s\sigma} c^-_{s\sigma} + c^-_{s\sigma} c^+_{s\sigma}), \quad : \hat{E} :\mid 1 >= k \mid 1 > \tag{2.120}$$

with

$$d\Sigma = \rho^3 \sin^2 \chi \sin \theta d\chi d\theta d\Phi.$$

As the gravitational field is not quantized, we have to substitute T^ν_μ on the right-hand side of equations (2.111) by the averages

$$<: \hat{T}^\nu_\mu :>=< 1 \mid: \hat{T}^\nu_\mu :\mid 1 >, \quad R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R + \delta^\nu_\mu \lambda = -æ <: \hat{T}^\nu_\mu :>, \tag{2.121}$$

where symbol ::, as usually, denotes the normal product.

In metric (2.114) tensor R^ν_μ has the following non-vanishing components:

$$R^1_1 = R^2_2 = R^3_3 = \frac{2}{\rho^2}, \tag{2.122}$$

and scalar curvature $R = 6/\rho^2$.

Since (2.113) gives $\hat{T}^\mu_\mu = m^2 \hat{\varphi}^2$, simplifying equations (2.121),

$$-R + 4\lambda = æm^2 <: \hat{\varphi}^2 :> \tag{2.123}$$

we obtain

$$< 1 \mid: \hat{\varphi}^2 :\mid 1 >= \text{const}$$

Using (2.116), we get

$$< 1 \mid: \hat{\varphi}^2 :\mid 1 >= \frac{(P^{s\sigma})^2}{\rho^3 k}. \tag{2.124}$$

¿From properties of harmonic polynomials and from (2.124) it follows that

$$S = 0, \quad P^{s\sigma} = P^{01} = \frac{1}{\sqrt{2}\pi}; \quad <: \hat{\varphi}^2 :>= \frac{1}{2\pi^2 \rho^3 k^0} = \frac{2}{\gamma^2},$$
$$k^0 = k(0) = \sqrt{m^2 + \frac{1}{\rho^2}}.$$

After rather simple calculations we have

$$<: \hat{T}^0_0 :>= 2{k^0}^2 \gamma^2; \quad <: \hat{T}^1_1 :>=<: \hat{T}^2_2 :>=<: \hat{T}^3_3 :>= -\frac{2}{3}\frac{\gamma^2}{\rho^2}. \tag{2.125}$$

Other component of $<: T^\nu_\mu :>$ vanish.

Substituting (2.122) and (2.125) into (2.121), we come to two non-coinciding equations

$$2æ\gamma^2 {k^0}^2 \rho^2 = 3 - \Lambda; \tag{2.126}$$
$$\frac{2}{3}æ\gamma^2 = \Lambda - 1, \tag{2.127}$$

where $\Lambda = \lambda\rho^2$; $\gamma = (4\pi^2 \rho^3 k^0)^{1/2}$. As the left parts of (2.126), (2.127) are positive,

$$1 < \Lambda < 3.$$

The set (2.126), (2.127) leads to a square equation with respect to k^0:

$$3æ\rho^2 k^{0^2} - 12\pi^2\rho^3 k^0 + æ = 0,$$
$$k^0 = \frac{1}{æ\rho}\left(2\pi^2\rho^2 \pm \sqrt{4\pi^4\rho^4 - \frac{1}{3}æ^2}\right). \tag{2.128}$$

The condition that energy k^0 is real

$$\rho \geq \rho_0 = \frac{\sqrt{æ}}{\pi\sqrt[4]{12}} \tag{2.129}$$

proves the existence of a minimal radius ρ_0. The corresponding value of Λ is $\Lambda_0 = 2$. When $\rho = \rho_0$, the solution for k^0 is unique,

$$k^0 = \pi\sqrt[4]{\frac{4}{3}}æ^{-1/2},$$

and a formula similar to the uncertainty relation

$$k^0(\rho_0)\cdot\rho_0 = \frac{1}{\sqrt{3}} \tag{2.130}$$

is valid. Calculations for this case give

$$\rho_0 \approx 4\cdot 10^{-33}cm; \quad k^0(\rho^0) \approx 0,5\cdot 10^{-6}g;$$
$$\lambda(\rho_0) \approx 1,2\cdot 10^{65}cm^{-2}. \tag{2.131}$$

It is of interest to discuss separately the case of a massless scalar field, $m = 0$. Then, by the definition of k^0, the condition

$$k^0\rho = \sqrt{m^2\rho^2 + 1} = 1 \tag{2.132}$$

is added to the set of equations (2.126), (2.127). Solving these three equations, we find:

$$\rho = \frac{\sqrt{æ}}{\pi\sqrt{3}} \approx 4,2\cdot 10^{-33}cm; \quad k^0 = \frac{\pi\sqrt{3}}{\sqrt{æ}} \approx 0,9\cdot 10^{-6}g;$$
$$\Lambda = 3/2; \quad \lambda \approx 0,8\cdot 10^{65}cm^{-2}. \tag{2.133}$$

Thus, basing on a consistent consideration of the scalar and gravitational fields interaction problem, without any additional assumptions (the uncertainty relation is got as a result) we obtained a solution in the form of Einstein static closed micro-universe with well known parameters: radius of the order 10^{-33} cm and mass about 10^{-6} g.

Similar results are obtained if we consider not conformally invariant scalar field, using instead of 1/6 before R arbitrary constant.

It should be noted that the use of a complex scalar field, both non-quantized and normalized as (2.115), and the one in the secondary quantization scheme, leads to the same results [29].

2.3.4 Now the question arises: is our object stable or not? Such an investigation based on the Lyapunov criterion was performed by K.Piragas [30]. He showed that the Einstein cosmological model is unstable if

$$4\pi + 3\frac{\partial\rho}{\partial\varepsilon}\Bigg|_{\varepsilon\,=\,\varepsilon_{\text{equilibrium}}} > 0,$$

P and ε denote pressure and energy dansity of matter respectively, and $\partial\rho/\partial\varepsilon$ is determined from the equation of state. When the inequality sign is inverse, both stability and instability are possible.

Evidently in our case we should put

$$\varepsilon =<: \hat{T}_0^0 :>= 2k^{0^2}\gamma^2; \quad p = - <: \hat{T}_1^1 :>= \frac{2}{3}\frac{\gamma^2}{\rho^2}.$$

Then equations (2.126), (2.127) take the form

$$\varepsilon æ\rho^2 = 3 - \Lambda; \quad \rho æ\rho^2 = \Lambda - 1.$$

Hence we get the "equation of state"

$$æ^4(\varepsilon + \rho)^4 - 192\pi^4\varepsilon\rho = 0.$$

and

$$\frac{d\rho}{d\varepsilon} = -\frac{3\varepsilon\rho - \rho^2}{3\varepsilon\rho - \varepsilon^2} = -\frac{(\Lambda - 1)(5 - 2\Lambda)}{(3 - \Lambda)(2\Lambda - 3)}.$$

The Piragas condition leads to the conclusion that our micro-universe is unstable for $3/2 < \Lambda < 3$. For the rest possible values of $\Lambda(1 < \Lambda \leq 3/2)$ further investigation is necessary.

2.3.5 Using the formalism of previous subsections, let us consider small partubations of our system. Now we take metric in the form

$$ds^2 = (1 + 2\alpha(t))[dt^2 - \rho^2 d\chi^2 - \rho^2 \sin^2\chi(d\theta^2 + \sin^2\theta d\Phi^2)] \tag{2.134}$$

where $\alpha(t) \ll 1$. Then operator $\hat{\varphi}$ (solution of (2.110)) will be written as

$$\hat{\varphi} = \frac{(1 - \alpha)}{\sqrt{2}\rho^{3/2}} \sum_{s=0}^{\infty} \sum_{\sigma=1}^{(s+1)^2} P^{s\sigma}(x^1, x^2, x^3)[u_s(t)c_{s\sigma}^{+} + u^*(t)c_{s\sigma}^{-}] \tag{2.135}$$

with functions u_s obeying equations

$$\ddot{u}_s + [\frac{(s + 1)^2}{\rho^2} + m^2(1 + 2\alpha)]u_s \equiv \ddot{u}_s + k^2 u_s + 2\alpha m^2 u_s = 0 \tag{2.136}$$

and condition

$$u_s\dot{u}_s^* - \dot{u}_s u_s^* = -2i \tag{2.137}$$

Terms of $0(\alpha)$ type are neglected. All the notation of section 2.3.3 is kept here. As before, we take the scalar field state in the form (2.119).

To examine equation (2.121) it is enough to write down R_0^0 and R in metric (2.134)

$$R_0^0 = 3\ddot{\alpha}, \quad R = \frac{6}{\rho^2} - \frac{12\alpha}{\rho^2} + 6\ddot{\alpha} \tag{2.138}$$

and averages $<: \hat{T}_0^0 :>$ and $<: \hat{T} :>$ for the state (2.119)

$$<: \hat{T}_0^0 :>= (1 - 4\alpha)k^{0^2}\gamma^2\dot{u}_0^*\dot{u}_0 + (k^{0^2} - 2\alpha k^{0^2} - \frac{2\alpha}{\rho^2})k^0\gamma^2 u_0^* u_0;$$
$$<: \hat{T} :>= 2m^2\gamma^2 k^0 u_0^* u_0(1 - 2\alpha). \tag{2.139}$$

Let

$$u_0 = \frac{\mathrm{e}ik^0t}{\sqrt{k^0}}(1 + \sigma); \quad \sigma(t) \ll 1.$$

Denote $2\text{Re}\sigma = X(t)$; $\quad 2\text{Im}\sigma = -Y(t)$.

Supposing $\alpha = 0$, $\sigma = 0$ (zero order approximation), we get the equation of section 2.3.3. Let us consider them to hold. Then, in the first order (when all higher order terms with respect to α, σ and their derivatives are neglected) equations (2.121), (2.136) give:

$$\frac{3\alpha}{\rho^2} = æ\gamma^2[3k^{0^2} + \frac{1}{\rho^2} + \frac{1}{2}k^0(-\dot{Y} + 2k^0X)], \tag{2.140}$$

$$-\frac{6\alpha}{\rho^2} + 3\ddot{\alpha} = m^2æ\gamma^2(X - 2\alpha), \tag{2.141}$$

$$\ddot{Y} - 2k^0\dot{X} = 0, \tag{2.142}$$

$$\ddot{X} + 2k^o\dot{Y} + 4m^2\alpha = 0. \tag{2.143}$$

¿From condition (2.137) it follows that

$$\dot{Y} - 2k^0X = 0, \tag{2.3.35a}$$

so that (2.142) is satisfied identically. As one can easily find, (2.140) also becomes an identity. Hence it is enough to consider (2.141) and

$$\ddot{X} + 4k^{0^2}X + 4m^2\alpha = 0, \tag{2.144}$$

which arises from (2.143) and (2.142), e.g. a set of two linear second order equations with constant coefficients. It is natural to regard the obtained micro-universe to be stable if function $\alpha(t)$ (the solution of this set) cannot contain an exponentially growing term. When $m = 0$, the solution of (2.141)

$$\alpha = c_1\, \mathrm{e}\sqrt{2}t/\rho + c_2\, \mathrm{e}-\sqrt{2}t/\rho; \quad c_1, c_2 = \text{const},$$

immediately brings us to a conclusion of the instability of our object. If $m \neq 0$, $\alpha(t)$ satisfies the equation

$$\dddot{\alpha} + 2A\ddot{\alpha} + B\alpha = 0, \tag{2.145}$$

where it is convenient to express coefficients A and B in terms of Λ and ρ:

$$A = \frac{2(3-\Lambda^2)}{3\rho^2(\Lambda-1)}; \quad B = \frac{8}{3}\cdot\frac{2\Lambda^2 - 6\Lambda + 3}{\rho^4(\Lambda-1)}. \tag{2.146}$$

By the above mentioned conception, micro-universe stability condition can be written in the form

$$A > 0; \quad A^2 > B > 0. \tag{2.147}$$

Expressions (2.146) allow us to conclude that for $1 < \Lambda < 3$ these conditions cannot be satisfied. In fact, from $A > 0$ it follows that $\mid \Lambda \mid < \sqrt{3}$ and from $B > 0$ we come either to $\Lambda < (3-\sqrt{3})/2$ or to $\Lambda > (3+\sqrt{3})/2$.

2.3.6 Thus, within our model the obtained static solutions are unstable. Treatment of many-particle φ field states of the form

$$\mid N >= \frac{1}{\sqrt{N!}}(c_{01}^{+})^N \mid 0 >$$

changes only parametres of "zero order approximation" ($æ$ is substituted by $Næ$), and stability investigation, similar to section 2.3.5, gives exactly expressions (2.145), (2.146).

However, it should be pointed out again that we considered a non-quantized gravitational field. It is quite probable that taking into account quantum gravitational effects, which apparently may arise at distances about 10^{-33} cm, would bring certain changes into the obtained results. Unfortunately, only the first attempts are made to create a quantum theory of gravitation.

Besides, it is of certain interest to treat an interaction of a gravitational field with vector, tensor and spinor fields in a similar way. It should be kept in mind that these are to be taken with $m \neq 0$, since for any zero rest mass field $T^{\alpha}_{\alpha} = 0$, and investigation of section 2.3.5 type leads immediately to (2.141) with vanishing right-hand side, i.e. to the instability of the Einstein model.

2.3.7

Above mensioned N-particle self consistent problem one may solve for the massless conformal scalar field in nonstatic closed isotropic model [61,62].

$$ds^2 = b^2(t)[dt^2 - \rho^2 d\chi^2 - \rho^2 sin^2\chi d\Omega^2] \ .$$

In this metric the system (2.110-2.111) is written as

$$\begin{aligned}
&(b\varphi)^{\cdot\cdot} + b\varphi/\rho^2 = 0 \ ; \\
&3(1/\rho^2 + \dot{b}^2/b^2)[1 - (x/3)\varphi^*\varphi] - \lambda b^2 = x[\dot{\varphi}^*\varphi + (\dot{b}/b)(\varphi^*\varphi)] \ ; \\
&\ddot{b} + b/\rho^2 - 2\lambda b^2/3 = 0 \ .
\end{aligned}$$

From Einstein equations we use only the time component and contraction. Other equations are consequences of these two. Only the integration constant had to be chosen from the first order Einstein equation. The solution for the scalar field is

$$\varphi = \varphi_0 \ exp(-it/\rho)/b(t) \ .$$

The sign in the exponent is chosen according to the nonrelativistic limit and normalization condition (2.115), but in this case

$$d\Sigma = b^2\rho^3 \ sin^2\chi \ sin\theta \ d\chi d\theta d\varphi_1 \ .$$

Then, $\varphi_0 = \sqrt{N/}2\pi\rho$.

The contraction of Einstein equations is easily integrated in the metric

$$ds^2 = d\tau^2 - a(\tau)[d\chi^2 + sin^2\chi d\Omega^2] \ ,$$

where $d\tau = b(t)dt$; $a(t) = \rho^2 b^2(t)$:

$$a(\tau) = \begin{cases} C_1 exp\left(\sqrt{\frac{4}{3}\lambda}\tau\right) + C_2 exp\left(-\sqrt{\frac{4}{3}\lambda}\tau\right) + \frac{3}{2\lambda} \ , \lambda > 0 \ ; \\ \\ C_1' cos\left(\sqrt{-\frac{4}{3}\lambda}\tau\right) + C_2' sin\left(\sqrt{-\frac{4}{3}\lambda}\tau\right) + \frac{3}{2\lambda} \ , \lambda < 0 \ . \end{cases}$$

$\lambda = 0$ case corresponds to the Friedmann solution for radiation type of matter. Such solutions are well known in cosmology.

We will be interested in solutions without singularities, i.e. for $C_1 \geq 0$ and $C_2 \geq 0$. Besides, the solution for $\lambda > 0$ corresponds to the positive vacuum energy if we interprete the cosmological constant as the vacuum energy.

If we substitute the solution to the first ordeer Einstein equation, we obtain the following relation between the integration constants:

$$C_1 C_2 = 9/16\lambda^2 - N\kappa/8\pi^2\lambda \ .$$

The condition of nonsingularity (positive C_1 and C_2) leads to the limit on particles number:

$$N \leq 9\pi^2/2\kappa\lambda \ ,$$

which for $\lambda \sim 10^{-56} cm$ gives as earlier $N_{max} \sim 10^{121}$. Among solutions obtained we may single out the stable ones according to the Lyapunov's criterion:

$$a(\tau) = N\kappa/3\pi^2 + C_2\ exp(-\sqrt{6\pi^2/N\kappa}\tau)\ , \lambda = 9\pi^2/2\ N\kappa\ ,$$

which for $\tau \to \infty$ tend to the static Einstein Universe with $\rho = \sqrt{a(\infty)} = \sqrt{N\kappa}/\pi\sqrt{3}$. It is interesting that for $N = 1$ ρ_{min} is of the order of the Planck length: $10^{-33} cm$. Other type of the solution is good for the estimation of the planckeon lifetime $(C_2 = 0, N = 1)$. This characteristic time scale is of the order of the Planck time: $10^{-43} s$.

2.4 Super-Heavy Particles in Scalar-Tensor Theory of Gravitation

2.4.1 In papers [29, 4, 31] an attempt was made to get particles with dimensions 10^{-33} cm and mass 10^{-5} gr (K.P.Stanukovich's plankeons [32], or M.A.Markov's maximons [25]) by solving a set of scalar and gravitational fields equations. It was essential there that the gravitational equations contained the λ − term which took very large values of the order ρ^{-2} where ρ is the microuniverse radius. As it was noted by Ja.B.Zel'dovich [33], the introduction of the λ − term may mean an assumption that the vacuum posesses energy density $\varepsilon = \lambda æ^{-1}$ (in the system of units where $c = h = 1$). As in [29, 4, 31] ρ is of the order $æ^{1/2}$, ε is of the order $æ^{-2}$. On the other hand, energy density ε within a closed microuniverse is $\varepsilon \sim k^0\rho^{-3} \sim æ^{-2}$, i.e. of the same order as ε_v. These considerations perhaps testify to the connection between properties of plankeons and those of "gravitational vacuum".

Here an alternative approach to the problem of the existense of super − heavy particles of closed microuniverse type is developed. Namely instead of introducing the λ − term we use a fundamental scalar field. The idea to include such a field into the theory of gravitation belongs to P.Jordan [34] and was developed by R.Dicke and his collaborators in various aspects [22]. In papers by N.A.Zaitsev and S.M.Kolesnikov [35] another variant of scalar − tensor theory was proposed. In their total Langrangian the sign of fundamental field Langragian is opposite to that of the matter. This approach allowed: 1) to calculate values of the classical gravitational effects; 2) to remove the gravitational paradox in a non-relativistic limit; 3) to give a satisfactory explanation to the observable value of the cosmological constant; 4) remove the singularity in homogeneous and isotropic universe models.

We introduce here a fundamental scalar field in the same manner as in [35] but instead of macroscopic matter we take the material scalar field.

2.4.2 Consider a system with a Langragian density

$$L = L_g + L_\varphi - L_\Phi, \tag{2.148}$$

where

$$L_g = \frac{R}{2æ} \tag{2.149}$$

$$L_\psi = \nabla^\mu \overset{*}{\psi} \nabla_\mu \psi - (m_\psi^2 + \beta_\psi R) \overset{*}{\psi} \psi, \tag{2.150}$$

R is a scalar curvature, æ is the gravitational constant. We use a metric tensor $g_{\mu\nu}$ with the signature $(+ - --)$ and choose the curvature tensor sign, so that

$$\nabla_\mu \nabla_\nu A_\sigma - \nabla_\nu \nabla_\mu A_\sigma = R^\alpha_{\sigma\mu\nu} A_\alpha$$

for any vector A_σ. Here and then $c = \hbar = 1$. The symbol ψ takes values φ and Φ, where φ denotes a material scalar field. Consequently, as it is noted in [27, 26], we must put $\beta_\varphi = \frac{1}{6}$. Φ is fundamental scalar field. A priori β_Φ is not fixed, but according to ZK, it is the most probable that $\beta_\Phi = \frac{1}{8}$. Still, we let β_Φ be arbitrary and write it as β further on.

Varying the action integral

$$S = \int L\sqrt{-g}d^4x; \quad d^4x = dx^0dx^1dx^2dx^3 \tag{2.151}$$

with respect to φ, Φ and $g^{\mu\nu}$, we obtain the equations of scalar and gravitational fields respectively:

$$(\Box + m_\psi^2 + \beta_\psi R)\psi = 0, \tag{2.152}$$

$$G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = -æT_\mu^\nu \tag{2.153}$$

where $\Box = \nabla^\alpha\nabla_\alpha$ – is the D'Alambert operator;

$$T_\mu^\nu = T_\mu^\nu(\varphi) - T_\mu^\nu(\Phi). \tag{2.154}$$

The energy-momentum tensor

$$T_\mu^\nu(\psi) = \nabla^\nu \overset{*}{\psi} \nabla_\mu\psi + \nabla\mu \overset{*}{\psi} \nabla^\nu\psi - \delta_\mu^\nu L_\psi - 2\beta_\psi[R_\mu^\nu + \nabla^\nu\nabla_\mu - \delta_\mu^\nu\Box] \overset{*}{\psi} \psi \tag{2.155}$$

posesses the following properties

$$T_{\mu\nu}(\psi) = T_{\nu\mu}(\psi); \quad \nabla_\alpha T_\mu^\alpha(\psi) = 0;$$
$$T(\psi) = T_\alpha^\alpha(\psi) = 2m_\psi^2 \overset{*}{\psi} \psi - 2(1 - 6\beta_\psi)L_\psi. \tag{2.156}$$

2.4.3 Further on we assume the metric to be found as that of the closed isotopic model:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = b^2(t)[dt^2 - \rho^2dx^2 - \rho^2\sin^2 x(d\theta^2 + \sin^2\theta d\varphi_1^2)] \tag{2.157}$$

with an arbitrary function $b(t)$; ρ is a factor with dimension of length, also unknown. In this case the non-diagonal components of G_μ^ν and T_μ^ν vanish and

$$G_0^0 = -\frac{3}{b^4}\left(\frac{b^2}{\rho^2} + \dot{b}^2\right), \quad G_1^1 = G_2^2 = G_3^3 = -\frac{1}{b^4}\left(\frac{b^2}{\rho^2} + 2b\ddot{b} - \dot{b}^2\right),$$
$$R = -G_\mu^\mu = \frac{6}{b^3}\left(\frac{b}{\rho^2} + \ddot{b}\right); \quad T_1^1 = T_2^2 = T_3^3,$$

where the dot means $\frac{d}{dt}$. Hence, among the Einstein equations there are two non-coinciding ones

$$G_0^0 = -æT_0^0, \tag{2.158}$$
$$G_1^1 = -æT_1^1 \tag{2.159}$$

The left-hand side of (2.158) contains only b and $\dot{b}$, whereas the left side of (2.159) includes b, $\dot{b}$ and $\ddot{b}$. In metric (2.157) (and also in metrics of open isotopic models written in proper coordinates) the condition

$$\nabla_\alpha A_0^\alpha = 0 \tag{2.160}$$

for tensor A_μ^ν may take the form

$$\frac{d}{dt}(b^3A_0^0) = 3\dot{b}b^2A_1^1, \text{ or } \frac{d}{dt}(b^4A_0^0) = \dot{b}b^3A_\alpha^\alpha. \tag{2.161}$$

As condition (2.160) is fulfilled for G_{μ}^{ν} as well as for T_{μ}^{ν}, equation (2.159) and the contraction of equations (2.153)

$$R = æT \tag{2.162}$$

in metric (2.157) follow from (2.158) and (2.160), except the case $\dot{b} \equiv 0$, i.e. that of the Einstein static model.

2.4.4 Let us consider this metric as the one to be found, putting b=1 in expession (2.157). Then ρ becomes a curvature radius. The field functions φ and Φ are assummed to be time-dependent only,

$$\varphi = \varphi(t); \quad \Phi = \Phi(t) \tag{2.163}$$

and for φ function the one-particle normalization condition

$$(\varphi, \varphi) = i \int\limits_{\Sigma} (\overset{*}{\varphi} \nabla_{\alpha} \varphi - \nabla_{\alpha} \overset{*}{\varphi} \varphi) d\Sigma^{\alpha} = 1 \tag{2.164}$$

is used, where Σ is an arbitrary space-like hyper-surface; $d\Sigma^{\alpha}$ is a vector $3-$square element on Σ. Then, equation (2.152) takes the simple form

$$\ddot{\psi} + \left(m_{\psi}^2 + \beta_{\psi} \frac{6}{\rho^2} \right) \psi = 0. \tag{2.165}$$

Their solution are

$$\psi = \psi_0^{i\Omega_{\psi} t}; \quad \Omega_{\psi} = \sqrt{m_r^2 + \frac{6\beta_{\psi}}{\rho^2}}. \tag{2.166}$$

For the constants in (2.166) introduce the notation:

$$\varphi_0 = \gamma; \quad \Phi_0 = \Gamma; \quad \Omega_{\varphi} = k; \quad \Omega_{\varphi} = \omega.$$

As follows from (2.164),

$$4\pi^2 \rho^3 k \gamma^2 = 1. \tag{2.167}$$

Evidently (cf.[4])

$$k = \int\limits_{\Sigma} T_{0\alpha} d\Sigma^{\alpha} \quad (t = \text{const})$$

is the energy of a material scalar field φ.

Equations (2.158) and (2.159) with (2.154) and (2.155) lead to

$$3 = 2x\rho^2(k^2\gamma^2 - \omega^2\Gamma^2), \tag{2.168}$$
$$3 = 2x(-\gamma^2 + 6\beta\Gamma^2), \tag{2.169}$$

which give

$$3 = æ\gamma^2(k^2\rho^2 - 1 - m_{\varphi}^2 \rho^2 \Gamma^2 \gamma^{-2}), \tag{2.170}$$

and hence, we come to the condition

$$k\rho > 1. \tag{2.171}$$

The set (2.167), (2.170) leads to an equation for k

$$æ\rho^2 sk^2 - 12\pi^2\rho^3 k - æs = 0 \tag{2.172}$$

and then

$$k = \frac{1}{æ\rho s}[6\pi^2\rho^2 + \sqrt{36\pi^4\rho^4 + æ^2 s^2}] \tag{2.173}$$

with

$$s = 1 + \frac{m_\varphi^2\rho^2}{12\beta + m_\varphi^2\rho^2}. \tag{2.174}$$

The dependence $k(\rho)$ given by (2.173) is much simplified if

$$m_\varphi = 0. \tag{2.175}$$

This assumption is in good agreement with the conclusions of [35]. Then, from (2.174) it follows that $s = 1$, and function (2.173) has a minimum equal to

$$k = k_0 = \frac{3\sqrt[4]{12}\pi}{\sqrt{æ}} \approx 7,5 \cdot 10^{-5} g \tag{2.176}$$

when

$$\rho = \rho_0 = \frac{\sqrt{æ}}{\pi\sqrt[4]{108}} \approx 0,8 \cdot 10^{-33} cm \tag{2.177}$$

The product $k_0\rho_0 = \sqrt{3}$, that agrees with inequality (2.171).

Thus, we obtained that within this formalism for a quantum scalar particle to be closed in its own gravitational field, its energy must be not less than $k_0 \approx 10^{-5}$ g. Then its localization domain is of the order of the Planck length $L = \sqrt{\frac{\hbar æ}{8\pi c}} \sim 10^{-33}$ cm.

2.4.5 Now, examine the question of stability of the obtained solutions. As far as the exact solutions of the whole problem in metric (2.157) are difficult to find, consider small deviations of $g_{\mu\nu}$, φ, Φ from the solutions of section 2.4.4 remaining within the frames of the closed isotropic model, i.e. the unknown functions are $\varphi(t)$, $\varphi(t)$ and $b(t)$ from expression (2.157). We shall use equations (2.152), (2.158), (2.162) to find them.

In general case of metric (2.157) equations (2.152) are of the form

$$\frac{d^2}{dt^2}\left(\frac{\psi}{b^{-1}}\right) + \left[\frac{r\beta_\psi}{\rho^2} + m_\psi^2 b^2 - \tilde{\beta}_\psi \frac{\ddot{b}}{b}\right]\psi b = 0 \tag{2.178}$$

with $\tilde{\beta}_\psi = 1 - 6\beta_\psi$ and equations (2.158) and (2.162)

$$-G_0^0 = \frac{3}{b^4}\left(\frac{b^2}{\rho^2} + \dot{b}^2\right) = æ[T_0^0(\varphi) - T_0^0(\varphi)], \tag{2.179}$$

$$R = \frac{6}{b^3}\left(\frac{b}{\rho^2} + \ddot{b}\right) = æ[T(\varphi) - T(\varphi)] \tag{2.180}$$

respectively, where according to (2.155) and (2.156)

$$T_0^0(\psi) = b^{-2}\,\dot{\overset{*}{\psi}}\,\dot{\psi} + m_\psi^2\,\overset{*}{\psi}\,\psi - 2\beta_\psi G_0^0\,\overset{*}{\psi}\,\psi + 6\beta_\psi b^{-3}\dot{b}\frac{d}{dt}(\overset{*}{\psi}\,\psi), \tag{2.181}$$

$$T(\psi) = 2m_\psi^2\,\overset{*}{\psi}\,\psi - 2\tilde{\beta}_\psi[b^{-2}\,\dot{\overset{*}{\psi}}\,\dot{\psi} - (m_\psi^2 + \beta_\psi R)\,\overset{*}{\psi}\,\psi]. \tag{2.182}$$

Denote

$$\psi b = \psi_0\, \mathrm{e} i\Omega_\psi t[1+\sigma_\psi(t)]; \quad 2\Re\sigma_\psi = x_\psi; \quad 2\Im\sigma_\psi = y_\psi. \tag{2.183}$$

We are looking for the solutions $b(t)$, $x_\psi(t)$, $y_\psi(t)$ as

$$b(t) = 1+\mu_1^\alpha(t)+\mu_2^{\alpha 2}(t)+\ldots, \quad x_\psi(t) = \mu_{1\psi}^x(t)+\mu_{2\psi}^{x2}(t)+\ldots,$$
$$y_\psi(t) = \mu_{1\psi}^y(t)+\mu_{2\psi}^{y2}(t)+\ldots \tag{2.184}$$

where μ is a small parameter. Further on in equalities concerning both φ and φ fields, we ommit the "ψ" index of the letters x, y, m, Ω.

According to section 2.4.3, equation (2.180) follows from (2.179). But for finding the solutions as expansions (2.194) to a definite power η of the parameter μ, we should use the set of equations (2.178), (2.180). In fact, when (2.179) is fulfilled to $O(\mu^n)$, then according to (2.161), (2.179) is valid to $O(\mu^{n-1})$, because $\dot{b} = O(\mu)$. On the contrary, from the validity of (2.180) to $O(\mu^{n-1})$ it follows (2.179) to $O(\mu^n)$ only if the integration constant is chosen properly. Thus, we have to solve the set (2.178), (2.180) and substituting its solution into (2.179), find an additional condition for integration constants which are present in this solution.

Besides, we take into account normalization condition (2.164) and put $m_\varphi = 0$ as before.

2.4.6 In zero order approximation, i.e. neglecting all the terms containing μ, we arrive again to the equations of section 2.4.4. Consider them to be fulfilled. Then, comparing terms linear in μ in (2.178), (2.179), (2.180) and taking into account (2.183), we obtain

$$\underset{1}{\ddot{x}} +2\Omega\dot{y}+4m^2 \underset{1}{\alpha} -2\tilde{\beta}_\psi \underset{1}{\ddot{\alpha}} = 0, \tag{2.185}$$

$$\underset{1}{\ddot{y}} -2\Omega \underset{1}{\dot{x}} = 0, \tag{2.186}$$

$$6\rho^{-2} \underset{1}{\alpha} = æ\gamma^2[2m_\varphi^2 \underset{1}{\alpha} -k \underset{1}{\dot{y}}_\varphi + 2k^2 \underset{1}{x}_\varphi] - æ\Gamma^2[-\omega\dot{y}_\varphi + 2\omega^2 \underset{1}{x}_\varphi], \tag{2.187}$$

$$3(\underset{1}{\ddot{\alpha}} +\rho^{-2}\alpha) = æ\gamma^2 m_\varphi^2(\underset{1}{\alpha} + \underset{1}{x}_\varphi) - \tilde{\beta}æ\Gamma^2[+\omega \underset{1}{\dot{y}}_\varphi + 6\beta \underset{1}{\ddot{\alpha}}] \tag{2.188}$$

respectively. Condition (2.164) leads to

$$\underset{1}{\dot{y}}_\varphi - 2k \underset{1}{x}_\varphi = 0. \tag{2.189}$$

To satisfy (2.187), we should use (2.189) and demand the fulfilment of a similar condition for Φ. Thus, in agreement with section 2.4.5, the temporal component of gravitational field equations gives only an additional condition on integration constants. Using (2.189) and its analogue for Φ, transform the set (2.185), (2.186), (2.188) to

$$\underset{1}{\ddot{x}}_\varphi + 4k^2 \underset{1}{x}_\varphi + 4m_\varphi^2 \underset{1}{\alpha} = 0, \tag{2.190}$$

$$\underset{1}{\ddot{x}}_\varphi + 4\omega^2 \underset{1}{x}_\varphi - 2\tilde{\beta} \underset{1}{\ddot{\alpha}} = 0, \tag{2.191}$$

$$3 \underset{1}{\ddot{\alpha}} -3\rho^{-2} \underset{1}{x}_\varphi + 6\beta\tilde{\beta}æ\Gamma^2(2\rho^{-2} \underset{1}{x}_\varphi + \underset{1}{\ddot{\alpha}}) = 0, \tag{2.192}$$

i.e. to a set of there linear homogeneous second order equations with constant coefficients. Solve it for the case (2.177) ($k = k_0$, $\rho = \rho_0$), i.e. for minimal energy of the material field φ. Then, for $\beta \neq \frac{1}{6}$:

$$\underset{1}{\alpha} = \sum_{i=1}^{3}(A_i^+\, \mathrm{e} i\omega_i t + A_i^-\, \mathrm{e}{-i\omega_i t}), \tag{2.193}$$

$$\underset{1}{x}_\varphi = -\frac{1}{8\tilde{\beta}(\tilde{\beta}+2)}[\rho^4(1+\tilde{\beta}) \underset{1}{\alpha}^{\cdots\cdot} +4\rho^2(\tilde{\beta}^2+3\tilde{\beta}+3) \underset{1}{\ddot{\alpha}} +8 \underset{1}{\alpha}],$$

$$\underset{1}{x}_\varphi = \rho^2(1+\tilde{\beta}) \underset{1}{\ddot{\alpha}} +2\tilde{\beta} \underset{1}{x}\varphi, \quad \underset{1}{y} = 2\Omega\int \underset{1}{x}\, dt, \tag{2.194}$$

where $A_i^{\pm}$ are integration constants, and for the function $\underset{1}{\alpha}$ to be real we have to put $\overset{*}{A_i^{+}} = A_i^{-}$. The values $z_i = -\omega_i^2$ are roots of the cubic equation

$$\rho^6(1+\tilde{\beta})z^3 + 4\rho^4(3\tilde{\beta}+4)z^2 + 56\rho^2 z + 32(1-\tilde{\beta}) = 0. \tag{2.195}$$

Evidently, if all z_i are real and negative, then the treatment of the first approximation is unsufficient for solving the stability problem, as the $O(\mu)$ terms in (2.184) have the form of superposition of harmonic oscilations with three frequencies $\omega_i = \sqrt{-z_i}$. This takes place when $0 < \beta < 0,24$ (the value 0,24 is calculated approximately). Expression (2.193) contains an exponentially growing term for $\beta > 0,24$ that proves the instability of these solutions.

As the development of the idea of the existence of planckeons is aimed at the creation of an elementary particles model, one cannot exclude the possibility that planckeon excitations in the form of oscillations with three different frequencies are related to quark hypothesis. Then, it may happen that only definite superpositions of these oscillations, corresponding to the observed particles, are realized. The elementary excitations themselves, interpreted as quarks, may be unobservable. Such a point of view differs from that advanced by M.A.Markov in [25], according to which maximons themselves may act as quarks. These considerations may be valid only assuming that the planckeon is not entirely closed.

In case $\beta = \frac{1}{6}$, $\tilde{\beta} = 0$, it is enough to use equations (2.190) and (2.192) to examine the behavior of the system. They give:

$$\begin{aligned}
&\underset{1}{\alpha} = \sum_{i=1}^{2}\left(A_i\, \mathrm{e} i\omega_i t + \overset{*}{A_i}\, \mathrm{e}{-i\omega_i t}\right), \\
&\underset{1}{x}{}_{\varphi} = \rho^2 \underset{1}{\ddot{\alpha}}, \quad \underset{1}{y}{}_{\varphi} = 2k\int \underset{1}{x}{}_{\varphi} dt, \\
&\omega_i^2 = 2k^2 \pm \sqrt{4(k^4 - \frac{m_{\varphi}^2}{\rho^2})} > 0
\end{aligned} \tag{2.196}$$

i.e. harminic oscillations with two frequencies.

2.4.7 So, for considering the question of microuniverse stability for some domain of parameter values, first order investigation is not sufficient. Consequently, examine the second order approximation, i.e. assuming the solution in form (2.184), and take into consideration $O(\mu^2)$ terms in equations (2.178), (2.179), (2.180). Let the first order equations (section 2.4.6) be fulfilled and suppose $\beta \neq \frac{1}{6}$, $0 < \beta < 0,24$. Then, equations (2.178) yield:

$$\underset{2}{\ddot{x}} + 2\Omega \underset{2}{\dot{y}} + 4\underset{2}{\alpha} m^2 - 2\tilde{\beta}_{\varphi} \underset{2}{\ddot{\alpha}} + 2m^2 \underset{1}{\alpha}(\underset{1}{\alpha} + \underset{1}{x}) + \tilde{\beta}_{\varphi} \underset{1}{\ddot{\alpha}}(2\underset{1}{\alpha} - \underset{1}{x}) = 0, \tag{2.197}$$

$$\underset{2}{\ddot{y}} - 2\Omega \underset{2}{\dot{x}} + 2m^2 \underset{1}{\alpha}\underset{1}{y} - \tilde{\beta}_{\psi} \underset{1}{\ddot{\alpha}}\underset{1}{y} = 0. \tag{2.198}$$

Since the continuity equation for scalar field

$$\nabla^{\alpha}(\overset{*}{\psi}\nabla_{\alpha}\psi - \psi\nabla_{\alpha}\overset{*}{\psi}) = 0$$

is valid and ψ functions are purely time dependent, equations (2.197) and (2.198) have an integral analogous to (2.189):

$$\underset{2}{\dot{y}} - 2\Omega \underset{2}{x} - \frac{1}{2}(\underset{1}{\dot{x}}\underset{1}{y} - \underset{1}{x}\underset{1}{\dot{y}} + \Omega \underset{1}{x}{}^2 + 2\underset{1}{y}{}^2) = C_{\psi} = \mathrm{const}. \tag{2.199}$$

Besides, from normalization condition (2.164) follows $C_{\varphi} = 0$. With (2.199) equation (2.179) takes the form

$$3\rho^{-2}\underset{1}{\alpha}{}^2 + 3\underset{1}{\dot{\alpha}}{}^2 = æ\gamma^2 \underset{2}{T}(\varphi) - æ\Gamma^2 \underset{2}{T}(\Phi)\ , \tag{2.200}$$

where

$$\underset{2}{T}(\psi) = \frac{1}{4}\,\underset{1}{\dot{x}}^{\,2} + m^2(2\,\underset{1}{\alpha}\,\underset{1}{x} + \underset{1}{\alpha}^{\,2}) + \tilde{\beta}_\psi(\underset{1}{\dot{\alpha}}^{\,2} - \underset{1}{\dot{\alpha}}\,\underset{1}{\dot{x}}) - C_\psi \Omega.$$

Using equations (2.190), (2.191), (2.192) one may prove that (2.200) is satisfied if one puts $C_\varphi = 0$. So in expression (2.199)

$$C_\psi = 0\ . \tag{2.201}$$

Write equation (2.197) substituting $\underset{2}{\dot{y}}$ from (2.201) and equation (2.180) to the order $O(\mu^2)$ respectively:

$$\underset{2}{\ddot{x}} + 4\Omega^2\,\underset{2}{x} + 4m^2\,\underset{2}{\alpha} - 2\tilde{\beta}_\psi\,\underset{2}{\ddot{\alpha}} = -2m^2\,\underset{1}{\alpha}(\underset{1}{\alpha} + \underset{1}{x}) + \tilde{\beta}_\psi\,\underset{1}{\ddot{\alpha}}(\underset{1}{x} - 2\,\underset{1}{\alpha}) - \tag{2.202}$$

$$\Omega\,\underset{1}{\dot{x}}\,\underset{1}{y} + \Omega^2(\underset{1}{x}^{\,2} - \underset{1}{y}^{\,2}),$$

$$3\,\underset{2}{\ddot{\alpha}} - 3\rho^2\,\underset{2}{x}_\varphi + 6\beta\tilde{\beta}æ\Gamma^2(2\rho^{-2}\,\underset{2}{x}_\varphi + \underset{2}{\ddot{\alpha}}) =$$

$$3\rho^{-2}[2\,\underset{1}{\alpha}\,\underset{1}{x}_\varphi + \frac{1}{4}(\underset{1}{x}^{\,2}_\varphi + \underset{1}{y}^{\,2}_\varphi)] + \tilde{\beta}æ\Gamma^2\left[\frac{1}{4}\,\underset{1}{x}^{\,2}_\varphi + 3\beta\rho^{-2}(\underset{1}{x}^{\,2}_\varphi - \underset{1}{y}^{\,2}_\varphi)\right.$$

$$\left. - \underset{1}{\dot{\alpha}}\,\underset{1}{\dot{x}}_\varphi - 6\beta(\underset{1}{\ddot{\alpha}}_\varphi - \underset{1}{\ddot{\alpha}}\,\underset{1}{\alpha} + \underset{1}{\dot{\alpha}}^{\,2})\right]. \tag{2.203}$$

In equation (2.203) we have taken into account (2.199) for Φ field.

Now, as in section *2.4.6,* we have obtained a set of three linear equations with constant coefficients for the unknown functions $\underset{2}{x}_\varphi$, $\underset{2}{x}_\varphi$, $\underset{2}{\alpha}$, where the coefficients of left sides of equations (2.202), (2.203) coinside exactly with those of (2.190), (2.191), (2.192). The only difference is that equations (2.202), (2.203) are reduced to that of the equation

$$P_6\left(\frac{d}{dt}\right)\underset{2}{\alpha} \equiv \rho^6(1+\tilde{\beta})\frac{d^6}{dt^6}\,\underset{2}{\alpha} + 4\rho^4(3\tilde{\beta}+4)\frac{d^4}{dt^4}\,\underset{2}{\alpha} +$$

$$56\rho^2\frac{d^2}{dt^2}\,\underset{2}{\alpha} + 32(1-\tilde{\beta})\,\underset{2}{\alpha} = f(t) \tag{2.204}$$

(when $k = k_0$), where the function $f(t)$ is a linear combination of the following functions:

$$\text{const};\quad \mathrm{e}\pm i\omega_i t;\quad \mathrm{e}\pm(\omega_i \pm \omega_j)t \tag{2.205}$$

and the values ω_i are defined according to section *2.4.6.* The general solution (2.204) (real) may be written as

$$\underset{2}{\alpha} = \sum_i\left(A_i\,\mathrm{e}i\omega_i t + \overset{*}{A}_i\,\mathrm{e}{-i\omega_i t}\right) + t\sum_i\left(B_i\,\mathrm{e}i\omega_i t + \overset{*}{B}_i\,\mathrm{e}{-i\omega_i t}\right) +$$

$$\sum_{i\geq j}\left[A_{ij}\,\mathrm{e}i(\omega_i+\omega_j)t + \overset{*}{A}_{ij}\,\mathrm{e}{-i(\omega_i+\omega_j)t}\right] +$$

$$\sum_{\substack{ij\\ \omega_i>\omega_j}}\left[B_{ij}\,\mathrm{e}i(\omega_i-\omega_j)t + \overset{*}{B}_{ij}\,\mathrm{e}{-i(\omega_i-\omega_j)t}\right] + C, \tag{2.206}$$

where A_i are integration constants. The constants B_i, A_{ij}, B_{ij}, C are expressed through integration constants appearing in $\underset{1}{\alpha}$, the coefficients of right sides of (2.202), (2.203) and the frequencies ω_i. Evidently, for the stability problem only the second term, describing oscillations with linear growing amplitude, is essential. The latter is appearing in solution (2.206) due to the functions $\underset{1}{y}$ in the right sides of equations (2.202), (2.203) and because $P_6(\pm i\omega_i) = 0$. According to (2.194), $\underset{1}{y}$ contains arbitrary additive constants, which enter the values B_i in such a way that B_i may take any sign, despite the values of constants $A_i^{\pm}$ characterizing $\underset{1}{\alpha}$. Therefore, among total solutions (2.184) there are ones containing oscillations with amplitude linear growing in time. This proves the instability of the solutions obtained in section *2.4.4*.

A similar treatment can be made completely for the case $\beta = \frac{1}{6}$, $\tilde{\beta} = 0$. Calculations which are much simplier in this case, lead to the same result.

2.4.8 It should be noted that here we considered non-quantized gravitational fiels. It is quit probable that taking into account quantum gravitational effects which perhaps may arise at distances of the order 10^{-33} cm, would bring certain changes into the obtained results.

We have not used secondary quantization formalism for scalar fields because, according to [29, 4, 31], the quantization of material scalar field brings no changes into the results, and as to the quantization of the fundamental field there arise certain difficulties connected with its interpretation.

Further we should like to discuss some questions concerning the formalism and the results of this paper. These questions are apparently of problematic character.

As in section 2.4.1, one may connect the fundamental scalar field with properties of a gravitational vacuum, which manifests itself locally here, whereas in the theory with λ-term it appears globally.

There may be two possible approached when interpreting the fundamental field Φ itself. First, it may be regarded as an additional component of a gravitational field, as it is done in [35]. Then, the question of Φ field contribution into total energy of the system is a part of a more general question of the gravitational field energy. Secondly, one may consider that Φ field does make a contribution to the total energy of matter. This contribution is negative (see e.g. (2.154)), and the total energy to be positive, we have to demand the existence of some matter, different from Φ field, with an energy momentum tensor $T_\mu^\nu(\mu)$, so that

$$T_0^0(\mu) \geq T_0^0(\Phi). \tag{2.207}$$

In our case the total energy

$$E = \int\limits_\Sigma \left[T_{0\alpha}(\varphi) - T_{0\alpha}(\Phi)\right] d\Sigma^\alpha = \frac{6\pi^2\rho}{æ} > 0 \quad (t = \text{const}),$$

that satisfies condition (2.207). But, despite these arguments, the first approach seems more preferable.

Note further that while staying within the frames of Einstein static model, energy $E_0(\varphi)$ is conserved, $E_0(\varphi) = k = \text{const}$. But, when we proceed to the perturbations of the system, the energy becomes time-dependent. In fact, in the first approximation

$$E(\varphi) = \int\limits_{\Sigma(t=\text{const})} T_{0\alpha}(\varphi) d\Sigma^\alpha = k + \mu\frac{k^2 - \rho^{-2}}{k}\underset{1}{\alpha}(t) = \underset{0}{E}(\varphi) + \Delta E(\varphi).$$

Taking into account expression (2.193) for $\underset{1}{\alpha}$, calculate the time average

$$\bar{E(\varphi)} = \underset{0}{E}(\varphi) = k$$

and the mean quadratic deviation

$$\sqrt{\Delta\bar{E}^2} = \mu \cdot \frac{k^2 - \rho^2}{k} \cdot \frac{\sqcap}{\sqcup},$$

where

$$\frac{\sqcap}{\sqcup} = \frac{1}{2\sqrt{2}}\sqrt{\sum_i |A_i^+|^2} = \text{const} \sim 1, \text{ and for } K = K_{\min} = K_0$$

$$\sqrt{\Delta \bar{E}^2} = \frac{2}{3}\mu k_0 \frac{\sqcap}{\sqcup}. \tag{2.208}$$

The obtained results may be interpreted according to the ideas advanced by K.P. Staniukovich [36, 37]. Supposing that a planckeon is not completely closed, one may prescribe fluctuative nature to its excitations, and as it is easily seen from (2.208), for μ of the order of 10^{-20}, planckeon excitation energy is of the order of elementary particles energy. Then, an amplitude growth in the second approximation is, perhaps, connected with an irreversible radiation of gravitons [37].

As for the place of the discussed closed microuniverses in the general physical picture, one can imagine them as a sort of cells of a physical vacuum. In unexcited state they are unobservable. Their excitations appear as elementary particles. Such an idea probably leads to the conception of spacetime of a complicated topology, considered for instance by J.A.Wheeler [38].

Our approach has some features similar to Markov's cosmological approach [39]. His considerations conserning the connection of electric charge with semi-closed universes may turn out to be fruitful. In [15] the case of Friedmann universe filled with dust is treated. In seems to us however that while considering small domains (in [39] near the semi-closed universe boundary with dimensions $\delta \ll 10^{-13}$ cm) microscopic (field) description of matter is necessary.

So, here field treatment to the microcosmological approach to the elementary particles was carried ont.

2.5 Instability of black holes with scalar charge [59]

The well-known statement "black holes have no hair" means that black holes can possess only gravitational and electromagnetic external fields. The only exception, found in [40, 17] and discussed in [41, 42], is a black hole with a conformal scalar field φ (equation: $\nabla^\alpha\nabla_\alpha\varphi + R\varphi/6 = 0$). This field is known to be a better curved-space version of a massless scalar field than the minimally coupled one, ψ (equation: $\nabla^\alpha\nabla_\alpha\psi = 0$); unlike ψ, it has a traceless energy-momentum tensor and a propagator concentrated on the null cone, see e.g. [26].

The black-hole solution with $\varphi \neq 0$ and Maxwell field $F_{\mu\nu}$ is:

$$\begin{gathered} ds^2 = (1 - m/r)^2 dt^2 - (1 - m/r)^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\xi^2), \\ \varphi = h\sqrt{6}/(r - m), \quad F_{\mu\nu} = q(\delta^1_\mu\delta^0_\nu - \delta^0_\mu\delta^1_\nu)/r^2, \quad m = (q^2 + h^2)^{1/2}, \end{gathered} \tag{2.209}$$

where m is the mass, q the electric charge and h the scalar charge. Here we prove that *such black holes are unstable.*

We study the stability of solution (2.209) under the simplest perturbations, namely, monopole (radial) ones: $\delta g_{\mu\nu}(r,t)$, $\delta F_{\mu\nu}(r,t)$, $\delta\varphi(r,t)$. Their existence is a distinctive feature of configurations with a scalar field. Besides, for other perturbations one generally obtains effective potentials with centrifugal barriers. Hence, if the instability, indeed, occurs, it should aprear most probably just under monopole perturbations.

Thus, the only nonstatic degree of freedom corresponds to monopole scalar waves. Consequently, we can express $\delta g_{\mu\nu}$ and $\delta F_{\mu\nu}$ in terms of $\delta\varphi$ using the Einstein and Maxwell equations. If we choose the perturbed space-time coordinates so that $\delta g_{22} = 0$ (the central frame of reference), separate the variables and bring the resulting ordinary differential equation to a normal Legendre form, i.e. if we put

$$\begin{gathered} \delta\varphi = \mathrm{e}i\omega t y(x) \cdot (u^3 + mh^2)u^{-2}r^{-1}|u^2 - h^2|^{-1/2}, \\ x = \int dr(l - m/r)^{-2} = u + 2m\log(u/m) - m^2/u + \mathrm{const}, \end{gathered} \tag{2.210}$$

where $u = r - m$, we arrive at the Schrödinger-type equation:

$$d^2y/dx^2 + [\omega^2 - V(x)]y = 0, \tag{2.211}$$

$$V(x) = \frac{u^4}{r^4}\left[\frac{2m}{r^2 u} - \frac{h^2}{(u^2 - h^2)^2} + 6h^2\frac{r^2(u^2 - h^2) - q^2u^2}{(u^2 - h^2)(u^3 + mh^2)^2}\right]. \tag{2.212}$$

The effective potential* $V(x)$ decreases ($V \sim 2m|x|^{-3}$) at $x \to +\infty$ (spatial infinity) and $x \to -\infty$ (the horizon $r = m$) and has one singularity at $x = x_0$ referring to $r = m + h(V \sim -1/4(x - x_0)^2)$. The corresponding quantum-mechanical boundary-value problem ($y \to 0$ for $x \to \pm\infty$) has a spectrum of eigenvalues ω^2, that is unbounded from below. Eigenfunctions $y_\omega(x)$ with $\omega^2 < 0$ correspond to perturbations which can grow exponentially with time. Hence, the instability is proved if such perturbations are physically acceptable. This means that $\delta g_{\mu\nu}$, $\delta F_{\mu\nu}$ and $\delta T_{\mu\nu}$ (energy-momentum tensor perturbation) should vanish at the infinity and be finite at the future horizon (perturbations are assumed to emerge in the static region $r > m$) in a frame in which the initial solution (2.209) is regular. Note that the space-time (2.209) has the same structure as an extreme Reissner-Nordström black hole, $q^2 = m^2$. In particular, we can use the null coordinates v_+, v_-, such that

$$2m \tan v_\pm = t \pm x, \text{ future horizon: } v_- \to \pi/2 - 0.$$

A direct calculation shows that all the perturbations in question vanish both at infinity and at the horizon due to the asymptotic behaviour $y_\omega \sim \exp(-|\omega x|)$ for $x \to \pm\infty$. Thus, the instability is proved.

As the increment $|\omega|$ is unlimited, we conclude that black holes with a scalar charge decay at an unlimited rate. Probably, if once formed, such a hole quickly looses its scalar field and turns into an ordinary Reissner-Nordstrom black hole.

Thus, indeed, black holes have no steady external scalar field. This can mean either (as is commonly thought) that the scalar field evaporates from the collapsing bodies, or the opposite, that it can prevent a black hole from being formed. To find out what is really the case, we must study the collapse of a scalar field source.

2.6. The Scenario for the Astrophysics with Scalar Field and the Cosmological Constant [57]

2.6.1 Introduction

Recently the existence of the cosmological constant becomes quite probable from the observation of the deep galaxy survey [43, 44]. While the dark matter is necessary in various observations such as the rotational curve of the spiral galaxy or the missing of the ordinary matter at the cosmological scale.

Here, we start from the theory of general relativity with the dark matter and the cosmological constant in order to study the standard scenario of the astrophysics, that is, the physics of the cosmological, the galactic or solar scale. Though the neutrino is the promising candidate of the dark matter, there is no established direct observation of the dark matter as the ordinary matter. There is another attempt to explain the rotation curves in the theory of the Brans-Dicke theory [45, 46], where the Newtonian force is modified by the effect of the scalar field. We consider the scalar field as a candidate of the dark matter [47] together with the cosmological constant and study their effects on a time development of the scale factor of the universe at the cosmological scale and the gravitational potential in the galactic or solar scale.

The famous scalar-tensor theory is the Brans-Dicke theory, but here the Einstein theory with the minimally coupled scalar field instead of the Brans-Dicke theory is adopted. The principle of the choice of the theory is the following. For the scalar-tensor gravity theory, we can transform one from the Jordan frame to the Einstein frame by the conformal transformation [48]. The Einstein frame is preferable because the post-Newtonian test of the general relativity such as the radar echo delay is quite stringent [49, 50].

*The potential $V(x)$ can be found also from a paper by Bronnikov and Khodunov where the stability of solutions with a minimal scalar field ψ is considered: one should put $k = 2h = 2C/\sqrt{6}$ and change $r \cosh hz \to r$. This can be done due to a conformal mapping which relates solutions with ψ and φ fields.

Also, the Einstein theory with minimally coupled scalar field is chosen because the scalar field as the dark matter means that the scalar field has no direct interaction of the gravitational field with the ordinary matter.

2.6.2 Einstein Theory with Minimally Coupled Scalar Field

The Brans-Dicke theory [45, 46] is the typical scalar-tensor theory of the gravity. The action of the Brans-Dicke theory with the cosmological term is given by

$$I_{\rm BD} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(\xi\varphi^2 R - 2\Lambda\varphi^n\right) - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \mathcal{L}_{\rm ordinary\ matter}\right].$$

Uehara-Kim [51] found the general solution for $n = 2$ and matter dominant case, and Fujii [52] has found the special solution for general n.

Putting $g_{\mu\nu}(x) = \Omega^{-2}(x)g_{*\mu\nu}(x)$ with $\Omega(x) = \sqrt{\xi}\varphi(x)$, one obtains the following action [48]

$$I_{\rm BD} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(R_* - 2\Lambda e^{(n-4)\zeta\varphi_*}\right) - \frac{1}{2}g_*^{\mu\nu}\partial_\mu\varphi_*\partial_\nu\varphi_* + \xi^{-2}e^{-4\zeta\varphi_*}\mathcal{L}_{*\rm ordinary\ matter}\right] ,$$

where $\varphi = \exp(\zeta\varphi_*)$ with $\zeta^{-1} = \sqrt{1/\xi + 3/4\pi G}$ and $\mathcal{L}_{*\rm ordinary\ matter}$ is obtained from $\mathcal{L}_{\rm ordinary\ matter}$ by replacing the metric part in the form $g_{\mu\nu} \to g_{\mu\nu}\xi^{-1}\exp(\zeta\varphi_*)$.

Fixing the theory comes from the following two principles: i) the kinetic part of the gravity is of the standard Einstein form, because of the stringent constraint of the post-Newtonian test such as the delay of the radar echo experiment, ii) the scalar field has no direct coupling to the ordinary matter nor gives the effect on the geodesic equation of the particle. ¿From these principles, the Brans-Dicke theory cannot adopted. In the following, the Einstein theory with the standard cosmological term and the minimally coupled scalar field is used.

By considering the minimally coupled scalar field as some kind of dark matter, the time development of the scale factor of the universe in the cosmological scale and the gravitational potential in the galactic or solar scale are studied.

In the Misner-Thorne-Wheeler notation [53] consider the Einstein action with the cosmological constant, the minimally coupled scalar field and an ordinary matter

$$I = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(R - 2\Lambda\right) - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \mathcal{L}_{\rm ordinary\ matter}\right] , \tag{2.213}$$

where G is the gravitational constant, R is the scalar curvature and φ is the minimally coupled scalar field. The equations of motion in this system are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G(T^\varphi_{\mu\nu} + T_{\mu\nu}), \tag{2.214}$$

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) = 0, \tag{2.215}$$

where $T^\varphi_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi$ and $T_{\mu\nu}$ is the energy-momentum tensor of the ordinary matter.

2.6.3 Cosmological Exact Solution

In order to study classical solutions in cosmology, let us substitute the homogeneous, isotropic and flat metric

$$ds^2 = -dt^2 + a(t)^2\left[dr^2 + r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\right] , \tag{2.216}$$

and the perfect fluid expression of the ordinary matter $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$ into equations of motion. ρ and p denote the density and the pressure of the perfect fluid respectively and $u_\mu = (1, 0, 0, 0)$ in the co-moving system. Then equations of motion to be solved become

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{\Lambda}{3} = \frac{8\pi G}{3}\left(\rho + \frac{\dot{\varphi}^2}{2}\right), \tag{2.217}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} - \Lambda = -8\pi G\left(p + \frac{\dot{\varphi}^2}{2}\right), \tag{2.218}$$

$$\frac{\ddot{\varphi}}{\dot{\varphi}} + 3\frac{\dot{a}}{a} = 0. \tag{2.219}$$

The perfect fluid is characterized by $p = \gamma\rho$, and one obtains the conservation law of the ordinary matter density by taking the linear combination of Eqs.(2.217), (2.218) and (2.219). From Eq.(2.219), another conservation law comes. Then we have the following two conservation laws

$$\rho = \rho_0 a^{-3(1+\gamma)}, \tag{2.220}$$

$$\dot{\varphi} = \frac{k}{a^3}, \tag{2.221}$$

where ρ_0 and k are integration constants. The equation to be solved becomes Eq.(2.217) with the conditions Eqs.(2.220) and (2.221). Substituting Eqs.(2.220) and (2.221) into Eq.(2.217), we have the equation of the form

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{\Lambda}{3} = \frac{8\pi G}{3}\left(\frac{\rho_0}{a^{3(1+\gamma)}} + \frac{k^2}{2a^6}\right). \tag{2.222}$$

This problem can be solved exactly in the $\gamma = 1$ and $\gamma = 0$ cases, but $\gamma = 1$ case is unphysical. Then we consider only the $\gamma = 0$ case, that is, the dust matter dominant case. In this case, we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{4\pi G}{3}\left(\dot{\varphi} + \frac{\rho_0}{k}\right)^2 = \frac{\Lambda}{3} - \frac{4\pi G{\rho_0}^2}{3k^2} \tag{2.223}$$

by using Eqs.(2.217), (2.220) and (2.221).

$k^2\Lambda > 4\pi G{\rho_0}^2$ case

In this case the cosmological and/or the scalar term are dominant, and we parametrize

$$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3} - \frac{4\pi G{\rho_0}^2}{3k^2}}\cosh\Theta, \tag{2.224}$$

$$\dot{\varphi} = -\frac{\rho_0}{k} + \sqrt{\frac{\Lambda}{4\pi G} - \frac{{\rho_0}^2}{k^2}}\sinh\Theta. \tag{2.225}$$

Substituting this parametrization into Eq.(2.219), we have

$$\dot{\Theta} + \sqrt{\frac{3\Lambda}{1 + A^2}}(\sinh\Theta - A) = 0, \tag{2.226}$$

where $A^{-1} = \sqrt{k^2\Lambda/4\pi G{\rho_0}^2 - 1}$. Then we obtain

$$\frac{1}{\sqrt{1 + A^2}}\log\left|\frac{-A\tanh(\Theta/2) - 1 + \sqrt{1 + A^2}}{-A\tanh(\Theta/2) - 1 - \sqrt{1 + A^2}}\right| = -\sqrt{\frac{3\Lambda}{1 + A^2}}(t - t_0) \tag{2.227}$$

by using the formula

$$\int \frac{d\Theta}{\sinh\Theta - A} = \frac{1}{\sqrt{1+A^2}} \log\left| \frac{-A\tanh(\Theta/2) - 1 + \sqrt{1+A^2}}{-A\tanh(\Theta/2) - 1 - \sqrt{1+A^2}} \right|. \tag{2.228}$$

And then we have the relation

$$\frac{-A\tanh(\Theta/2) - 1 + \sqrt{1+A^2}}{-A\tanh(\Theta/2) - 1 - \sqrt{1+A^2}} = \exp\left(-\sqrt{3\Lambda}\,(t-t_0)\right), \tag{2.229}$$

which gives the relation

$$\tanh(\Theta/2) = -\frac{1}{A} + \frac{\sqrt{1+A^2}}{A\tanh\left(\sqrt{3\Lambda}\,(t-t_0)/2\right)}. \tag{2.230}$$

Using this relation, we can write $\cosh\Theta$ and $\sinh\Theta$ in the form

$$\cosh\Theta = \frac{1+\tanh^2(\Theta/2)}{1-\tanh^2(\Theta/2)} = \frac{(1+A^2)\cosh(T-T_0) - \sqrt{1+A^2}\sinh(T-T_0)}{-A^2 - \cosh(T-T_0) + \sqrt{1+A^2}\sinh(T-T_0)}, \tag{2.231}$$

$$\sinh\Theta = \frac{2\tanh(\Theta/2)}{1-\tanh^2(\Theta/2)} = \frac{A\left(1-\cosh(T-T_0) + \sqrt{1+A^2}\sinh(T-T_0)\right)}{-A^2 - \cosh(T-T_0) + \sqrt{1+A^2}\sinh(T-T_0)}, \tag{2.232}$$

where $T = \sqrt{3\Lambda}t$ and $T_0 = \sqrt{3\Lambda}t_0$. Introducing Θ_0 through the relation $\cosh\Theta_0 = \sqrt{1+A^2}/A$, $\sinh\Theta_0 = 1/A$, we can simplify the above expression in the form

$$\cosh\Theta = \frac{\sqrt{1+A^2}\cosh(T-T_0-\Theta_0)}{\sinh(T-T_0-\Theta_0) - A}, \tag{2.233}$$

$$\sinh\Theta = A\left(1 + \frac{(A+1/A)}{\sinh(T-T_0-\Theta_0) - A}\right). \tag{2.234}$$

Using Eqs.(2.224) and (2.233), we have

$$\begin{aligned} \log a &= \int \frac{da}{a} = \sqrt{\frac{\Lambda}{3} - \frac{4\pi G{\rho_0}^2}{3k^2}} \int dt\cosh\Theta \\ &= \frac{1}{3}\sqrt{1 - \frac{4\pi G{\rho_0}^2}{k^2\Lambda}}\sqrt{1+A^2}\int dT \frac{\cosh(T-T_0-\Theta_0)}{\sinh(T-T_0-\Theta_0) - A} \\ &= \frac{1}{3}\log|\sinh(T-T_0-\Theta_0) - A| + \text{const.}, \end{aligned} \tag{2.235}$$

where we use the relation $4\pi G{\rho_0}^2/k^2\Lambda = A^2/(1+A^2)$. Therefore, we have

$$a(t) = a_0\Big(\sinh(T-T_0-\Theta_0) - A\Big)^{1/3}, \tag{2.236}$$

where a_0 is the constant.

Similarly, from Eqs.(2.225) and (2.234), we have

$$\begin{aligned} \varphi &= \int dt\left(-\frac{\rho_0}{k} + \sqrt{\frac{\Lambda}{4\pi G} - \frac{{\rho_0}^2}{k^2}}\sinh\Theta\right) = -\frac{\rho_0}{k\sqrt{3\Lambda}}\int dT\left(1 - \frac{\sinh\Theta}{A}\right) \\ &= \frac{\rho_0(1+A^2)}{kA\sqrt{3\Lambda}}\int \frac{dT}{\sinh(T-T_0-\Theta_0) - A} \\ &= \varphi_0 + \frac{1}{\sqrt{12\pi G}}\log\left| \frac{A\tanh\Big((T-T_0-\Theta_0)/2\Big) + 1 - \sqrt{1+A^2}}{A\tanh\Big((T-T_0-\Theta_0)/2\Big) + 1 + \sqrt{1+A^2}} \right| \\ &= \varphi_1 + \frac{1}{\sqrt{12\pi G}}\log\left| \frac{\exp(T-T_0-\Theta_0) - A - \sqrt{1+A^2}}{\exp(T-T_0-\Theta_0) - A + \sqrt{1+A^2}} \right|, \end{aligned} \tag{2.237}$$

where φ_0 is the constant and φ_1 is given by $\varphi_1 = \varphi_0 + \frac{1}{\sqrt{12\pi G}} \log \left| \frac{1+A-\sqrt{1+A^2}}{1+A+\sqrt{1+A^2}} \right|$.

The integration constant a_0 is not the independent integration constant but it can be expressed by k and ρ_0. ¿From Eqs.(2.221), (2.236) and (2.237), we have

$$\begin{aligned}\dot{\varphi} &= \frac{\rho_0(1+A^2)}{kA\Big(\sinh(T-T_0-\Theta_0)-A\Big)} \\ &= \frac{\rho_0(1+A^2){a_0}^3}{kAa^3} = \frac{k}{a^3}.\end{aligned} \tag{2.238}$$

which gives the relation $a_0^3 = k^2A/\rho_0(1+A^2)$. This can be written in the form

$$k^2\Lambda - 4\pi G{\rho_0}^2 = \frac{\Lambda^2 {a_0}^6}{4\pi G}. \tag{2.239}$$

Then we can obtain

$$a_0 = \Big(4\pi G(\frac{k^2}{\Lambda} - \frac{4\pi G\rho_0^2}{\Lambda^2})\Big)^{1/6}. \tag{2.240}$$

Therefore we have the exact solution in the form

$$a(t) = \Big(4\pi G(\frac{k^2}{\Lambda} - \frac{4\pi G\rho_0^2}{\Lambda^2})\Big)^{1/6}\Big(\sinh(T-T_1)-A\Big)^{1/3}, \tag{2.241}$$

$$\varphi(t) = \varphi_1 + \frac{1}{\sqrt{12\pi G}} \log \left| \frac{\exp{(T-T_1)} - A - \sqrt{1+A^2}}{\exp{(T-T_1)} - A + \sqrt{1+A^2}} \right|, \tag{2.242}$$

where

$$T = \sqrt{3\Lambda}t, \quad T_1 = T_0 - \Theta_0 = \sqrt{3\Lambda}t_1 = \text{const.},$$
$$A^{-1} = \sqrt{\frac{k^2\Lambda}{4\pi G\rho_0^2} - 1}, \quad \varphi_1 = \text{const.}.$$

$4\pi G{\rho_0}^2 > k^2\Lambda$ case

In this case the ordinary matter is dominant, and we parametrize

$$\frac{\dot{a}}{a} = \sqrt{\frac{4\pi G{\rho_0}^2}{3k^2} - \frac{\Lambda}{3}}\sinh\Theta, \tag{2.243}$$

$$\dot{\varphi} = -\frac{\rho_0}{k} + \sqrt{\frac{{\rho_0}^2}{k^2} - \frac{\Lambda}{4\pi G}}\cosh\Theta. \tag{2.244}$$

Substituting this parametrization into Eq.(2.219), we have

$$\dot{\Theta} + \sqrt{\frac{3\Lambda}{B^2-1}}(\cosh\Theta - B) = 0, \tag{2.245}$$

where $B^{-1} = \sqrt{1 - k^2\Lambda/4\pi G{\rho_0}^2}$. Then we obtain

$$\frac{1}{\sqrt{B^2-1}} \log \left| \frac{1-B+\sqrt{B^2-1}\tanh(\Theta/2)}{1-B-\sqrt{B^2-1}\tanh(\Theta/2)} \right| = -\sqrt{\frac{3\Lambda}{B^2-1}}(t-t_0), \tag{2.246}$$

by using the formula

$$\int \frac{d\Theta}{\cosh\Theta - B} = \frac{1}{\sqrt{B^2-1}} \log \left| \frac{1 - B + \sqrt{B^2-1}\tanh(\Theta/2)}{1 - B - \sqrt{B^2-1}\tanh(\Theta/2)} \right| .$$

Then we have the relation

$$\frac{1 - B + \sqrt{B^2-1}\tanh(\Theta/2)}{1 - B - \sqrt{B^2-1}\tanh(\Theta/2)} = -\exp\Big(-\sqrt{3\Lambda}(t-t_0)\Big), \tag{2.247}$$

where we take the branch of the logarithm in such a way, so that the scale factor of the universe behaves as the power law in time at the very early age of the universe. Then we have the relation

$$\tanh(\Theta/2) = \sqrt{\frac{B-1}{B+1}} \tanh\Big(\sqrt{3\Lambda}(t-t_0)/2\Big). \tag{2.248}$$

Using this relation, one can write $\sinh\Theta$ and $\cosh\Theta$ in the form

$$\sinh\Theta = \frac{2\tanh(\Theta/2)}{1-\tanh^2(\Theta/2)} = \frac{\sqrt{B^2-1}\sinh(T-T_0)}{\cosh(T-T_0)-B}, \tag{2.249}$$

$$\cosh\Theta = \frac{1+\tanh^2(\Theta/2)}{1-\tanh^2(\Theta/2)} = \frac{B\cosh(T-T_0)-1}{\cosh(T-T_0)-B}, \tag{2.250}$$

where $T = \sqrt{3\Lambda}t$ and $T_0 = \sqrt{3\Lambda}t_0$.

Using Eqs.(2.243) and (2.249), we have

$$\begin{aligned}
\log a &= \int \frac{da}{a} = \sqrt{\frac{4\pi G {\rho_0}^2}{3k^2} - \frac{\Lambda}{3}} \int dt \sinh\Theta \\
&= \frac{1}{3}\sqrt{\frac{4\pi G{\rho_0}^2}{k^2\Lambda} - 1} \int dT \frac{\sqrt{B^2-1}\sinh(T-T_0)}{\cosh(T-T_0)-B} \\
&= \frac{1}{3}\log|\cosh(T-T_0) - B| + \text{const.},
\end{aligned} \tag{2.251}$$

where we used the relation $4\pi G{\rho_0}^2/k^2\Lambda = B^2/(B^2-1)$. Therefore, we have

$$a(t) = a_0\Big(\cosh(T-T_0) - B\Big)^{1/3}, \tag{2.252}$$

where a_0 is the constant and $(T-T_0) = \sqrt{3\Lambda}(t-t_0)$.

Similarly, from Eqs.(2.244) and (2.250), we have

$$\begin{aligned}
\varphi &= \int dt \left(\sqrt{1 - \frac{k^2\Lambda}{4\pi G{\rho_0}^2}}\cosh\Theta - 1\right) = \frac{\rho_0}{k\sqrt{3\Lambda}} \int dT \left(\frac{\cosh\Theta}{B} - 1\right) \\
&= \frac{\rho_0(B^2-1)}{kB\sqrt{3\Lambda}} \int \frac{dT}{\cosh(T-T_0) - B} \\
&= \varphi_0 + \frac{\rho_0\sqrt{B^2-1}}{kB\sqrt{3\Lambda}} \log\left| \frac{1 - B + \sqrt{B^2-1}\tanh\Big((T-T_0)/2\Big)}{1 - B - \sqrt{B^2-1}\tanh\Big((T-T_0)/2\Big)} \right| \\
&= \varphi_1 + \frac{1}{\sqrt{12\pi G}} \log\left| \frac{\exp(T-T_0) - B - \sqrt{B^2-1}}{\exp(T-T_0) - B + \sqrt{B^2-1}} \right|,
\end{aligned} \tag{2.253}$$

where φ_0 is the constant and φ_1 is given by $\varphi_1 = \varphi_0 + \frac{1}{\sqrt{12\pi G}} \log\left|\frac{1+B-\sqrt{B^2-1}}{1+B+\sqrt{B^2-1}}\right|$.

The integration constant a_0 is not an independent integration constant but it can be expressed by k and ρ_0. ¿From Eqs.(2.221), (2.252) and (2.253), we have

$$\begin{aligned}\dot{\varphi} &= \frac{(B^2-1)\rho_0}{kB\Big(\cosh(T-T_0)-B\Big)} \\ &= \frac{(B^2-1)\rho_0 {a_0}^3}{kBa^3} = \frac{k}{a^3},\end{aligned} \tag{2.254}$$

which gives the relation $k^2 = (B^2-1)\rho_0 {a_0}^3/B$. This can be written in the form

$$4\pi G{\rho_0}^2 - k^2\Lambda = \frac{\Lambda^2 {a_0}^6}{4\pi G}. \tag{2.255}$$

Then one can obtain

$$a_0 = \Big(4\pi G(\frac{4\pi G{\rho_0}^2}{\Lambda^2} - \frac{k^2}{\Lambda})\Big)^{1/6}. \tag{2.256}$$

Therefore we have the exact solution in the form

$$a(t) = \Big(4\pi G(\frac{4\pi G{\rho_0}^2}{\Lambda^2} - \frac{k^2}{\Lambda})\Big)^{1/6}\Big(\cosh(T-T_0)-B\Big)^{1/3}, \tag{2.257}$$

$$\varphi(t) = \varphi_0 + \frac{1}{\sqrt{12\pi G}}\log\left|\frac{\exp(T-T_0)-B-\sqrt{B^2-1}}{\exp(T-T_0)-B+\sqrt{B^2-1}}\right|, \tag{2.258}$$

where

$$T = \sqrt{3\Lambda}t, \quad T_0 = \sqrt{3\Lambda}t_0 = \text{const.},$$
$$B^{-1} = \sqrt{1-\frac{k^2\Lambda}{4\pi G{\rho_0}^2}}, \quad \varphi_0 = \text{const.}\ .$$

$4\pi G{\rho_0}^2 = k^2\Lambda$ case

For the completeness of the solution, we give the exact solution in this case. As the method to solve the equation is similar, we give only the result. The solution is given by

$$a(t) = \Big(\frac{4\pi G\rho_0}{\Lambda}\Big)^{1/3}\Big(\exp(T-T_0)-1\Big)^{1/3}, \tag{2.259}$$

$$\varphi(t) = \varphi_0 + \frac{1}{\sqrt{12\pi G}}\log\Big|1-\exp\Big(-(T-T_0)\Big)\Big|, \tag{2.260}$$

where

$$T = \sqrt{3\Lambda}t, \quad T_0 = \sqrt{3\Lambda}t_0 = \text{const.}, \quad \varphi_0 = \text{const.}\ .$$

Special Limiting Cases

i) $\rho_0 = 0$ case (no ordinary matter)

In case there is no ordinary matter $\rho_0 \to 0$, which corresponds to $A \to \sqrt{4\pi G{\rho_0}^2/k^2\Lambda}$, we have the expression

$$a(t) = \left(\frac{4\pi G k^2}{\Lambda}\right)^{1/6} \Big(\sinh(T - T_1)\Big)^{1/3}, \tag{2.261}$$

$$\varphi(t) = \varphi_1 + \frac{1}{\sqrt{12\pi G}} \log\left|\tanh\Big((T - T_1)/2\Big)\right|. \tag{2.262}$$

ii) $k = 0$ case (no scalar matter): Lemaître universe [54]

In case there is no scalar matter $k \to 0$, which corresponds to $B \to 1$, we have the expression

$$a(t) = \left(\frac{4\pi G\rho_0}{\Lambda}\right)^{1/3} \Big(\cosh(T - T_1) - 1\Big)^{1/3}, \tag{2.263}$$

$$\varphi(t) = \varphi_1. \tag{2.264}$$

iii) $\Lambda = 0$ case (no cosmological constant)

In case there is no cosmological term $\Lambda \to 0$, which corresponds to $B \to 1 + k^2\Lambda/8\pi G{\rho_0}^2$, we have the expression

$$\begin{aligned} a(t) &= \lim_{\Lambda\to 0} \left(\frac{4\pi G\rho_0}{\Lambda}\right)^{1/3} \left(\frac{3\Lambda\,(t - t_0)^2}{2} - \frac{k^2\Lambda}{8\pi G{\rho_0}^2}\right)^{1/3} \\ &= (6\pi G\rho_0)^{1/3} \left((t - t_0)^2 - \frac{k^2}{12\pi G{\rho_0}^2}\right)^{1/3} \end{aligned} \tag{2.265}$$

$$\begin{aligned} \varphi(t) &= \lim_{\Lambda\to 0} \left\{\varphi_0 + \frac{1}{\sqrt{12\pi G}} \log\left|\frac{\sqrt{3\Lambda}(t - t_0) - k\sqrt{\Lambda}/\sqrt{4\pi G\rho_0^2}}{\sqrt{3\Lambda}(t - t_0) + k\sqrt{\Lambda}/\sqrt{4\pi G\rho_0^2}}\right|\right\} \\ &= \varphi_0 + \frac{1}{\sqrt{12\pi G}} \log\left|\frac{(t - t_0) - k/\sqrt{12\pi G\rho_0^2}}{(t - t_0) + k/\sqrt{12\pi G\rho_0^2}}\right|. \end{aligned} \tag{2.266}$$

2.6.4 Effect on the Gravitational Potential

Here we calculate the effect of the cosmological constant and the scalar matter to the gravitational potential. For the special case of i) no scalar matter or ii) no cosmological term, the exact solutions are well-known.

Exact solutions for special cases

i) No scalar matter case

In this case, we take the standard metric in the form

$$ds^2 = -h(r)^2dt^2 + f(r)^2dr^2 + \left[r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\right] \ , \tag{2.267}$$

and the exact solution is given by [55]

$$h(r)^2 = 1 - r_0/r - \Lambda r^2/3, \tag{2.268}$$

$$f(r)^2 = \frac{1}{1 - r_0/r - \Lambda r^2/3}. \tag{2.269}$$

ii) No cosmological term case

In this case, we take the isotropic metric in the form

$$ds^2 = -h_1(r)^2dt^2 + f_1(r)^2\left[dr^2 + r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\right] \ , \tag{2.270}$$

and the exact solution is given by [56]

$$\varphi(r) = \varphi_0 \log\left(\frac{r-r_0}{r+r_0}\right), \tag{2.271}$$

$$h_1(r)^2 = \left(\frac{r-r_0}{r+r_0}\right)^{2C}, \tag{2.272}$$

$$f_1(r)^2 = \left(1-\frac{{r_0}^2}{r^2}\right)^2\left(\frac{r+r_0}{r-r_0}\right)^{2C}, \tag{2.273}$$

where

$$\varphi_0 = \sqrt{\frac{2(1-C^2)}{8\pi G}}, \quad C = \text{const.}\ .$$

Cosmological term and scalar matter co-existing case

When the cosmological term and scalar matter co-exist, we cannot solve analytically, and we calculate the effect on the gravitational potential approximately. For this purpose, we take the standard metric Eq.(2.267) and the equations of motion Eqs.(2.214) and (2.215) are given by

$$\frac{4rf^{'}}{f^2} + 2f - \frac{2}{f} - 2\Lambda r^2 f = \frac{8\pi r^2 G{\varphi^{'}}^2}{f}, \tag{2.274}$$

$$-\frac{4rh^{'}}{f^2} - \frac{2h}{f^2} + 2h - 2\Lambda r^2 h = -\frac{8\pi r^2 Gh{\varphi^{'}}^2}{f^2}, \tag{2.275}$$

$$\left(\frac{r^2h\varphi^{'}}{f}\right)^{'} = 0. \tag{2.276}$$

These can be rewritten into the form

$$\frac{f^{'}}{f} - \frac{h^{'}}{h} + \frac{f^2}{r} - \frac{1}{r} = \Lambda r f^2, \tag{2.277}$$

$$\frac{f^{'}}{f} + \frac{h^{'}}{h} = 4\pi G r{\varphi^{'}}^2, \tag{2.278}$$

$$\left(\frac{r^2h\varphi^{'}}{f}\right)^{'} = 0. \tag{2.279}$$

¿From Eq.(2.279), we have $\varphi^{'} = \alpha f/r^2h$ with constant α. Then we have

$$(\log f/h)^{'} = \frac{f^{'}}{f} - \frac{h^{'}}{h} = -\frac{f^2}{r} + \frac{1}{r} + \Lambda r f^2, \tag{2.280}$$

$$(\log fh)^{'} = \frac{f^{'}}{f} + \frac{h^{'}}{h} = \frac{4\pi G\alpha^2 f^2}{r^3h^2}. \tag{2.281}$$

We introduce the new variables X, Y in the form $\exp(X) = fh$, $\exp(Y) = f/h$, then the above equation becomes in the form

$$Y' = -\frac{\exp(X+Y)}{r} + \frac{1}{r} + \Lambda r \exp(X+Y), \tag{2.282}$$

$$X' = \frac{4\pi G\alpha^2 \exp(2Y)}{r^3}. \tag{2.283}$$

For $\alpha = 0$ case (no scalar matter case), we have the solution

$$X = 0, \quad \exp(Y) = \frac{1}{1 - r_0/r - \Lambda r^2/3}, \tag{2.284}$$

which is the exact solution to Eqs.(2.268) and (2.269). Then we calculate the gravitational potential by considering the region of r where $r_0/r, \Lambda r^2, 4\pi G\alpha^2/r^2 \ll 1$. In this approximation, we have

$$Y \approx \frac{r_0}{r} + \frac{\Lambda r^2}{3}, \tag{2.285}$$

$$X \approx 2\pi G\alpha^2 \left(\frac{1}{{r_1}^2} - \frac{1}{r^2}\right) \tag{2.286}$$

from Eqs.(2.283) and (2.284) where r_1 is constant.

In order to find the solution for $\alpha \neq 0$, we put $r_0(=\text{const.}) \to r_0(r)$ (function of r). Then Eq.(2.282) becomes

$$\begin{aligned} Y' &= \left(\frac{r_0(r)}{r} + \frac{\Lambda r^2}{3}\right)' = -\frac{r_0}{r^2} + \frac{{r_0}'}{r} + \frac{2\Lambda r}{3} \\ &\approx -\frac{r_0}{r^2} + \frac{2\Lambda r}{3} - \frac{X}{r} \end{aligned} \tag{2.287}$$

which gives ${r_0}' = -X$. Using Eq.(2.286), we have

$$r_0(r) = r_2 - 2\pi G\alpha^2 \left(\frac{r}{{r_1}^2} + \frac{1}{r}\right), \tag{2.288}$$

where r_2 is constant.

The gravitational potential Φ is given by

$$\begin{aligned} g_{00} &= -(1+2\Phi) = -\exp(X-Y) \\ &\approx -(1 - \frac{r_0(r)}{r} - \frac{\Lambda r^2}{3} + X) \\ &\approx -(1 + \frac{4\pi G\alpha^2}{r_1^2} - \frac{r_2}{r} - \frac{\Lambda r^2}{3}), \end{aligned} \tag{2.289}$$

which gives $\Phi = 2\pi G\alpha^2/r_1^2 - r_2/2r - \Lambda r^2/6$. Therefore the scalar matter does not contribute to the gravitational force $F_r = -\partial\Phi/\partial r = -r_2/2r^2 + \Lambda r/3$ within our approximation. The cosmological term contribute to the repulsive force within the approximation.

2.6.5 Summary and Discussion

We considered here the scalar field as the candidate for the dark matter. In order to give the standard scenario of the astrophysics, we studied the Einstein theory with minimally coupled scalar field and the cosmological constant. Various classical solutions with minimally coupled field scalar and the cosmological term in the cosmological, galactic or solar scale, where the scale factor expands as the power law in the main order and then expands exponentially at the cosmological scale. In the galactic or solar scale, the exact solution cannot be found, and we examined the contribution from the scalar field to the gravitational potential and found that the scalar field does not contribute to the gravitational force within the approximation. In this way, at the cosmological scale, the scalar field plays the role of the dark matter in some sense. While, in the galactic or solar scale, the scalar field does not play the role of the dark matter.

2.7 Brans-Dicke cosmology with the cosmological constant [51]

Brans and Dicke [58] proposed a tensor-scalar theory of the gravitational field based on Mach's principle. This theory is consistent with experiment as long as the Dicke constant ω is about equal to or greater than 2000. In the limit $\omega \to \infty$, the BD theory reduces to the Einstein theory for a constant BD scalar field φ. In order to determine the solutions of the BD equations for cosmology (when applied to cosmology with the cosmological principle) for a given value of ω, it is always necessary to have one more initial condition than needed for determination of the solutions of the Friedmann equations. The BD equations in the absence of the cosmological constant λ can be solved analytically in the case of zero space curvature $(k=0)$ and zero pressure $(p=0)$ [60]. In general, the BD equation for the case of $k \neq 0$ and $\lambda \neq 0$ uniquely determines the scale factor $a(t)$, the matter density $\rho(t)$, and the BD scalar field $\varphi(t)$ for all t provided the present values of five variables, e.q., $\rho_0, H_0, q_0, \varphi_0$, and $\dot{\varphi}$, as well as equations of state and the constants ω and k are given. Hubble's constant H_0 and the deceleration parameter q_0 are defined by

$$H_0 = \left[\frac{\dot{a}}{a}\right]_0, q_0 = \left[-\frac{a\ddot{a}}{\dot{a}^2}\right]_0, \tag{2.290}$$

where the subscript zero denotes the present value and an overdot denotes a derivative with respect to t. The solutions, however, may not be expressed in terms of elementary functions (recall that the Friedmann equations with $k \neq 0$ and $\lambda \neq 0$ lead to elliptic integrals). Also, the case $k \neq 0$ and $\lambda = 0$ cannot be solved analytically even for $p = 0$. In this case the present values of four variables are necessary to determine the solutions.

We present analytic solutions of the BD equations with the nonvanishing cosmological constant for the case $k = 0$ and $p = 0$. The solutions for $a(t), \rho(t)$, and $\varphi(t)$ are determined by ρ_0, H_0, q_0, and φ_0. Hence, in this case the theory has a predictive power for $\dot{\varphi}_0$ which in turn provides the present changing rate of the gravitational "constant" for a given value of ω. As in the case $k = 0$ for the Friedmann equations (with or without λ), the multiplicative factor for $a(t)$ cannot be determined in terms of other observable quantities.

We start with the following Lagrangian density for the BD theory with the cosmological constant λ [†]

$$\mathcal{L} = \sqrt{-g}\left[-\varphi(R+2\lambda) + \omega\frac{g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi}{\varphi}\right] + 16\pi\mathcal{L}_M, \tag{2.291}$$

[†] As is seen when φ is transformed to χ^2, this Lagrangian density is nothing but the one which represents a scalar field interacting with gravitation. (However, the scalar field is not supposed to contribute to the matter so that the equivalence principle is satisfied.) In this version, the cosmological term corresponds to a mass term unless some invariance is assumed, such as invariance under conformal transformation for which the Dicke constant is also uniquely determined.

where $\mathcal{L}_M$ denotes the Lagrangian density of the matter. We follow the "Landau-Lifshitz timelike convention." The Euler-Lagrange equations of motion for $g_{\mu\nu}$ and φ are

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda g_{\mu\nu} \\ = \ & \frac{8\pi}{\varphi} T_{M\mu\nu} + \frac{\omega}{\varphi^2}(\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2} g_{\mu\nu}\varphi_{,\rho}\varphi^{,\rho}) + \frac{1}{\varphi}(\varphi_{,\mu;\nu} - g_{\mu\nu}\Box\varphi), \end{aligned} \tag{2.292}$$

$$\Box\varphi - \frac{2\lambda}{3+2\omega}\varphi = \frac{8\pi}{3+2\omega} T_M{}^{\mu}{}_{\mu}, \tag{2.293}$$

where the energy-momentum tensor of the matter $T_{M\mu\nu}$ is defined by

$$T_{M\mu\nu} = 2(-g)^{-1/2} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}}. \tag{2.294}$$

The equivalence principle ($T_{M\mu\nu}{}^{;\mu} = 0$) is satisfied. This can be easily checked by using Eqs.(2.292) and (2.293), and the Bianchi identities.

We now apply the theory to cosmology where the universe is smeared out into a homogeneous isotropic distribution of the matter. The metric is then given by Robertson-Walker form

$$d\tau^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{2.295}$$

and the energy-momentum tensor is that of the perfect fluid

$$T_{M\mu\nu} = -p(t) g_{\mu\nu} + [\rho(t) + p(t)] u_\mu u_\nu, \tag{2.296}$$

where ρ is the mass density, p is the pressure, and

$$u_\mu = g_{\mu\nu} u^\nu = g_{\mu\nu}\frac{dx^\nu}{d\tau}.$$

Then, Eqs.(2.292) and (2.293) reduce to the following three equations:

$$\frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} - \lambda = \frac{8\pi}{\varphi}\rho + \frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi^2} - \frac{3\dot{a}}{a}\frac{\dot{\varphi}}{\varphi}, \tag{2.297}$$

$$-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} + \lambda = \frac{8\pi}{\varphi} p + \frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\ddot{\varphi}}{\varphi} + \frac{2\dot{a}}{a}\frac{\dot{\varphi}}{\varphi}, \tag{2.298}$$

$$\frac{\ddot{\varphi}}{\varphi} + 3\left[\frac{\dot{a}}{a}\right]\left[\frac{\dot{\varphi}}{\varphi}\right] = \frac{2\lambda}{3+2\omega} + \frac{8\pi}{\varphi}\frac{\rho - 3p}{3+2\omega}. \tag{2.299}$$

These are the BD equations with the nonvanishing λ. It can be checked that Eqs.(2.297)-(2.299) lead to the continuity equation

$$\dot{\rho} + (\rho + p)\frac{3\dot{a}}{a} = 0, \tag{2.300}$$

which is consistent with the principle of equivalence. The gravitational "constant" is given by

$$G = \left[\frac{2\omega + 4}{2\omega + 3}\right]\frac{1}{\varphi} \tag{2.301}$$

and $G_0 \equiv G_N$ is the Newtonian gravitational constant.

We now set $k = 0$ in Eqs.(2.297)–(2.299) and consider the matter-dominant universe where the pressure can be neglected. Defining

$$\epsilon = -\frac{\dot{\varphi}}{\varphi} \tag{2.302}$$

and using Eq.(2.290), we rewrite Eq.(2.297) as

$$3H^2 - \frac{\omega}{2}\epsilon^2 - 3\epsilon H = 8\pi\frac{\rho}{\varphi} + \lambda. \tag{2.303}$$

Eliminating $\ddot{\varphi}$ from Eqs.(2.298) and (2.299) and using Eqs.(2.290) and (2.302), we obtain

$$3qH^2 - \omega\epsilon^2 - 3\epsilon H = 8\pi\frac{\rho}{\varphi}\frac{\omega+3}{2\omega+3} - \frac{2\omega}{2\omega+3}\lambda. \tag{2.304}$$

We eliminate ϵ from Eqs.(2.303) and (2.304) to find a quadratic equation for λ:

$$\begin{aligned} & \frac{4\omega(\omega+1)^2}{(2\omega+3)^2}\lambda^2 \\ + \quad & \frac{2}{2\omega+3}\left[2\omega(\omega+1)qH^2 - (2\omega+1)(2\omega+3)H^2 + \frac{16\pi\omega(\omega+1)^2}{2\omega+3}\frac{\rho}{\varphi}\right]\lambda \\ + \quad & \omega q^2H^4 + 2(2\omega+3)(1-q)H^4 + \frac{16\pi\omega}{2\omega+3}[(\omega+1)(q-2)-1]\frac{\rho}{\varphi} \\ + \quad & \omega\left[\frac{8\pi(\omega+1)}{2\omega+3}\frac{\rho}{\varphi}\right]^2 = 0. \end{aligned} \tag{2.305}$$

This equation determines the values of λ provided we are given the present values of H_0, q_0, ρ_0, and φ_0 as well as ω.

Next, using λ obtained by solving Eq.(2.305) and combining Eqs.(2.303) and (2.304), we can express ϵ in terms of H, q, ρ, and φ. The result is

$$\epsilon = \frac{1}{3H}\left[-3qH^2 + 6H^2 + 8\pi\left[\frac{\omega+3}{2\omega+3} - 2\right]\frac{\rho}{\varphi} - \left[\frac{2\omega}{2\omega+3} - 2\right]\lambda\right], \tag{2.306}$$

where λ is given by Eq.(2.305). Equation(2.306) determines ϵ_0 in terms of the observable parameters H_0, ρ_0, q_0, and φ_0 for a given value of ω.

Since both λ and ϵ_0 can be expressed in terms of ρ_0, H_0, q_0, and φ_0, the solutions of the BD equations will be written in terms of λ and ϵ_0 as well as the other four initial values. This enables us to write down the solutions in a compact way.

We now proceed to solve Eqs.(2.297)–(2.299) for $a(t)$. The combination, Eq.(2.297) –Eq.(2.298)–$(\frac{1}{3})$Eq.(2.299), gives the following equation for φa^3:

$$\frac{d^2}{dt^2}(\varphi a^3) - \eta^2\lambda(\varphi a^3) = 4\pi\eta^2\rho_0 a_0^3, \tag{2.307}$$

where

$$\eta^2 = \frac{2(4+3\omega)}{3+2\omega}$$

and $\rho_0{a_0}^3$ is an integration constant of eq.(2.300):

$$\rho(t)a^3(t) = \rho_0{a_0}^3. \tag{2.308}$$

Equation(2.307) can easily be solved. Observing that the left-hand side of Eq.(2.299) is $(\varphi a^3)^{-1}d(\dot{\varphi}a^3)/dt$, we can rewrite Eq.(2.299) as

$$\frac{d}{dt}(\varphi a^3) = 8\pi\frac{\rho_0 {a_0}^3}{3+2\omega} + \frac{2\lambda}{3+2\omega}f(t), \tag{2.309}$$

where $f(t)=\varphi(t)a^3(t)$ is the solution of Eq.(2.307), expressed in terms of elementary functions. Defining the solution of Eq.(2.309) by $h(t)=\dot{\varphi}(t)a^3(t)$, and taking the time derivative of $f(t)=\varphi(t)a^3(t)$, we obtain

$$\dot{f}(t) = h(t) + 3f(t)\left[\frac{\dot{a}}{a}\right]. \tag{2.310}$$

It is elementary to solve Eq.(2.310) for $a(t)$. However, since Eqs.(2.297)–(2.299) are coupled equations, the consistency of the solution with all equations must be checked. This consistency leads to a constraint among the integration constants and the correct number of the integration constants is recovered. The exact solutions with initial conditon $a(t)=0$ at $t=0$ are the following.

(1)Positive λ:

$$a(t) \propto \left[A\cosh\eta\sqrt{\lambda}(t-t_c) - \frac{4\pi}{\lambda}\right]^{\alpha(\omega)}$$
$$\left[\frac{\left[\frac{4\pi}{\lambda}+A\right]\tanh\frac{\eta\sqrt{\lambda}(t-t_c)}{2} - [(4\pi/\lambda)^2 - A^2]^{1/2}}{\left[\frac{4\pi}{\lambda}+A\right]\tanh\frac{\eta\sqrt{\lambda}(t-t_c)}{2} + [(4\pi/\lambda)^2 - A^2]^{1/2}}\right]^{\delta\beta(\omega)} \quad for B>0, \tag{2.311}$$

$$a(t) \propto \left[A'\sinh\eta\sqrt{\lambda}(t-t'_c) - \frac{4\pi}{\lambda}\right]^{\alpha(\omega)}$$
$$\left[\frac{\left[\frac{4\pi}{\lambda}\right]\tanh\frac{\eta\sqrt{\lambda}(t-t'_c)}{2} + A' - [(4\pi/\lambda)^2 + A'^2]^{1/2}}{\left[\frac{4\pi}{\lambda}\right]\tanh\frac{\eta\sqrt{\lambda}(t-t'_c)}{2} + A'[(4\pi/\lambda)^2 + A'^2]^{1/2}}\right]^{\delta\beta(\omega)} \quad for B>0, \tag{2.312}$$

where

$$B = \left[\frac{4\pi}{\lambda}\right]^2 - \frac{3}{2\lambda}\frac{1}{4+3\omega}\left[\frac{\varphi_0}{\rho_0}\right]^2[(1+\omega)\epsilon_0 + H_0]^2. \tag{2.313}$$

(2)Negative λ:

$$a(t) \propto \left[A\sin\eta\sqrt{-\lambda}(t-t_c\prime\prime) - \frac{4\pi}{\lambda}\right]^{\alpha(\omega)}$$
$$\left[\frac{-\frac{4\pi}{\lambda}\tan\frac{\eta\sqrt{-\lambda}(t-t_c\prime\prime)}{2} + A - [A^2 - (4\pi/\lambda)^2]^{1/2}}{-\frac{4\pi}{\lambda}\tan\frac{\eta\sqrt{-\lambda}(t-t_c\prime\prime)}{2} + A + [A^2 - (4\pi/\lambda)^2]^{1/2}}\right]^{\delta\beta(\omega)}. \tag{2.314}$$

In Eqs.(2.311),(2.312), and (2.314)

$$A=\sqrt{B},\ A'=\sqrt{-B},\ \alpha(\omega)=\frac{1+\omega}{4+3\omega},\ \beta(\omega)=\frac{1}{4+3\omega}\left[\frac{3+2\omega}{3}\right]^{1/2} \tag{2.315}$$

and $\delta = \mathrm{sgn}[(1+\omega)\epsilon_0 + H_0]$, and $t_c, t_c\prime$, and $t_c\prime\prime$ are given by

$$\begin{aligned} t_c &= -\frac{2}{\eta\sqrt{\lambda}}\tanh^{-1}\left[\frac{4\pi/\lambda - A}{4\pi/\lambda + A}\right]^{1/2}, \\ t_c\prime &= -\frac{2}{\eta\sqrt{\lambda}}\tanh^{-1}\frac{[(4\pi/\lambda)^2 + A'^2] - A'}{(4\pi/\lambda)}, \\ t_c\prime\prime &= -\frac{2}{\eta\sqrt{-\lambda}}\tan^{-1}\frac{[A^2 - (4\pi/\lambda)^2]^{1/2} - A}{(-4\pi/\lambda)}. \end{aligned} \tag{2.316}$$

As mentioned earliar, multiplicative factor for $a(t)$ in Eqs.(2.311),(2.312), and (2.314) cannot be determined in terms of $H_0, \rho_0, q_0,$ and φ_0.

In the limits $\epsilon_0 = 0$ and $\omega \rightarrow \infty$, the solution in Eqs.(2.311) and (2.314) become, respectively,

$$a(t) \propto \sinh^{2/3}\left[\frac{\sqrt{3\lambda}}{2}t\right] \;\; for \lambda > 0, \;\; a(t) \propto \sin^{2/3}\left[\frac{\sqrt{-3\lambda}}{2}t\right] \;\; for \lambda < 0. \tag{2.317}$$

These are nothing but the exact solutions to the Friedmann equations with $k = p = 0$ and $\lambda \neq 0$.

The present age of the universe is

$$t_0 = \begin{cases} \frac{1}{\eta\sqrt{\lambda}}\left[\cosh^{-1}\left[\frac{\varphi_0/\rho_0+4\pi/\lambda}{A}\right] - 2\tanh^{-1}\left[\frac{4\pi/\lambda - A}{4\pi/\lambda + A}\right]^{1/2}\right] & \text{for} \lambda > 0, B > 0, \\ \frac{1}{\eta\sqrt{\lambda}}\left[\sinh^{-1}\left[\frac{\varphi_0/\rho_0+4\pi/\lambda}{A}\right] - 2\tanh^{-1}\left[\frac{-A' + [(4\pi/\lambda)^2 + A'^2]^{1/2}}{4\pi/\lambda}\right]\right] & \\ & \text{for} \lambda > 0, B < 0, \\ \frac{1}{\eta\sqrt{-\lambda}}\left[\sin^{-1}\left[\frac{\varphi_0/\rho_0+4\pi/\lambda}{A}\right] - 2\tan^{-1}\left[\frac{-A + [A^2 - (4\pi/\lambda)^2]^{1/2}}{-4\pi/\lambda}\right]\right] & \text{for} \lambda < 0. \end{cases} \tag{2.318}$$

The corresponding age of the universe for the Friedmann equations with $k = 0, \lambda \neq 0$ is

$$t_0 = \begin{cases} \frac{2}{\sqrt{3\lambda}} \coth^{-1}(\sqrt{3/\lambda} H_0) & for \lambda > 0, \\ \frac{2}{\sqrt{-3\lambda}} \cot^{-1}(\sqrt{3/-\lambda} H_0) & for \lambda < 0, \end{cases} \tag{2.319}$$

In conclusion, analytic expressions of $a(t)$ for the BD equations in the $k = p = 0$ and $\lambda \neq 0$ case have been presented. The solutions are determined, apart from multiplicative factors, by four values $H_0, \rho_0, q_0,$ and φ_0 (or G_N) for a given value of ω. For some typical values of these initial conditions, the predicted values of λ, age of the universe and $\epsilon_0 = (\dot{G}/G)_0$ may be calculated.

Finally, we present the solution of $a(t)$ for the BD equations in the $k = p = \lambda = 0$ case. The solution is well known [60]. However, our present formulation enables us to write it in the following compact way ‡

$$a(t) \propto [t(t - 2t_c)]^{\alpha(\omega)} \left[\frac{t}{t - t_c}\right]^{\delta\beta(\omega)}, \tag{2.320}$$

where

$$t_c = -\frac{1}{4\pi\eta^2}\left[\frac{3}{3 + 2\omega}\right]^{1/2} \frac{\varphi_0}{\rho_0} \mid (1 + \omega)\epsilon_0 + H_0 \mid . \tag{2.321}$$

‡Compare this solution with the solution in Ref.60 where the physical meaning of t_c is still unsettled.

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Chapter 3

Scalar, Eletromagnetic, and Gravitational Fields Interaction: Particlelike Solutions

Scalar, Electromagnetic, and Gravitational Fields Interaction: Particlelike Solutions

3.1 Particle-Like Criterion [35]

The necessity of treatment of extended particle models is dictated, on one head, by modern experimental data on complicated internal structure of particles and, on the other hand, by the difficulties of the theories operating with point objects. Classical particle models (instantons, solitons, etc.) based on nonlinear field equations are at present widely discussed. However, in these investigations the gravitational field is, in general, not taken into account.

We will discuss particle models based on scalar, electromagnetic, and gravitational field interaction [35]. A self-consistent inclusion of gravity is important due to the universal character of gravitational interaction, which is always present and cannot be shielded. This interaction induces a nonlinearity even into systems with linear material field. Treatment of their direct of their direct interactions broadens the possibilities for model construction although the difficulties connected with the choice and interpretation of interactions remain the same as in the case of flat space-time.

It is to be notes that the nonlinearity should be essential mainly at distances where quantum effects are of importance. Therefore, classical nonlinear field theory should be considered as a tentative, though necessary, stage on the way to quantum models.

Sets of equations describing real physical processes are very complicated and, as a rule, it is impossible to find exact solutions for them. On the other hand, perturbation theory does not allow one to clear out many principal features of the interacting fields. Thus, special interest is gained by nonlinear model systems admitting exact solutions.

A study of such models was initiated by a treatment of interacting conformal scalar and gravitational field [1]. A starting point was the conjecture by Staniukovich [2] and Markov [3] on the possible existence of fundamental superheavy particles, planckeons (or maximons), with parameters of the order of Planck mass ($10^{-5}g$) and length (10^{-33}cm). Such objects could serve as elementary particle cores or provide a model for scalar-gravitational vacuum [4]. Later, exact spherically symmetric solutions were found with a conformal scalar field [5], with scalar and electromagnetic fields [6, 7] and, moreover, including direct scalar-electromagnetic interaction in general relativity (GR) [8, 9, 10]. The case of a bare mass source was also treated [11, 12].

In flat space-time the concept of a particlelike solution includes, in general, the following basic requirements:

(i) localization in space;

(ii) staticity or stationarity (conservation of shape, no spreading);

(iii) finite total energy.

If gravity is included, i.e., the space-time is curved, requirement (iii) causes certain difficulties. It is insufficient to consider only the material fields' total energy

$$E_f = \int T_0{}^0 (-g)^{1/2} d^3x \tag{3.1}$$

and, at the same time, there is no gravitational field energy definition which would be fully satisfactory. Though, in an asymptotically flat space-time (requirement (i)) the total energy of the system including gravitation ($E_{tot} = m_I c^2$, where m_I is the inertial mass) can be computed using the total canonical energy-momentum tensor. In the static case [13]

$$E_{tot} = m_I c^2 = \frac{1}{2\kappa}\int d^3x \frac{\partial}{\partial x^k}\left\{\frac{g^{00}}{(-g)^{1/2}}\frac{\partial}{\partial x^i}[(-g)g^{00}g^{ik}]\right\};\ i,k = 1,2,3 \tag{3.2}$$

where integration is carried out over the whole three-space in asymptotically Cartesian coordinates (otherwise the integral may formally diverge). The asymptotic form of the metric can be written as

$$\begin{aligned} &ds^2 = e^{2\gamma(R)} dx^{0^2} - e^{2\lambda(R)} dx^i dx^i, \quad R = (x^i x^i)^{1/2}; \\ &e^{2\gamma} = 1 - 2Gm/c^2R + o(R^{-1}); \qquad e^{2\lambda} = 1 + 2Gm^*/c^2R + o(R^{-1}), \end{aligned} \tag{3.3}$$

where R is the so-called isotropic radius, m is the active gravitational (Schwarzshild) mass, and m^* coincides with m_I if the space-time is regular and one can transform (3.2) into an integral over a distant two-surface. The equality $m = m_I$ (the equivalence principle) is a priori not guaranteed and can be violated in the two cases: (a) in some scalar-tensor theories (STT) of gravity [14, 15] due to $m \neq m^*$; we confine ourselves (unless specially indicated) to GR where always the Schwarzschild asymptotic takes place, $m = m^*$; (b) if there are space-time singularities and integration over a distant surface is insufficient.

Thus we call a solution to be *particlelike* if it is

(i) asymptotically flat (with material fields properly decreasing also);

(ii) static or stationary;

(iii) singularity free.

The regularity requirement combined with (i) guarantees both finiteness of E_f and m_I and the equality $m = m_I$. Besides, existence of singularities manifests a limitation upon the validity domain for a given model, or even for the classical concept of space-time. However, requirement (iii) cannot be fulfilled not only by systems with linear matter fields (see Section 3.3), but also by a broad class of systems containing directly interacting fields (see Section 3.5 and 3.6). Therefore, along with criterion (i) – (iii), hereafter refered to as "*the strong criterion*," it is worthwhile to introduce an alternative criterion for particlelike solutions (hereafter refered to as "*the weak criterion*"), in which requirements (i) and (ii) remain unchanged but (iii) is replaced by (iii.a) $E_f \ < \ \infty$

(iii.b) : $m \ = \ m_I$.

Later we shall see that the *weak criterion* is satisfied by some singular configuration.

The problems of constructing self-gravitating particle models have been considered also in Refs. [16, 17, 18, 19]. However, the results of these papers are either based on approximate solutions, or contain at least one of such undesired features as thin shells, surface changes, negative matter densities, or pressures. We discuss exact solutions possessing no such features (exact Section 3.4.5 where a negative density is introduced).

In Section 3.2 we write down the metric and the basic equations and formulate explicitly the requirements upon particlelike solutions, namely, the boundary conditions at the infinity, regularity conditions for the center, and the finiteness requirement for the material fields' total energy. Besides, we present formulas for the gravitational and inertial masses and, moreover, describe a way of transforming solutions of GR into solutions for a class of STT. In particular, this enables us to consider problems with a conformal scalar field in GR.

In Section 3.3 we describe static spherically symmetric solutions to equations describing the gravitational, linear massless scalar and linear electromagnetic fields. It is shown that all these solutions are and thus violate the *strong criterion.* Besides, in all cases when horizons are absent, the total material fields' energy E_f is infinite, therefore the *weak criterion* is also violated. When there is a horizon (the Reissner-Nordström solution), E_f integrated over the static region is finite and formally the *weak criterion* is fulfilled.

In Section 3.4 we discuss one of the ways to obtain singularity-free models, namely, construction of extended sources of the fields in the form of electrically and scalarly charged incoherent matter distributions. It is shown that if an external scalar field is absent, the strong criterion is satisfied only for $Gm^2 = q^2$. Electrically neutral or weakly changed models are possible only with a repulsive scalar field, strongly changed ones with an attractive scalar field. A concrete example of a singularity-free charged particle models in GR is given. It is also shown that, for this class of regular models, the source radius cannot be less than the particle classical radius $r_{\rm cl}$, whatever the arbitrary functions of the solutions are chosen.

Further, we formulate a number of limitations upon the possible nonsingular model parameters in GR with a conformal scalar field and in the Brans-Dicke STT. Besides, we construct a model for a particle-antiparticle pair of a Wheeler-handle type in GR, containing no scalar field: the internal region containing a throat, is matched to two external regions, each of them described by the Reissner-Nordström solution, with equal masses and charges of equal magnitudes and opposite signs. In constructing this model, we have to abandon the requirement that the matter density is nonnegative; nevertheless, gravitational masses corresponding to both asymptotics and the total energy E_{tot} are positive.

In Section 3.5 we consider nonlinear electrodynamics in GR with the Lagrangian depending arbitrarily on the electromagnetic field invariant. We prove that the *strong criterion* cannot be fulfilled, whatever this dependence is. General conditions are given for the fulfillment of the *weak criterion.* The fact that the latter can be really satisfied is demonstrated explicitly for the Einstein-Born-lnfeld system.

In Section 3.6 we consider various types of direct interaction of massless scalar and electromagnetic fields in GR. For the interaction of the form $L_{\rm int}$
$= \sigma F^{\alpha\beta} A_\alpha \psi_{,\beta}$ the exact solution is singular at the center but satisfies the weak criterion. When the charge equals that of an electron, the total mass of the system is of the order of the Planck mass. The material fields' energy diverges in the flat-space limit, thus gravitation plays a regularizing role. For the interaction of the form $(e^{\sigma\psi} - 1) F_{\alpha\beta} F^{\alpha\beta}$, the solution also satisfies the *weak criterion*, though both the metric and the ψ field are singular at the center. The field energy is finite both in the flat-space limit and for $\sigma \to 0$ (note that $\sigma = 0$ corresponds to the Reissner-Nordstr6m field with an event horizon) and the total mass

is again of the order of the Planck mass. If real particle parameters are taken, the direct interaction is dominant and the gravitational contribution to the mass is extremely small.

Finally, for the interaction of the form $F_{\alpha\beta}F^{\alpha\beta}[\sin B(\psi-\psi_0)]^{-2}$ we succeed in constructing a fully regular field model, satisfying the strong criterion. In this model the mass and charge densities vanish at the center and have a maximum at $r=\frac{1}{4}r_{\rm cl}$, reproducing the principal features of the experimentally obtained proton charge and mass distributions.

3.2 The Model

3.2.1 Consider a system with the Lagrangian

$$L = R/2\kappa - F^{\alpha\beta}F_{\alpha\beta}/16\pi + \frac{1}{2}ng^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} + L_m + L_{\rm int}, \tag{3.4}$$

where the first term describes the Einstein gravitational field (R is the scalar curvature, $\kappa = 8\pi Gc^{-4}$ the gravitational constant), the second and the third ones describe free electromagnetic ($F_{\alpha\beta}$) and scalar (ψ) fields, L_m refers to bare mass (incoherent matter) and $L_{\rm int}$ describes a direct interaction of scalar and electromagnetic fields either with each other or, in case of $L_m \neq 0$, with the bare mass. The metric tensor $g_{\alpha\beta}$ has the signature $(+---)$ and $\psi_{,\alpha} \equiv \partial\psi/\partial x^\alpha$. The factor n equals ± 1; $n=+1$ corresponds to a usual attractive scalar field with positive energy density, $n=-1$ to a repulsive field which can appear, e.g., in some STT; see further.

Let the system be static spherically symmetric:

$$\begin{aligned} &ds^2 = e^{2\gamma(\xi)}dx^{0^2} - e^{2\alpha(\xi)}d\xi^2 - e^{2\beta(\xi)}d\Omega^2, \\ &d\Omega^2 = dx^{2^2} + \sin^2 x^2 dx^{3^2}; \end{aligned} \tag{3.5}$$

$$\psi = \psi(\xi);\; F_{01} = -F_{10}(\xi);\; \text{the other } F_{\mu\nu} = 0 \tag{3.6}$$

(we use also the notation r for the curvature radius of coordinate spheres: $r = e^\beta$). Consequently, in the Einstein equations

$$G_\mu^{\ \nu} \equiv R_\mu^{\ \nu} - \frac{1}{2}\delta_\mu^{\ \nu}R = -\kappa[T^\nu_{{\rm em}\ \mu} + nT^\nu_{s\ \mu} + T^\nu_{m\ \mu} + T^\nu_{{\rm int}\ \mu}] \tag{3.7}$$

one can express three of the four $T_\mu^{\ \nu}$ (energy-momentum tensor for the corresponding) term in (3.4) as follows (prime denote $d/d\xi$):

$$T^\nu_{{\rm em}\ \mu} = -(F^{01}F_{10}/8\pi){\rm diag}(1,1,-1,-1); \tag{3.8}$$

$$T^\nu_{s\ \mu} = (\frac{1}{2}e^{-2\alpha}\psi'^2){\rm diag}(1,-1,1,1); \tag{3.9}$$

$$T^\nu_{m\ \mu} = \rho_m c^2 \delta_{\mu 0}\delta^{\nu 0}. \tag{3.10}$$

The structure of $T^\nu_{{\rm int}\ \mu}$ and the form of field equations depend essentially on the form of $L_{\rm int}$.

Now let us formulate requirements (i) and (iii) for the particular case of static (requirement (ii)) spherically symmetric systems.

The boundary condition at spatial infinity is that the spacetime is flat and the fields ψ and $F_{\mu\nu}$ vanish (requirement (i)), i.e., at some $\xi = \xi_\infty$

$$e^\gamma = 1;\quad e^\beta = \infty;\quad e^{-2\alpha+2\beta}\beta'^2 = 1;\quad \psi = 0;\quad E = 0, \tag{3.11}$$

where E is the electric field intensity: $|E| = (F_{01}F^{10})^{1/2}$.

The condition that the configuration has a regular center (requirement (iii)) means that at some $\xi = \xi_c$

$$e^\beta = 0;\quad \gamma| < \infty;\quad e^{-2\alpha+2\gamma}\gamma'^2 = 0;\quad e^{-2\alpha+2\beta}\beta'^2 = 1;\quad E = 0, \tag{3.12}$$

b i.e., the space-time is locally flat and there is no force acting upon a test charge.

Condition (iii.a) that the total field energy E_f is finite means that the integral

$$E_f = E_s + E_e + E_{\rm int} = 4\pi \int d\xi e^{\alpha+2\beta+\gamma}(nT^0_{s\ 0} + T^0_{\rm em\ 0} + T^0_{\rm int\ 0}) \tag{3.13}$$

over the whole three-space should converge.

The gravitational mass m for the asymptotically flat metric (3.5) is calculated by the formula

$$m = (c^2/G)(\gamma'/\beta')e^{\beta}|_{\xi\to\xi_\infty}. \tag{3.14}$$

In order to compute the inertial mass, let us take into account not only the contribution of a distant surface, but also that of a surface enclosing a possible singularity and tending to it as a limit. In the isotropic coordinates, in which an arbitrary metric (3.5) has the form (3.3), we get

$$m_I = m^* + \triangle m = m^* + \frac{c^2}{G}R^2\frac{d\lambda}{dR}e^{\lambda+\gamma}|_{R\to R_s(\text{singularity})} \tag{3.15}$$

(remember that in GR $m^* = m$). The coordinates (3.3) are not only asymptotically Cartesian, but also locally Cartesian in a regular center if it exists. in the latter case automatically $m_I = m^*$; however, this condition may hold at singularities as well.

3.2.2 It should be noted that some problems of STT of gravitation are reduced to the equations corresponding to (3.4), i.e., the field system of GR. Indeed, let the Lagrangian of STT (metric $\tilde{g}_{\mu\nu}$, scalar field φ) have the form

$$L_{\rm STT} = A(\varphi)\tilde{R} + B(\varphi)\tilde{g}^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + L^*, \tag{3.16}$$

where a tilde denotes quantities obtained using $\tilde{g}_{\mu\nu}$ and L^* is the Lagrangian for all the remaining matter. Then, after the conformal mapping [20]

$$\begin{aligned} &\tilde{g}_{\mu\nu} = F(\hat{\psi})g_{\mu\nu},\ F(\psi) = 1/A(\varphi(\psi));\\ &\frac{d\varphi}{d\hat{\psi}} = A\cdot\left|AB + \frac{3}{2}\left(\frac{dA}{d\varphi}\right)^2\right|^{-1/2}, \end{aligned} \tag{3.17}$$

the Lagrangian takes the quasi-Einsteinian form

$$L = R + ng^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} + F^2(\psi)L^*, \tag{3.18}$$

$$n = \text{sign}[2AB + 3(dA/dvarphi)^2], \tag{3.19}$$

and further we should prescribe a concrete form of L^* (the dimensionless field $\hat{\psi}$ is connected with ψ form (3.4) by $\hat{\psi} = \psi\kappa^1{}_e{}^2$).

The theory (3.16) includes as particle cases:
GR with a massless (minimally coupled) scalar field, theory (3.4):

$$A \equiv 1, \quad B = n\kappa = \pm\kappa, \quad F \equiv 1, \quad \varphi \equiv \psi; \tag{3.20}$$

GR with a conformal scalar field:

$$\begin{aligned} &A = 1 - \tfrac{1}{6}\kappa\varphi^2;\ \ B \equiv \kappa;\ \ n = +1;\\ &F(\hat{\psi}) = \cosh^2(\hat{\psi}/\sqrt{6});\ \kappa^{1/2}\varphi/\sqrt{6} = \tanh(\hat{\psi}/\sqrt{6}); \end{aligned} \tag{3.21}$$

the Brans-Dicke STT:

$$A = \varphi; \quad B = \omega/\varphi; \quad \omega = \text{const} \neq -\tfrac{3}{2};$$
$$n = \text{sign}(3 + 2\omega); \; F(\hat{\psi}) = 1/\varphi = \exp(\hat{\psi}|\omega + \tfrac{3}{2}|^{-1/2}). \tag{3.22}$$

3.2.3 In the following we shall frequently use a special choice of the radial coordinate ξ, namely, such that in the metric (3.5)

$$\alpha(\xi) = 2\beta(\xi) + \gamma(\xi). \tag{3.23}$$

Now we shall give some general formulas in these coordinates. For the Einstein tensor $G_\mu^{\ \nu}$ we have

$$e^{2\alpha} G_1^{\ 1} = -e^{2\beta+2\gamma} + (\beta' + \gamma')^2 - \gamma'^2; \tag{3.24}$$
$$e^{2\alpha}(G_1^{\ 1} + G_2^{\ 2}) = -e^{2\beta+2\alpha} + \beta'' + \gamma''. \tag{3.25}$$

This implies that if the total matter energy-momentum tensor, i.e., the right-handside of Eqs. (3.7), satisfies the relation

$$T_1^{\ 1} + T_2^{\ 2} = 0, \tag{3.26}$$

then, integrating the $\binom{1}{1} + \binom{2}{2}$ component of (3.7), we arrive at the metric

$$ds^2 = e^{2\gamma(\xi)} dx^{02} - \frac{e^{-2\gamma(\xi)}}{s^2(k,\xi)} \left[\frac{d\xi^2}{s^2(k,\xi)} + d\Omega^2 \right],$$
$$k = \text{const}, \; s(k,\xi) = \begin{cases} k^{-1} \sinh k\xi, & k > 0; \\ \xi, & k = 0; \\ k^{-1} \sin k\xi, & k < 0. \end{cases} \tag{3.27}$$

This metric is flat at spatial infinity ($\xi = 0$) provided that $\gamma(0) = 0$. With no loss of generality we take always $\xi > 0$.

Moreover, it can be easily shown that the center regularity condition for a system with metric (3.27) is fulfilled only if

$$\xi_c = \infty; \quad k = 0; \quad \gamma' = o(\xi^{-2}) \quad \text{as} \;\; \xi \to \infty. \tag{3.28}$$

3.3. Scalar-Electrovacuum Solutions [7]

3.3.1 Now the shall outline the properties of static spherically symmetric solutions for linear source-free scalar and electromagnetic fields in GR. For convenience, we present a solution for a more general problem of finding an external field due to a charged body in STT (Lagrangina (3.16), $L^* = Gc^{-4}\tilde{g}^{\alpha\gamma}\tilde{g}^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta}$). The transformation (3.17) reduces the problem to that of GR with a minimally coupled ψ field (Lagrangian (3.4), $L_m = L_{\text{int}} = 0$), in which Eq. (3.26) is valid and hence the metric has the form (3.27). The free-field equations

$$\nabla^\alpha \nabla_\alpha \psi = 0, \; \nabla_\alpha F^{\alpha\mu} = 0 \tag{3.29}$$

in coordinates (3.23) under conditions (3.11) give

$$\psi = C\xi, \; F_{01} = qe^{2\gamma}, \tag{3.30}$$

where C is the scalar charge and q is the electric charge. The function e^{γ} is easily found from the $\left({1 \atop 1}\right)$ Einstein equation. FInally, mapping (3.17) leads to the following solution in STT:

$$ds^2 = F(\hat{\psi})\left\{e^{2\gamma}dx^{0^2} - \frac{e^{-2\gamma}}{s^2(k,\xi)}\left[\frac{d\xi^2}{s^2(k,\xi)} + d\Omega^2\right]\right\}, \tag{3.31}$$

$$\hat{\psi} = \kappa^{1/2}\psi = \kappa^{1/2}C\xi;\ F_{01} = qe^{2\gamma};\ |E| = |q|/r^2 \tag{3.32}$$

where for $q \neq 0$

$$e^{2\gamma} = \frac{s^2(h,\xi_1)}{s^2(h,\xi+\xi_1)};\quad s^2(h,\xi_1) = \frac{c^{\gamma}}{Gq^2};\ h^2\text{sign } h = k^2\text{sign } k - \frac{1}{2}n\kappa C^2 \tag{3.33}$$

and for $q = 0$

$$e^{2\gamma} = e^{-2h\xi},\quad h^2 = k^2\text{sign } k - \frac{1}{2}n\kappa C^2 \tag{3.34}$$

(remember that $r = (-g_{22})^{1/2} = F^{1/2}e^{-\gamma}/s(k,\xi)$ is the spherical radius).

The mass m is defined by the formula

$$2Gm/c^2 = -\kappa^{1/2}C\left.\frac{dF}{d\hat{\psi}}\right|_{\hat{\psi}=0} - 2\gamma'\big|_{\xi=0}. \tag{3.35}$$

Solution (3.31) − (3.34) satisfies boundary conditions (3.11) and contains three independent integration constants: q, C, and h (or m). (With no less of generality we we suppose for the function F, which determines the choice of STT, that $F(0) = 1$; this can always be reached by multiplying the Lagrangian (3.16) by a constant.) The constant ξ_1 is is determined up to changes preserving the value of $s^2(h,\xi_1)$. The ξ coordinate is difined in the region $0 < \xi < \xi_{\max}$, where $\xi_{\max}$ is determined from the behavior of the functions F, γ, and s in (3.31) and can be either finite or infinite.

Now let us discuss some particular cases.

3.3.2 In GR with a minimal ψ field, (3.31) − (3.34) with $F \equiv 1$ is valid. Due to $dF/d\hat{\psi}(0) = 0$, we get the relation

$$G(Gm^2 - q^2) = h^2\text{sign } h = k^2\text{sign } k - \frac{1}{2}n\kappa C^2, \tag{3.36}$$

and the condition $m > 0$ imposes the limitations

$$\begin{array}{ll} & \tan(-h\xi_1) > 0 \quad \text{for } h < 0; \\ \xi_1 > 0 \quad \text{for } q \neq 0,\ h \geqslant 0; & h > 0 \ \text{ for } q = 0. \end{array} \tag{3.37}$$

Under hte physically plausible choice $n = +1$ the solution has a naked singularity at the center ($\xi \to \xi_{\max}$, $r \to 0$) for all the values of integration constants if just $C \neq 0$. If $h > 0$, then $\xi_{\max} = \infty$ and the singularity is attractive ($g_{00} \to 0$). One may call it "a scalar-dominated singularity" because $\psi \to \infty$ and the total scalar field energy $E_s = \infty$, whereas the electromagnetic field energy $E_e < \infty$. This case is referred to as that of small electric charge ($q^2 \leqslant Gm^2$). The corresponding electrovacuum Reissner-Nordström solution has horizons; we see that they vanish due to the the presence of the scalar field. Note that the present solution in the case $q = 0$ is the well-known Fisher solution [21]; the scalar-electrovacuum solution has been found by Penney [22].

If $h < 0$ (big charge, $q^2 > Gm^2$), then $\xi_{\max}$ coincides with the closest zero of $s(h, \xi+\xi_1) = h^{-1}\sin h(\xi+\xi_1)$ and describes a repulsive "electric" singularity of Reissner-Nordström type: $g_{00} \to \infty$, $\psi \to \infty$; $E_s < \infty$, $E_e = \infty$.

Thus these solutions contain either horizons, or singularities, with diverging energy E_f. Moreover, these singularities cause a nonzero correction $\triangle m$ in expression (3.15): $\triangle m = \infty$ for electric-type singularities and $\triangle m = (k-h)c^2/G > 0$ for scalar-dominated ones. If we apply (3.15) to black holes ($C = 0$, $q^2 \leqslant Gm^2$) with R_s corresponding to the boundary of the static region, the horizon, we obtain $\triangle m = 0$. Thus, the canonical energy definition (3.2) implies that the entire energy of a black hole is distributed in the static region out of the horizon.

3.3.3 The solution with a conformal field φ in GR is described by Eqs. (3.31) – (3.34) under the substitution (3.21), $F(\hat\psi) = \cosh^2(\kappa^{1/2}C\xi/6^{1/2})$. As $dF/d\hat\psi(0) = 0$, conditions (3.36) and (3.37) are again vaild. However, the conformal factor F essentially alters the character of singularities of the solution.

If $h < 0$, then $\xi = \xi_{\max}$ ($\xi_{\max} = \pi/|h| - \xi_1$, $0 < \xi_1 < \pi/2|h|$ with no less of generality) describes a naked repulsive singularity ($r = 0$, $g_{00} = \infty$) analogous to the Reissner-Nordström singularity ($q^2 > Gm^2$). However, in this case both the electric (E_e) and scalar ($E_s^{\rm conf}$) energy diverge. $E_s^{\rm conf}$ is determined by integration of the $\binom{0}{0}$ component of the tensor [23]

$$T_{s\ \mu}^{\nu\ \rm conf} = \varphi_{,\mu}\varphi_{,}{}^{\nu} - \frac{1}{2}\delta_\mu{}^\nu \varphi_{,\alpha}\varphi_{,}{}^{\alpha} - \frac{1}{6}(G_\mu{}^\nu + \nabla^\nu\nabla_\mu - \delta_\mu{}^\nu\nabla^\alpha\nabla_\alpha)\varphi^2. \tag{3.38}$$

Thus even for big charges the singularity is determined by both fields.

The case $0 \leqslant h < h_0 \stackrel{\rm def}{=} |C|(\frac{1}{6}\kappa)^{1/2}$ differs from the one considered in that $\xi_{\max} = \infty$ and $E_e < \infty$. Again there is a repulsive singularity at the center ($\xi \to \infty$, $r \to 0$, $g_{00} \to \infty$) but now it is a scalar-dominated singularity: $E_s^{\rm conf} = \infty$.

In the case $h > h_0$, or $Gm^2 - q^2 > 4\pi C^2/3c^4$, again $\xi_{\max} = \infty$. The quantify r is nowhere zero, thus the space has no center. At $\xi \to \infty$ we have $r \to \infty$, $g_{00} \to 0$, i.e., a naked attractive singularity of a type called "a space pocket" (Jordan). The infinity $\xi = 0$ and the singularity are separated by a throat, a sphere where the function $r(\xi)$ has its minimum. The energy E_e is finite but $E_s^{\rm conf} = -\infty$. This anomaly is accounted for by the absence of energy positive-definiteness in the tensor $T_{s\ \mu}^{\nu\ \rm conf}$.

Finally, in the case $h = h_0$ ($Gm^2 = q^2 + 4\pi C^2/3c^4$), the sphere $\xi = \infty$ is regular (though $\psi = \infty$, the quantities φ, r, and g_{00} are finite) and it is convenient to carry out a transition beyond this sphere in the curvature coordinates ($x^1 = r$). Thus we obtain [5, 6] (see also [24])

$$ds^2 = (1 - Gm/c^2r)^2 dx^{0^2} - (1 - Gm/c^2r)^{-2}dr^2 - r^2d\Omega^2, \tag{3.39}$$

$$\varphi = C(r - Gm/c^2)^{-1}, \quad F^{10} = F_{01} = q/r^2. \tag{3.40}$$

This metric coincides with the special Reissner-Nordström metric for $Gm^2 = q^2$ and has a horizon at $r = r_g = Gm/c^2$. This solution represents a black hole with scalar charge, see [5, 6, 7, 25, 26]. Though $\varphi = \infty$ at $r = r_g$, the corresponding energy-momentum tensor is nonsingular and the singularity of metric occurs due to the frame of reference chosen. In the region $r > r_g$ noth field energies E_e and $E_s^{\rm conf}$ are finite.

The inertial mass for the solutions with a conformal scalar field has to be calculated using Eq. (3.2) and the mectic $G_{\mu\nu}$ corresponding to the quasi-Einsteinian Lagrangian (3.18) [14], i.e., by Eq. (3.31) with $F \equiv 1$. Consequently, the energy is the same as that for the system with a minimal ψ field, except for the black-hole case ((3.39) and (3.40)). In this case the region $r < r_g + h$ is obtained by means of a transition beyond the sphere which corresponds to the center in the $F \equiv 1$ solution; here in order order to calculate the contribution of this region, an additional study is necessary. One can show, however, that solution (3.39), (3.40) is unstable under monopole perturbations (Bronnikov and Kireyev [34]), thus we do not consider it any more.

We will not describe the properties of the Brans-Dicke theory electrovacuum solutions; we just note that, for $\omega > -3/2$, the solution contains a naked singularity for all possible choices of integration constants (excluding, certainly, $C = 0$). The inertial mass calculation is reduced to that for (3.31) with $F \equiv 1$, as before.

Thus the solutions with linear fields satisfy neither the *strong criterion* for particlelike solutions, nor the weak one, with the exception of Reissner-Nordström black holes for which the static region formally satisfies the *weak criterion*.

3.4 Particle Models with a Bare Mass

3.4.1 Apparently the simplest way of obtaining singularity-free particle models is to consider extended sources of the vacuum fields described in Section 3.3, in the form of electrically and scalarly charged matter distributions. Solutions to field equations for such distributions in the generalized STT (3.16) can be obtained by reducing the problem to the corresponding problem of GR by transformation (3.17). Assume that in a space-time $\tilde{V}_4$ with the metric $\tilde{g}_{\mu\nu}$ a matter distribution is given with zero pressure and the mass, electric and scalar charge densities $\tilde{\rho}_m$, $\tilde{\rho}_e$, and $\tilde{\rho}_s$, respectively. The field equations for φ and $F_{\mu\nu}$ are

$$2B(\varphi)\tilde{\nabla}^\alpha\tilde{\nabla}_\alpha\varphi - \left(\frac{dA}{d\varphi}\right)R + \left(\frac{dB}{d\varphi}\right)\tilde{g}^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} = -8\pi\rho\varphi, \tag{3.41}$$

$$\tilde{\nabla}^\alpha F_{\alpha\mu} = -4\pi\tilde{\rho}_e\tilde{g}_{\mu\alpha}\tilde{u}^\alpha, \tag{3.42}$$

where u^μ is the matter four-velocity. Then after mapping (3.17) we obtain a matter distribution in the space-time V_4 with the metric $g_{\mu\nu}$, having the densities and the four-velocity

$$\begin{aligned} &\rho_m = F^2(\hat{\psi})\tilde{\rho}_m;\ \rho_e = F^{3/2}\tilde{\rho}_e;\ u^\mu = F^{1/2}\tilde{u}^\mu; \\ &\rho_s = \frac{nG}{c^2}\ F\ \frac{dF}{d\hat{\psi}}\ \tilde{\rho}_m + F^2\frac{d\varphi}{d\hat{\psi}}\ \rho_\varphi \end{aligned} \tag{3.43}$$

and the field equations for ψ and $F_{\mu\nu}$

$$\nabla^\alpha\nabla_\alpha\psi = -4\pi\rho_s;\ \nabla_\alpha F^{\alpha\mu} = -4\pi\rho_e u^\mu. \tag{3.44}$$

The metric $g_{\mu\nu}$ obeys the Einstein equations (3.7) with $T^{\nu}_{\text{int}\ \mu} = 0$. As usual, the dynamic variables are chosen so that L_{int}, creating the right-hand sides of Eqs. (4.4), gives no contribution to $T_\mu{}^\nu$, i.e., $L_{\text{int}}g^{1/2}$ dose not depend on $g_{\mu\nu}$. Thus in the static spherically symmetric case, the right-hand side of (3.7) is the sum of the tensors (3.8) − (3.10) satisfying condition (3.26). Consequently, the metric $g_{\mu\nu}$ has the form (3.27) and $\tilde{g}_{\mu\nu}$ the form (3.31). Form Eqs. (4.4) and (3.7) it follows that

$$\hat{\psi}'(\xi) = 4\pi\int d\xi{\cdot}\rho_s e^{-2\gamma}/s^4(k,\xi); \tag{3.45}$$

$$e^{-\gamma}F_{01} = Q(\xi) = 4\pi\int d\xi{\cdot}\rho_e e^{-3\gamma}/s^4(k,\xi);\ |E| = |Q|/r^2; \tag{3.46}$$

$$4\pi G\rho_m/c^2 = s^4(k,\xi)e^{2\gamma}(\gamma'' - GQ^2e^{2\gamma}/c^4); \tag{3.47}$$

$$GQ^2e^{2\gamma}/c^4 = \gamma'^2 + \frac{1}{2}n\psi' - k^2\text{sign}\ k. \tag{3.48}$$

The quantity $Q(\xi)$ is the integral electric inside a sphere $\xi = \text{const}$.

Equations (3.31) and (4.5) – (4.8) completely determine the general static spherically symmetric solution for scalarly and electrically charged dust in the class of STT (3.16). The solution contains two arbitrary functions $\gamma(\xi)$ and $\psi(\xi)$ and one arbitrary constant k.

The scalar-electrovacuum solution of Section 3.3 is obtained by putting all the densities equal to zero. On the other hand, if $\hat{\psi} \equiv 0$, we arrive at charged dust distributions in GR and the external Reissner-Nordström field. Moreover, all the results of Ref. [16] are easily reproduced. If we "switch" ψ and $F_{\mu\nu}$ simultaneously, we automatically get $\rho_m \equiv 0$ (static neutral dust distributions in GR are impossible) and the field is reduced to the Schwarzschild one.

3.4.2 To construct an extended isolated source if the fields considered, let us identify some sphere $\xi = \xi_0$ in the vacuum and matter solutions (out: $\xi < \xi_0$; in: $\xi > \xi_0$) and impose certain matching conditions at $\xi = \xi_0$. In our case they are reduced to the requirements that $g_{\mu\nu}$, $g'_{\mu\nu}$, and the normal components of the electric field intensity and the scalar field gradient are continuous at $\xi = \xi_0$. This implies that the constants and the arbitrary functions boundary values should obey the conditions

$$k_{\rm in} = k_{\rm out};\ Q(\xi_0) = q;\quad [\gamma] = [\gamma'] = [\psi] = [\psi'] = 0, \tag{3.49}$$

where, for any $f(\xi)$, $[f] = f_{\rm out}(\xi_0) - f_{\rm in}(\xi_0)$. In GR with a minimal ψ field, the requirement $[\psi] = 0$ does not occur because in the external solution only the derivative ψ' is essential.

Conditions for regular center existence [requirement (iii) for particlelike solutions] (3.12) for all three theories (3.20) – (3.22) are fulfilled if and only if [11, 12]

$$\xi_c??b = \infty,\quad k = 0;\qquad \gamma' = o(\xi^{-2}),\quad \hat{\psi}' = o(\xi^{-2})\qquad \text{for}\ \ \xi \to \infty. \tag{3.50}$$

For theory (3.20) these conditions are obvious. For theories (3.21) and (3.22) we omit the proof of (4.10) because it is rather cumbersome.

3.4.3 Consider particlelike models with a regular center in GR, both with a minimal (3.20) and conformal (3.21) scalar field. In both cases Eq. (3.36) for the external solutions is valid and the regularity condition k = 0 implies that

(A) *Regular models having no external scalar field* ($C = 0$) *are possible only if* $Gm^2 = q^2$ (a well-known result in GR; we extend it for configurations having an internal scalar field).

(B) *Electrically neutral or weakly charged* ($q^2 < Gm^2$) *regular models are possible only with a repulsive scalar field* ($n = -1$)*, strongly charged ones* ($q^2 > Gm^2$) *with an attractive one* ($n = +1$).

In the latter case, if $\rho_m \geq 0$, in regular models $\gamma_{\rm in} \leq 0$ and matching with $\gamma'_{\rm out}$ is possible only if $\xi_0 < \pi/2|h| - \xi_1$. This implies:

(C) *In regular models of GR with* $n = +1$ *and nonnegative matter density, the source radius value is bounded from below:*

$$r_0 > d = G^{1/2}|q|(1-\delta^2)^{1/2}/c^2 \arcsin\delta,\quad \delta^{\rm def} = G^{1/2}m/|q|. \tag{3.51}$$

If δ is not too close to unity, d is of the order of the classical radius q^2/mc^2. For real particle parameters $\delta \ll 1$ (e.g., for an electron $\delta \sim 10^{-21}$) and (4.11) yields

$$r_0 > q^2/mc^2 = r_{\rm cl}. \tag{3.52}$$

It is worth emphasizing that limitations (4.11) and (4.12) are independent on the choice of the arbitrary functions $\gamma(\xi)$ and $\hat{\psi}(\xi)$.

A concrete example of a singularity-free particle model can be obtained if we adopt γ and $\hat{\psi}$ in the form [11]

$$\gamma_{\rm in}(\xi) = \gamma_0 + a/\xi^2; \qquad \hat{\psi}_{\rm in} = \hat{\psi}_0 + b/\xi^2, \tag{3.53}$$

where $\gamma_0, \psi_0, a,$ and b are constants which can be expressed in terms of $m, q,$ and ξ using the matching conditions. Assuming. $\delta \ll 1$, the gravitational field is weak, $\gamma(\xi) = O(\delta^2)$ and the difference between the models of theories (3.20) and (3.21) is of the same order. One can show that

$$\begin{gathered} 4\pi c^2 (r_0^4/q^2)\rho_m(r) = (r_0/r_{\rm cl} - 1) - r^2/r_0^2, \\ \rho_e = 3q/4\pi r_0^3, \quad \rho_s = 3C/4\pi r_0^3. \end{gathered} \tag{3.54}$$

The density ρ_m is a decreasing function of r and $\rho_m \geq 0$ implies

$$r_0 > r_{\rm min} = 4/3 r_{\rm cl}; \quad \rho_m < \rho_{\rm max} = (3/4)^4 m^4 c^6/4\pi q^6. \tag{3.55}$$

(The central density has its maximum if $r_0 = r_{\rm min}$; this limiting case is rejected because at $r = r_0 = r_{\rm min}$ the ratios ρ_e/ρ_m and ρ_s/ρ_m are infinite.) Numerical estimates give

$$\begin{aligned} \text{for electrons} \ &: \ r_{\rm min} \approx 4 \times 10^{-23}{\rm cm}, \ \ \rho_{\rm max} \approx 0.9 \times 10^9 {\rm g/cm^{-3}}; \\ \text{for protons} \ &: \ r_{\rm min} \approx 2 \times 10^{-16}{\rm cm}, \ \ \rho_{\rm max} \approx 10^{22} {\rm g/cm^{-3}}. \end{aligned}$$

These models with a regular center satisfy the strong criterion for particlelike solutions; in particular, $m = m_I$.

3.4.4 In STT with $F \not\equiv 1$ the effective scalar charge density ρ_s, is nonzero even if $\rho_\varphi \equiv 0$ (see (4.9)), i.e., when there is *no charge due to direct scalar field-matter interaction* ; such an assumption is quite usually adopted in STT. Equation (4.3) in this case gives a relation between the functions $\hat{\psi}(\xi)$ and $\gamma(\xi)$

$$nF\hat{\psi}'' \left(\frac{dF}{d\hat{\psi}}\right)^{-1} + \frac{1}{2} n \hat{\psi}'^2 = \gamma'' - \gamma'^2 + k^2 {\rm sign} k \tag{3.56}$$

so that solution (**??**) and (4.5)-(4.8) contains only one arbitrary function. Let us enumerate some limitations arising in theories (3.21) and (3.22) due to such a reduction of arbitrariness.

For *uncharged matter*, $Q = \rho_e \equiv 0$ and $\rho_\varphi \equiv 0$:

(D) *In GR with a conformal scalar field there are no dust distributions with nonnegative matter density.*

(E) *In the Brans-Dicke STT no static dust distributions exist, except for* $\omega = -2$. In theory (3.22),$\omega = -2$, the function $\hat{\psi}(\xi)$ is arbitrary; if $\omega \neq -2$, automatically ???0. For all the other $F(\hat{\psi})$ there are no arbitrary functions and $\gamma(\xi)$ and $\hat{\psi}(\xi)$ are found in quadratures:

$$e^{2\gamma} = \frac{\rm const}{F}; \quad \left(\frac{d\xi}{d\psi}\right)^2 = \frac{1}{4k^2 {\rm sign}\ k} \left[\left(\frac{d\ln F}{d\hat{\psi}}\right)^2 + 2n\right]. \tag{3.57}$$

The sign of ρ_m coincides with the sign of the expression

$$-\left(2 + n\frac{d\ln F}{d\hat{\psi}}\right) \frac{d^2 \ln F}{d\hat{\psi}^2}. \tag{3.58}$$

particular, for theory (3.21), $\rho_m < 0$. *For charged matter*, with $\rho_\varphi \equiv 0$:

(F) *In GR with a conformal scalar field there are no models having a regular center and* $\rho_m \geq 0$.

(G) In the Brans-Dicke STT with $\omega > -3/2$ models with a regular center and $\rho_m \geq 0$ can exist only if

$$1 \leq Gm^2/q^2 < (2\omega + 4)/(2\omega + 3) \qquad (3.59)$$

(for convenience we use the initial constant G which differs in STT ¿from the gravitational constant G_N measured far from massive bodies in Cavendish-type experiments; in the Brans-Dicke theory, $G_N/G = (4 + 2\omega)/(3 + 2\omega)$). In the limit $C \to 0$ or $\omega \to \infty$ we get the regularity condition for charged dust balls in GR:$Gm^2 = q^2$. As modern observations yield the estimate $\omega \gtrsim 30$, we see that ratio (4.19) can vary only from 1 to about 1.016. Thus, using the example of charged dust balls as a basis, we can conclude that *the Brans-Dicke theory faintly differs from GR as far as the existence of equilibrium configurations is concerned.*

Condition (4.19) forbids the existence of regular models with real particle parameters. Such models are certainly possible if one rejects the limitation $\rho_\varphi \equiv 0$.

3.4.5 Besides models with a regular center, metric (3.31) admits one more type of singularity-free models, those having no center, but with a second spatial infinity, like the ones considered by Wheeler [27] and Ellis [28]. Metric (3.31) behaves just in this way for: $k < 0$ and regular $F(\hat{\psi})$ and $\gamma(\xi) : r(\xi) = 0$ both at $\xi = 0$ and $\xi = \pi/|k|$. Such models of extended bodies (we will call them *Wheeler models*) should be constructed from three solutions: out: $0 < \xi \leq \xi_0$; in: $\xi_0 \leq \xi \leq \bar{\xi}_0$; $\overline{\text{out}} : \bar{\xi} \leq \xi < \pi/|k|$. The constant k is the same for all the three solutions though C, q, and m can differ in the out and $\overline{\text{out}}$ solutions, meaning different mass and charge values for observers separated by a throat. However, it may be stated that [12]

(H) *Wheeler models with a regular function $F(\hat{\psi})$ and nonnegative matter density are possible only in theories with $n = -1$* (i.e., with a repulsive scalar field).

In particular, there are no Wheeler models for charged dust balls in GR (having no scalar field or an attractive one).

However, if we admit $\rho_m < 0$ (like Ref. [16]), it becomes possible to create a regular Wheeler model for a pair of charged particles with $E_{\text{tot}} > 0$.Indeed, (3.36) is compatible with the conditions $k < 0$ and $Gm^2 < q^2$, for instance, if $C = 0$.

Let us give an example of such a model in GR *without scalarfield.* Thus the internal solution possesses only one arbitrary function $\gamma_{\text{in}}(\xi)$. Further, let the matter be neutral. Hence, the only solution for γ_{in} with $\rho_m \not\equiv 0$ is $\gamma_{\text{in}} = \text{const}$. Moreover, $Q^2 = \text{const}\ q^2$, i.e., charges in both external Reissner-Nordström solutions are equal in magnitude and opposite in sign. It can be shown that the asymptotic values of the function $g_{00} = e^{2\gamma(\xi)}$ at the two spatial infinities $\xi = 0$ and $\xi = \pi/|k|$ are equal if and only if the masses m calculated at these asymptotics *are equal.* Such a model is constructed ??? the following way:

$$\begin{aligned} \text{out} &: \quad 0 < \xi < \xi_0 = (\arcsin \delta)/|k|; \\ \text{in} &: \quad \xi_0 \leq \xi \leq \bar{\xi}_0 = \pi/|k| - \xi_0; \quad e^{2\gamma_{\text{in}}} = 1 - \delta^2; \\ \overline{\text{out}} &: \quad \xi_0 \leq \xi < \pi/|k|, \end{aligned}$$

where $\delta = G^1 j^2 m/|q| < 1$. The solutions are matched at two spheres of equal radii at which the function $e^{2\gamma(\xi)}$ of the external solutions has its minimum. The electric field exists without any sources and its lines of force go ¿from one spatial infinity to the other. Thus Wheeler's idea of "charge without charge" is realized. The model describes a particle-antiparticle pair. Due to the constancy of $g_{00} = e^{2\gamma}$ in the internal region, there is no force acting upon the matter elements. The function $e^{\gamma(\xi)}$, as a whole, is symmetric with respect to the value $\xi = \pi/2|k|$, as well as $\sin k\xi$. Thus the regions "out" and "$\overline{\text{out}}$" are identical; in particular, conditions (3.11) are satisfied at both infinities.

The spherical radius r and the density ρ_m in the internal region change according to

$$r_{\text{in}} = r_{\min}/\sin |k|\xi, \qquad \rho_m = -\rho_{\max} \sin^4 k\xi, \qquad (3.60)$$

where $r_{\min}$ and $\rho_{\max}$ refer to the throat and depend only on q:

$$r_{\min} = G^{1/2}|q|/c^2, \qquad \rho_{\max} = c^6/4\pi G^2 q^2, \tag{3.61}$$

they have approximately Planck values if q is the electron charge:

$$r_{\min} \approx 1.4 \times 10^{-34}\text{cm}, \qquad \rho_{\max} \approx 5.7 \times 10^{94}\text{g cm}^{-3}.$$

The parameters of the matter-vacuum boundaries (radius r_0 and density $\rho_{m(0)}$, equal for both boundaries) are uniquely determined by the mass and the charge and do not contain the constant G:

$$r_0 = r_{\min}/\delta = q^2/mc^2 = r_{\text{cl}}; \qquad -\rho_{m(0)} = \rho_{\max}\delta^4 = m^4c^6/4\pi q^6. \tag{3.62}$$

the electron mass and charge ($\delta \approx 0.5 \times 10^{-21}$)

$$r_0 = 2.8 \times 10^{-13}\text{cm}, \qquad |\rho_{m(0)}| \approx 3.3 \times 10^9\text{g cm}^{-3}.$$

The total nongravitational energy consists of four parts: the electric field energy in ??th external regions ($E_{\overline{e(\text{out})}} = E_{e(\text{out})} = \frac{1}{2}mc^2$) and the internal region ($E_{e(\text{in})}$), as the negative matter energy E_m:

$$E_m = -2E_{e(\text{in})} = -\left(\frac{1}{2}|q|c^2G^{-1/2}\right)(1-\delta^2)^{1/2}(\pi - \arcsin\delta). \tag{3.63}$$

small δ the negative quantity $c^{-2}(E_m + E_{e(\text{in})})$ has the order of Planck's mass $|q|G^{-1/2}$ (for $q =$(electron charge) and the energy $E_e + E_m$ is also negative. At the same time, the total energy with gravitation taken into account is

$$E_{\text{tot}} = m_I c^2 = 2mc^2, \tag{3.64}$$

because the integration region in (3.2) is bounded by two identical Reissner-Nordström asymptotics. The quantity m_I is the sum of particle and antiparticle inertial masses, so that there is no contradiction to the equivalence principle. The purely gravitational contribution to E_{tot} is positive for any δ (because $E_e + E_m < mc^2$), but for small δ it should be of the order $|q|G^{-1/2}$ in order to compensate $E_e + E_m < 0$. Thus in this model, although m can be small (say, the electron mass), the gravitational field plays a decisive role in the energy balance. This model for a particle-antiparticle system satisfies the *strong criterion* for particlelike solutions.

It should be noted that for the above discussion it is unessential whether the two flat-space asymptotics belong to the same physical space or different ones.

3.5 Nonlinear Electrodynamics in General Relativity Exact Solutions [35]

Now we take in consideration the systems with nonlinear and interacting fields in GR. First, we shall discuss the case of nonlinear gauge-invariant electrodynamics. The total Lagrangian is

$$L = R/2\kappa + \Phi(I), \qquad I = -F_{\alpha\beta}F^{\alpha\beta}, \tag{3.65}$$

where $\Phi(I)$ is an arbitrary smooth function with Maxwell asymptotic:

$$\Phi(I) \sim \frac{I}{16\pi}, \qquad \frac{d\Phi}{dI} \to \frac{1}{16\pi} \qquad \text{for} \quad I \to 0.$$

A static spherically symmetric solution for this system is [29, 30]

$$ds^2 = f(r)dx^{0 2} - dr^2/f(r) - r^2 d\Omega^2; \tag{3.66}$$

$$f(r) = 1 - \frac{\kappa}{r}\int T_0{}^0 r^2 dr,\ T_0{}^0 = 2I\frac{d\Phi}{dI} - \Phi, \tag{3.67}$$

$$\left(\frac{d\Phi}{dI}\right) F^{01} = \frac{q^2}{r^2}, \qquad q = \text{const(charge)}. \tag{3.68}$$

A special case of (3.65), the Born-lnfeld electrodynamics,

$$\Phi(I) = [1 - (1 - \sigma I)^{1/2}]/8\pi\sigma \tag{3.69}$$

($\sigma =$const is the nonlinearity parameter), has been treated in [31]. Solution (3.66)-(3.68) satisfies condition (i) for particlelike solutions, for the $r \to \infty$ asymptotic is that of Reissner-Nordström. However [32], *requirement* (iii) interpreted as that *of regular center existence, cannot be fulfilled, whatever the function* $\Phi(I)$ *is.*

Indeed, Eq. (3.68) implies

$$I\left(\frac{d\Phi}{dI}\right)^2 = \frac{2q^2}{r^4}. \tag{3.70}$$

Let us transform the expression for energy density $T_0{}^0$. Equation (3.70) gives

$$\Phi(I) = 8^{1/2}|q| \int_0^{1^{1/2}} d(I^{1/2})/r^2, \tag{3.71}$$

where the lower limit corresponds to $r = \infty$. Thus, integrating by parts, we obtain for $T_0{}^0$:

$$T_0{}^0 = 4(2)^{1/2}|q| \int_r^{\infty} r^{-3} I^{1/2} dr. \tag{3.72}$$

If there is a regular center, then from conditions (3.12) for metric (3.66) it follows, in particular, that $f(0) = 1$. Thus in Eq.(3.67) integration should be carried out from 0 to r, so that

$$r^{-1}\int_0^r T_0{}^0 r^2 dr \to 0 \quad \text{for } r \to 0. \tag{3.73}$$

This is possible only if $T_0{}^0 r^2 \to 0$ for $r \to 0$ and by (3.72) we come to the requirement that $I \to 0$ for $r \to 0$. This contradicts (3.70).

Thus the strong criteria (i)-(iii) cannot be fulfilled. Occurence of a singularity of the metric at $r = 0$ is related to an infinite field energy density.

Let us consider conditions under which requirements (iii.a) and (iii.b) of the weak criterion are satisfied. Requirement (iii.a) means that the integral in Eq.(3.67) taken from 0 to ∞, converges, whereas (iii.b) takes the form

$$r f^{1/2}(f^{1/2} - 1) \to 0 \qquad r \to 0. \tag{3.74}$$

By Eq.(3.67), this requirement is satisfied if the same integral converges at $r = 0$. Thus, fulfilment of (iii.a) implies fulfilment of (iii.b).

The integral E_f converges if at $r \to 0$ either $I < \infty$ or $I \to \infty$ but $Ir^2 \to 0$. In the first case from (3.68) and the expression for $T_0{}^0$ it follows that for $r \to 0$

$$\frac{d\Phi}{dI} \to \infty; \quad \Phi(I) r^2 \to 0. \tag{3.75}$$

In the second case we see that $d\Phi/dI$ should increase faster than $I^{1/2}$ when $I \to \infty$, i.e., $\Phi(I)$ increases faster than $I^{3/2}$.

In both cases the necessity of $d\Phi/dI \to \infty$ emphasizes the essentially nonlinear character of the electromagnetic field.

For the Born-Infeld case (3.69) the solution is

$$F^{01} = \bar{q}(r^4 + 2\sigma\bar{q}^2)^{-1/2}, \qquad \bar{q} = 16\pi q;$$
$$I = 2\bar{q}^2(r^4 + 2\sigma\bar{q}^2)^{-1}; \qquad T_0{}^0 = [(r^4 + 2\sigma\bar{q}^2)^{1/2} - r^2]/8\pi\sigma r^2. \qquad (3.76)$$

It exists for all $r \in (0, \infty)$ when the nonlinearity parameter $\sigma > 0$. The weak criterion for particlelike solutions is satisfied. The form of the solution illustrates the general conclusions made above.

3.6 Systems with a Direct Interaction of Scalar and Electromagnetic Fields [35]

Now let us consider some examples of systems including direct scalar-electromagnetic interaction in GR, satisfying the weak (Sections 3.6.1 and 3.6.2) and strong (Section 3.6.3) criteria for particlelike solutions.

3.6.1 Let in the Lagrangian (3.4)

$$L_{\rm int} = \sigma F^{\alpha\beta} A_\alpha \Psi_{,\beta}/(4\pi)^{1/2}, \qquad (3.77)$$

where A_μ is the electromagnetic four-potential ($F_{\mu\nu} = 2A_{[\mu,\nu]}$). The set of field equations includes

$$\nabla_\alpha(g^{\alpha\beta}\psi_{,\beta} + \sigma F^{\alpha\beta}A_\beta/2\pi^1 j^2) = 0, \qquad (3.78)$$
$$(4\pi)^{-1/2}\nabla_\alpha F^{\alpha\mu} - \sigma\nabla_\alpha(A^\alpha g^{\mu\beta}\psi_{,\beta} - A^\mu g^{\alpha\beta}\psi_{,\beta}) + 6F^{\mu\alpha}\psi_{,\alpha} = 0 \qquad (3.79)$$

and the Einstein equations (3.7) with $T^\mu_{m\mu} = 0$ and

$$\underset{\rm int}{T_\mu{}^\nu} = (4\pi)^{-1/2}\sigma[2F^{\nu\alpha}(A_\mu\psi_{,\alpha} - A_\alpha\psi_{,\mu}) - \delta_\mu{}^\nu F^{\alpha\beta}A_\alpha\psi_{,\beta}]. \qquad (3.80)$$

In the static spherically symmetric case $T^\nu_{{\rm int}\ \mu}$, satisfies (3.26); thus, for the metric in coordinates (3.23) we obtain expression (3.27). Equation (3.78) with (3.27) is easily integrable:

$$\psi' = (4\pi)^{-1/2}\sigma e^{-2\gamma}A_0A_0' + C, \qquad C = \text{const}. \qquad (3.81)$$

The remaining equations can be written in the from

$$(e^{-2\gamma}A_0')' + \sigma(4\pi)^{1/2}A_0(e^{-2\gamma}\psi')' = 0, \qquad (3.82)$$
$$\gamma'^2 = k^2 \text{sign}\, k - \frac{1}{2}\kappa C^2 + (\kappa/8\pi)e^{-2\gamma}{A_0'}^2(1 + \sigma^2 e^{-2\gamma}{A_0}^2). \qquad (3.83)$$

The constant C has the meaning of a scalar charge due to a source which is external in respect to the system (e.g., localized at a singularity). If $C = 0$, then the scalar field is created fully by the electromagnetic field interacting with it. In the following we confine ourselves to the case $C = k = 0$ (the condition $k = 0$ not only simplifies the equations, but also is necessary, though not sufficient, for a regular center existence, see (3.28)).

For Eqs. (3.82) and (3.83) the conditions at the infinity are

$$\gamma \to 0, \quad A^2 \stackrel{\rm def}{=} A_0A^0 \to 0, \qquad E^2 = \xi^4 {A_0'}^2 \to 0, \qquad (3.84)$$

where $A = e^{-\gamma}A_0$ is the electrostatic potential.

Under these conditions the solution to Eqs. (3.82) and (3.83) can be written in the parametric form [9]

$$\begin{aligned} A^2(x) &= \bar\kappa/(x^2-x_0^2), \quad \bar\kappa \stackrel{\text{def}}{=} \kappa/8\pi; \quad x_0 \stackrel{\text{def}}{=} (\bar\kappa\sigma^2)^{1/2};\\ e^{2\gamma(x)} &= (x^2-x_0^2)(x-x_+)^{-1+\eta}(x-x_-)^{-1-\eta},\\ \xi(x) &= \left[\frac{xx_0}{x^2-x_0^2}+\coth^{-1}\frac{x}{x_0}\right]\frac{1}{2|q\sigma|} \end{aligned} \tag{3.85}$$

with

$$2x_\pm = -\bar\kappa \pm [\bar\kappa(\bar\kappa+4\sigma^2)]^{1/2}, \quad \eta = (1+4\sigma^2/\bar\kappa)^{-1/2}.$$

For the scalar field ψ and the effective electric charge $Q(x)$ within a sphere $x = \text{const}$, we get

$$\psi'(x) = \sigma Q(x) A_0 (4\pi)^{-1/2} = \frac{\sigma\kappa q}{4(2)^{1/2}\pi}\frac{x^2-x_0^2}{x(x-x_-)(x-x_+)}, \tag{3.86}$$

$$Q(x) = qx^{-1}(x^2-x_0^2)(x-x_+)^{-(1+\eta)/2}(x-x_-)^{-(1-\eta)/2}. \tag{3.87}$$

The constant $q = Q(\infty)$ is the total electric charge which determines the electric field at the asymptotic.

The variable x is defined for $x_0 < x < \infty$; moreover, $x_- < 0 < x_+ < x_0$. When $x \to x_0$, the quantities $g_{00} = e^{2\gamma}$ and $-g_{22} = r^2$ tend to zero, thus the metric has a singularity at the center. For all $x > x_0$ the solution is regular.

The Schwarzschild mass m in this solution is unambiguously connected with the charge q: $m = G^{1/2}|q|$, being of the Planck order if q is the electron charge.

The material fields energy E_f (3.13) has the finite value

$$E_f = \frac{|q|}{2\bar\kappa^{1/2}}\left[\frac{\bar\kappa+6\sigma^2+x_0}{\bar\kappa+4\sigma^2} + \frac{|\sigma|x_0}{(\kappa+4\sigma^2)^{3/2}}\log\left(\frac{x_0-x_+}{x_0-x_-}\right)\right]. \tag{3.88}$$

Further, with (3.15) one can verify that $m = m_I$. Thus, in spite of a singular center, the solution is particlelike according to the *weak criterion.*

When the interaction is switched off $(\sigma \to 0)$, the scalar field vanishes and the solution tends to the special Reissner-Nordström field with one horizon. The region $x > x_0$ turns into the external region $r > r_{\text{horizon}} = Gm/c^2$. Accordingly, E_f tends to the energy of the linear field $F_{\mu\nu}$, out of the horizon, $\frac{1}{2}mc^2$. However, for arbitrarily small but finite σ, the quantity (3.88), being close to $\frac{1}{2}mc$, describes the field energy in the whole space.

In the other limiting case $\kappa \to 0$ (vanishing gravity), by (3.88), $E_f \to \infty$. The similar problem solved in flat space-time, also gives $E_f = \infty$. Thus in this example of the interaction, the gravitational field plays a regularizing role, which enables us to obtain a particlelike solution.

3.6.2 Consider another example of direct interaction in (3.4):

$$L_{\text{int}} = [1-\Psi(\psi)]F_{\alpha\beta}F^{\alpha\beta}/16\pi, \quad \Psi(0) = 1, \tag{3.89}$$

where $\Psi(\psi)$ is unspecified so far. Unlike (3.77), this interaction preserves the gradient invariance of the $F_{\mu\nu}$ field. That such a kind of interaction is possible is indicated, e.g.,

$$\nabla_\alpha\nabla^\alpha\psi + (16\pi)^{-1}F_{\alpha\beta}F^{\alpha\beta}\left(\frac{d\Psi}{d\psi}\right) = 0, \tag{3.90}$$

$$\nabla_\alpha(\Psi F^{\alpha\beta}) = 0 \tag{3.91}$$

The total energy-momentum tensor

$$T_\mu{}^\nu = -\frac{\Psi}{4\pi}[F_{\mu\alpha}F^{\nu\alpha} - \frac{1}{4}\delta_\mu{}^\nu F_{\alpha\beta}F^{\alpha\beta}] + \psi_{,\mu}\psi^{,\nu} - \frac{1}{2}\delta_\mu{}^\nu\psi_{,\alpha}\psi^{,\alpha} \tag{3.92}$$

with (3.5) and (3.6) satisfies (3.26), so that the metric has the final form (3.27); Eq.(3.91) gives

$$F^{01} = q\Psi^{-1}e^{-2\alpha} \tag{3.93}$$

(q =const is the electric charge). The functions $\psi(\xi)$ and $\gamma(\xi)$ are determined by the equations

$$\psi'' - q^2 e^{2\gamma}\Psi^{-2}\left(\frac{d\Psi}{d\psi}\right)(8\pi)^{-1} = 0, \tag{3.94}$$

$$\gamma'^2 - k^2 \text{sign } k = \kappa q^2 e^{2\gamma}\Psi^{-1}/8\pi - \frac{1}{2}\kappa\psi'^2, \tag{3.95}$$

which can be solved only if an explicit form of $\Psi(\psi)$ is chosen. If we take for instance [35]

$$\Psi(\psi) = e^{\sigma\psi}, \quad \sigma = \text{const.}, \tag{3.96}$$

we obtain the solution satisfying conditions (3.11) at the infinity:

$$\begin{aligned} e^{2\gamma} &= \left[\frac{8\pi e^{\sigma C\xi}}{\kappa q^2\Sigma s^2(h,\xi+\xi_1)}\right]^{1/\Sigma}, \quad \Sigma \stackrel{\text{def}}{=} 1+\sigma/2\kappa; \\ \psi(\xi) &= \Sigma^{-1}\left\{C\xi + \left(\frac{\sigma}{2\kappa}\right)\ln\left[\frac{\kappa q^2\Sigma s^2(h,\xi+\xi_1)}{8\pi}\right]\right\}, \end{aligned} \tag{3.97}$$

where the metric and the function s are defined by (3.27) and the constants C (scalar charge), q, h, k, and ξ_1 are related by

$$s^2(h,\xi_1) = 8\pi/\kappa q^2\Sigma; \quad \Sigma k^2 \text{sign } k = h^2 \text{sign } h + \frac{1}{2}\kappa C^2.$$

Three integration constants are independent: the charges C and q and also h which can be related to the mass.

The solution is singular at the center for all values of integration constants but $\kappa C = -\sigma h$, $h > 0$, $\xi_1 > 0$. In the latter case there is a Schwarzschild-like event horizon at $\xi = \infty$: $e^\beta \to \text{const} > 0, e^\gamma \to 0$. Nevertheless, there exists such a choice of the constants that the total material energy E_f is finite: $C = h = 0$, $\xi_1 > 0$. In this case

$$ds^2 = \left(\frac{\xi_1}{\xi+\xi_1}\right)^{2/\Sigma} dx^{0 2} - \frac{1}{\xi^2}\left(\frac{\xi+\xi_1}{\xi_1}\right)^{2/\Sigma}\left[\frac{d\xi^2}{\xi^2} + d\Omega^2\right], \tag{3.98}$$

$$E_f = |q|c^2(G\Sigma)^{-1/2}(\kappa+\sigma^2)/(2\kappa+\sigma^2). \tag{3.99}$$

At the center ($\xi \to \infty$, $r = 0$) the electric field is finite ($|E| = c^8/\Sigma^2G^2q^2$) but the metric and the scalar field are singular ($e^\gamma \to 0$, $\psi \to \infty$). Note that, like Section 3.6.1, the ψ field exists along with zero scalar charge due to the interaction. It can be easily shown that the inertial and gravitational masses coincide: $m = m_I = |q|(G\Sigma)^{-1/2}$.

Thus it is a particlelike solution by the *weak criterion*.

For m and q of the order of particle parameters, $\Sigma = q^2/Gm^2 \gg 1$ (remember that, e.g., for an electron $\Sigma^{-1/2} \sim 10^{-21}$). Hence, the fields localization domain and the length at which the interaction is essential are both of the order of $1/\xi_1 = r_{\text{cl}} = q^2/mc^2$. Indeed. if $E_f(r)$ and $Q(r)$ are, respectively, the field energy and the effective charge within a sphere ξ =const (i.e., r =const), then $E_f(r) = E_f(1+r_{\text{cl}}/r)^{-1}$ and $Q(r) = q(1+r_{\text{cl}}/r)^{-2}$ (Q is defined by $|E| = |Q|/r^2 = |q|\Psi^{-1}/r^2$). The space-time curvature is essential only at extremely small lengths of the order of $r_{\text{cl}}\exp(-q^2/Gm^2)$, so that the space-time is practically flat almost everywhere. Accordingly, for $\Sigma \gg 1$ the mass of the system is almost entirely of nongravitational origin ($E_f \approx mc^2$).

Unlike the solution of Section 3.6.1, in this case the energy E_f remains finite in both limiting cases $\sigma \to 0$ and $\kappa \to 0$. The limiting form of the solution for $\sigma \to 0$ again coincides with the Reissner-Nordström solution with one horizon and $E_f \to \frac{1}{2}mc^2$. The transition $\kappa \to 0$ yields a field configuration in flat space-time, with the finite energy $E_f = 4\pi|q/\sigma|$ which diverges for $\sigma \to 0$. A regularizing role of gravitation is contained in the fact that expression (3.99) for E_f is limited for arbitrarily small σ by the value $\frac{1}{2}|q|c^2G^{-1/2}$.

3.7 Nonsingular Field Model of a Particle

Now we shall give an example of a particlelike solution by the *strong criterion* [10]. Let us take the interaction Lagrangian in the form (3.89). It can be shown that the center regularity condition (3.12) can be satisfied only if $\Psi(\psi)$ is infinite for some finite ψ. We can choose for instance (note that it should be $\Psi(0) = 1$)

$$\Psi(\psi) = [\sin^2 B\psi_0 / \sin^2 B(\psi - \psi_0)]; \quad B, \psi_0 = \text{const.} \tag{3.100}$$

In this case we obtain a special solution to Eqs. (3.94) (3.95) satisfying both boundary conditions (3.11) and (3.12) [10]:

$$e^{-\gamma} = \frac{1 - \delta^2 e^{-b\xi}}{1 - \delta^2}; \quad \psi = \psi_0 - B \arcsin(\delta e^{-b\xi/2}) \tag{3.101}$$

Note that $\delta < 1$ and ψ can change from 0 to ψ_0.

In case $\delta \ll 1$ the space-time is flat up to δ^2 (see (3.101)) and the coordinate ξ equals $1/r$ with the same accuracy. In this approximation the effective matter density $\rho_m = T_0{}^0/c^2$ and the effective charge Q inside a sphere $r =$const (defined as before) are

$$\rho_m = (q^2 e^{-r_{\rm cl}/r})/4\pi c^2 r^4; \quad Q = q e^{-r_{\rm cl}/r}. \tag{3.102}$$

The ratio of the effective charge and matter densities is constant and equals q/m. Both densities vanish at the center and take maximum values at $r = \frac{1}{4} r_{\rm cl}$. This reproduces the basic features of proton charge and mass distributions obtained experimentally [33].

As the solution is regular, the inertial and gravitational masses manifestly coincide and equal m.

Like the other particlelike solutions of this paper, this solution exists only for $m < |q|G^{-1/2}$, i.e., $m < 1.8 \times 10^{-6}$ g (of the order of the Planck mass), if q is the electron charge.

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Chapter 4

Quantum Cosmology

4.1 Problems of Quantum Cosmology [1]

Quantum cosmology, i.e. canonical quantization of cosmological models, started to attract attention after the famous paper by B. DeWitt [2], where the canonical general scheme quantization, and of a simple closed isotropic Friedmann universe with a matter in the form of noninteracting particles (dust) as well, was done.

Here, we describe basic trends in these studies, some problems and results, concerning scalar fields and the role of the cosmological constant.

Quantum cosmology occupies an intermediate place between different approaches to unification of gravitational and quantum physics, namely between a quantization of different physical fields on a given classical gravitational background and a quantization of the gravitational field. ¿From the first approach it differs by the fact that quantization is made of not only other than gravitational fields, but also of simple variants of the gravitational (cosmological) field and, what is important, in a self-consistent manner. In comparison with the full quantization of the gravitational field it is a more simple problem as only a finite number of gravitational field degrees of freedom are quantized and, in essence, the quantum field theory is replaced by a quantum mechanical one.

But, from this fact comes an advantage to solve this problem both at classical and quantum levels. Of course, one should take the results of quantum cosmology to a certain extent cautiously, as they are not results of a full theory as in a classical cosmology. Really, in those degrees of freedom of the gravitational field, which are suppressed, there had to be vacuum fluctuations, in other words, the Heisenberg uncertainty relations are violated for coordinates and momenta of these degrees of freedom. But, probably, some main features of these cosmological models quantization will remain in the full quantum theory as well and we will be able to find the way to unification of macro and micro worlds.

There are physical, methodological and technical problems in quantum cosmology. Singular or nonsingular initial states, creation of the universe, initial perturbations and so on are physical problems of quantum cosmology. Other problem of a physical character related to the quantum cosmology is a problem of a final state of the gravitational collapse of a massive body.

As methodological and technical problems, we may mention such as ordering problem of noncommuting operators in a Hamiltonian of cosmological models; comparison of different methods of quantization, related to use of zero and monzero Hamiltonians (Dirac and ADM approaches); reduction of the Heisenberg uncertainty relation in the quantum cosmology; physical interpretation of the wave function of the universe, describing quantum states of cosmological models; sensible criterion of collapse in quantum cosmology; (in)dependance of results of quantum cosmology on a choice of canonical variables, being excluded from constrants, on initial coordinate conditions etc.

4.2 Comparison of Dirac and ADM Methods for Quantization of Geometrical Optics Equation

Let us dwell upon different approaches to quantum cosmology where quantum mechanics of a dynamical system with constraints is used. Dirac [3] and Armowitt, Deser, Misner (ADM) worked out two methods of canonical quantization of such dynamical systems.

Dirac's method based on treatment of constraints leads to Klein-Gordon-Fock equation for the wave function Ψ describing the quantum state of a system.

ADM method starts (before quantization) with elimination of extra dynamical variables by choosing coordinate conditions and solving the constraint equations.Then, the Schrödinger type of equation with the Hamiltonian like a square root of a differential operator is obtained for the wave function Ψ.

Let us compare Dirac's and ADM method on an example of a quantization of the system of rays in an optically nonhomogeneous media [4]. This system has something in common with a system of geodesics in GR. For these rays the quantization by Dirac and ADM methods may be done to the end. The problems of operators ordering, physical interpretation of the wave function, definition of Hamiltonian as a square root of the differential operator are solved also.

Geometric optics equations follow from the Fermat variational principle with the action

$$S = \int n(x_1, x_2, x_3)\sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}dt. \tag{4.1}$$

Here $x_i (i = 1, 2, 3)$ are Cartesian space coordinates, $n(x_1, x_2, x_3)$ is a refraction index of a media, t is a parameter along a ray trajectory, point means the differentiation with respect to time t.

Lagrangian $L(x, \dot{x})$ corresponding to the action integral (4.1) is a homogeneous first order function of generalized coordinates $\dot{x}_i$. For such a Lagrangian the usual Hamiltonian $H = p_i\dot{x}_i - L$ is equal to zero and momenta variables $p_i = \partial L/\partial\dot{x}_i$ satisfy the primary constraint equation

$$\varphi(x, p) \equiv p^2 - n^2 = 0. \tag{4.2}$$

Other primary or secondary constraints are absent for the system of rays.The constraint equation (4.2) and the equation of motion for some function of dynamical variables x_i, p_i with the total Lagrangian $H_T - u\varphi$, where u is an arbitrary function of the parameter t give the description of the system of rays by the Dirac method in a classical case.

Due to the ADM method the action integral may be representation in a canonical form with the solving of constraint equation (4.2) and using the coordinate condition $t = x_1$. We have two values for the action:

$$S^{\pm} = \int (p_2x_2' - p_3x_3' - H_1^{\pm})dx_1, \tag{4.3}$$

where prime means the derivative with respect to x_1. The values of the non null Hamiltonian are

$$H_1^{\pm} = \pm\sqrt{n^2 - p_2^2 - p_3^2}. \tag{4.4}$$

So, in the ADM method we come to the devision of a ray trajectory into parts with "positive" and "negative" Hamiltonians H_1^+ and H_1^-. The part of a trajectory with H_1^+ corresponds to negative values of $x_1 (p_1 < 0)$ and the part of trajectory with H_1^- — to positive $x_1 (p_1 > 0)$. Points of trajectory deviding parts corresponding to H_1^+ and H_2^- are peculiar points in the ADM method under the used coordinate condition and are called the turning points of a ray. The trajectory in the turning points are perpendicular to the x_1 axes $(p_1 = 0)$.

In a quantum case momenta variables p_i become operators $\hat{p}_i$ which in a coordinate representation take the form

$$\hat{p}_j = -(i/k_0)\partial/\partial x_j. \tag{4.5}$$

Here, instead of a usual Planck constant $\hbar$ there is an inverse value of some constant wave number k_0. If Ψ is a wave function of a ray quantum state, then the constraint equation in the Dirac quantum formalism is written as a wave Helmholtz equation of the type:

$$(\Delta + k_0^2 n^2)\psi = 0. \tag{4.6}$$

For the equation (4.6) one may set a boundary problem of scattering waves with the condition of radiation at infinity. In this approach the physical interpretation of a wave function is quite clear and is given by the density flow vector which satisfies the continuity equation.

Let us consider first in the Dirac method a case of a plane layers medium which refraction index depends only on one coordinate x_1, *i.e.* $n = n(x_1)$. Within this, in a Fourier representation

$$\psi(x_1, x_\perp) = \int \exp(ik_\perp x_\perp) d^2k_\perp \tilde{\psi}(x_1, k_\perp) \tag{4.7}$$

over transverse coordinates $x_\perp = (x_2, x_3)$ the Helmholtz equation (4.6) is reduced to a one dimensional wave equation

$$\partial^2\tilde{\psi}/\partial x_1^2 + \varkappa^2(x_1, k_\perp)\tilde{\psi} = 0. \tag{4.8}$$

Here $\varkappa(x_1, k_\perp)$ denotes a variable "longitudal" wave number:

$$\varkappa = \begin{cases} \sqrt{k_0^2 n^2 - k_\perp^2}, & k_\perp < k_0 n; \\ i\sqrt{k_\perp^2 - k_0^2 n^2}, & k_\perp > k_0 n. \end{cases} \tag{4.9}$$

It is real in a classically accessible region $k_\perp < k_0 n$ and pure imaginary in a classically forbidden region $k_\perp > k_0 n$.

For the ADM method for the quantization of rays one should start from (4.4). Denoting $p_1 = -H_1^\pm$ and changing momenta p_i to operators (4.5) we come to two independent equations:

$$(i/k_0)\partial\psi^\pm/\partial x_1 = \pm\sqrt{n^2 + \Delta_\perp/k_0^2}\ \psi^\pm. \tag{4.10}$$

Each of them may be considered as a Schrödinger equation with the Hamiltonian in the form of a square root of a differential operator with a nonhomogeneous term equal to refraction index squared and the role of time is played by the first space coordinate.

For the plane layers media the square root of a differential operator of the Schrödinger equation (4.10) may be found using the Fourier transformation (4.7):

$$\sqrt{k_0^2 n^2 + \Delta_\perp^2}\ \psi^\pm(x, x_\perp) = \int \exp(ik_\perp, x_\perp) d^2k \cdot \varkappa(x_1, k_\perp)\tilde{\psi}^\pm(x_1, k_\perp). \tag{4.11}$$

Here a longitudal variable part of the wave number $\varkappa(x_1, k_\perp)$ satisfies (4.9). Taking into account (4.11), Schrödinger equations (4.11) are solved exactly in the Fourier representation and their solutions are running waves

$$\tilde{\psi}^\pm(x_1, k_\perp) = C^\pm(k_\perp)\exp[\mp i \int \varkappa(x_1, k_\perp) dx_1] \tag{4.12}$$

with constant amplitudes $C^\pm(k_\perp)$.

Let us compare results of the ray system quantization according to Dirac and ADM methods. It is easy to see that solutions in the form of running waves (4.12) do not satisfy the wave function (4.8) in general. But they may be obtained as an approximate solution of this wave equation (4.8) using the WKB method. And only in the variant of this method where the effect of change of the wave intensity due to increasing or decreasing of rays space separation is not taken into account.

As it is seen in this case the quantization by the ADM method may be considered as a WKB approximation to the Dirac's quantization method.

4.3 Uncertainty Relation Within the Quantization of a Friedman Cosmological Model

Obtaining the Heisenberg uncertainty relation for canonically conjugate variables is interesting, first of all, for the study of a singular state in cosmological models. Let us get this relation for the Friedman cosmological model with dust [5].

Metric of the closed Friedman universe in a comoving frame of reference is:

$$ds^2 = c^2 d\tau^2 - a^2(\tau)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{4.13}$$

The scale factor $a(\tau)$ of this model satisfies the Einstein equation

$$(3\pi c^2/4G)(a\dot{a}^2 + c^2 a) = Mc^2, \tag{4.14}$$

where M is a total dust mass, point means a derivative with respect to the proper time τ.

B. De Witt suggested the following wave equation for the vector $\Phi = \Phi(x)$ of the quantum state of the model:

$$(-d^2/dx^2 + \pi x^{2/3})\Phi = (4\pi/3)(E/m_L c^2)\Phi. \tag{4.15}$$

Here $E = Mc^2$ is a total energy of dust, $x = (a/L)^{3/2}$.

The wave equation (4.15) is considered in the Hilbert space $L_2(0,\infty)$ at the semiaxes $x \geq 0$ with a norm

$$\int_0^\infty |\Phi(x)|^2 dx = 1$$

and the null boundary condition:

$$\Phi(0) = 0. \tag{4.16}$$

Under this condition the operator in the left side of equation (4.15) is a self-adjoint operator in the space $L_2(0,\infty)$ [6].

Let us denote $\hat{p} = -id/dx$ as a momentum operator canonically conjugate to x variable and connected with it by the standard commutation relations:

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i. \tag{4.17}$$

¿From this relation using known formal transformations (see, for example [7]) the Heisenberg uncertainty relation is obtained:

$$\langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle \geq 1/4. \tag{4.18}$$

We note here that operator $\hat{p}$ defined on functions with boundary conditions (4.16) is Hermitian (symmetric) but not self-adjoint [8]. But for obtaining the uncertainty relation (4.18) on functions $\varphi(x)$ with null boundary condition (4.16) it is enough to have $\hat{p}$ operator to be Hermitian. ¿From relation (4.18) one may easily get an inequality for the mean volume over quantum states of the closed Friedman universe as:

$$\langle a^3 \rangle \geq (3/16\pi) L^3 m_L / \langle M \rangle. \tag{4.19}$$

If we put in the r.h.s. of (4.19) $\langle M \rangle = m_L \approx 10^{-5} g$, then we obtain

$$\langle a^3 \rangle^{1/3} \geq 0,\ 4L,$$

where $L \sim 10^{-33} cm$ (Planck length). Only for $M \to \infty$ one gets
$\langle a^3 \rangle^{1/3} \to 0$.

4.4 Quantization of Friedman Universe Filled with a Scalar Field [17,1]

Let us address some questions related to physical problems of quantum cosmology. We dwell upon one of the approaches to the study of a singular state on an example of the Friedman closed model with matter in the form of a scalar field (conformal and with minimal coupling). This problem, which is of course of a general interest, was investigated first in [11] (conformal scalar field) and in [9] for the minimally coupled scalar field. The treatment of matter in the form of fields is more correct than the dust-like matter, as near the singularity, as it will be shown farther, the dust approximation is not valid. Here we give both classical and quantum treatment of a problem. For m = 0 it will be an exact one.

Let us start from the classical analysis of the system of scalar and gravitational fields in a closed isotropic model. We use only a free scalar field (without interaction) here. As a GR generalization of the scalar field in a curved space-time we adopt the Penrose-Chernicov-Tagirov equation (PCT)[11], which is conformally invariant for m = 0:

$$(\Box + m^2c^2/\hbar^2 + R/6)\ \psi = 0, \tag{4.20}$$

where $\Box = -\nabla^\alpha\nabla_\alpha$ is a covariant D'Alembert operator, R is a scalar curvature.

Let us consider the system with the Lagrangian

$$\mathcal{L} = -\frac{m^2c^3}{2\hbar}[\psi^*_\sigma\psi^\sigma + \left(\frac{m^2c^2}{\hbar^2} + \frac{R}{6}\right)\psi\psi^*] + \frac{c^2}{2\varkappa}R, \tag{4.21}$$

where ψ^* is a complex-conjugate field, $\varkappa = 8\pi G/c^2$ is the Einstein gravitational constant.

Here we use the metric tensor $g_{\mu\nu}$ with signature (-+++). Variation of the action

$$S = \frac{1}{c}\int\sqrt{-g}\ \mathcal{L}\ d^4x \tag{4.22}$$

with respect to ψ and $g^{\mu\nu}$ gives the scalar field (4.20) and the Einstein equations correspondingly

$$R^\nu_\mu - (R/2)\ \delta^\nu_\mu = -(\varkappa/c^2)\ T^\nu_\mu, \tag{4.23}$$

where

$$\begin{aligned} T^\nu_\mu \quad = \quad & \frac{m^2c^3}{2\hbar}\{-(\psi^*_\mu\psi^\nu + \psi^{*\nu}\psi_\mu) + \delta^\nu_\mu\ [\psi^*_\sigma\psi^\sigma + \left(\frac{m^2c^2}{\hbar^2} + \frac{R}{6}\right)\psi\psi^*] \\ & -\frac{1}{3}(R^\nu_\mu - \nabla_\mu\nabla^\nu + \delta^\nu_\mu\ \Box)\psi\psi^*\} \end{aligned} \tag{4.24}$$

is an energy-momentum tensor of matter (scalar field) with properties:

$$\nabla_\sigma T^\sigma_\mu = 0;\ T_{\mu\nu} = T_{\nu\mu};\ T \equiv T^\nu_\nu = (m^4c^5/\hbar^3)\ \psi\psi^*. \tag{4.25}$$

The metric for a closed Friedman model has the form:

$$ds^2 = -c^2dt^2 + a^2(t)d^3\Omega, \tag{4.26}$$

where $a(t)$ is the Universe radius (scale factor) and

$$d^3\Omega = d\chi^2 + \sin^2\chi\ (d\theta^2 + \sin^2\theta\ d\varphi^2) \tag{4.27}$$

is a unit 3-sphere.

If the scalar field wave function ψ depends only on time t, i.e. $\psi = \psi\ (t)$, then the scalar field energy-momentum tensor in a curved space-time is diagonal and has the following non null components:

$$\left.\begin{aligned} T_0^0 &= \frac{m^2c}{2\hbar}\ [\dot\psi\dot\psi^* + \frac{\dot a^2}{a^2}\psi\psi^* + \omega^2(1+\frac{\lambda^2}{a^2})\ \psi\psi^*]; \\ T_1^1 = T_2^2 = T_3^3 &= \frac{m^2c}{2\hbar}\ [-\frac{1}{3}\ (\dot\psi\dot\psi^* + \frac{\dot a^2}{a^2}\psi\psi^* \\ &+\frac{\omega^2}{3}\ (1-\frac{\lambda^2}{a^2})\psi\psi^*], \end{aligned}\right\} \tag{4.28}$$

where λ is the Compton wave length and $\omega = mc^2/\hbar$ is the Compton frequency.

Then, the self-consistent system of equations (4.20) and (4.23) for the scale factor $a(t)$ and scalar field ψ in the metric (4.26) takes the form:

$$-(2\ddot a a + \dot a^2 + c^2)/a^2c^2 = -(\varkappa/c^2)\ T_1^1; \tag{4.29}$$

$$-3\ (\dot a^2 + c^2)/a^2c^2 = -(\varkappa/c^2)\ T_0^0; \tag{4.30}$$

$$\ddot\psi\ + 3\ \frac{\dot a}{a}\dot\psi\ +\ (\omega^2 + \frac{a\ddot a + \dot a^2 + c^2}{a^2})\ \psi = 0. \tag{4.31}$$

This system may be solved exactly only for $m = 0$. Due to the complexity of the system for $m \neq 0$ we shall study two limiting cases which may be solved exactly.

Nonrelativistic approximation. Let us study the system (4.29)-(4.31) at the time range when the scale factor is a slowly varying function at the scale of a particle Compton wavelength:

$$|\dot a|/a \ll \omega\ (m\neq 0). \tag{4.32}$$

Then, we may transfer from the relativistic T-C-P equation to the nonrelativistic Schrödinger equation. Solution of (4.31) we search in the form:

$$\psi(t) = f(t)\ \exp(i\omega t), \tag{4.33}$$

where $f(t)$ is a slowly varying amplitude. Substitution of (4.33) into (4.31) leads us to the equation for $f(t)$:

$$\begin{aligned} &\ddot f + 2i\omega\dot f - \omega^2 f + 3\ (\dot a/a)\ (\dot f + i\omega f) + f\omega^2\ (1+\lambda^2/a^2) \\ &\qquad +f(\ddot a a + \dot a^2)/a^2 = 0, \end{aligned} \tag{4.34}$$

where the second and the fifth terms are the leading ones. So, in the lowest approximation neglecting higher order terms, we come to the integral of motion:

$$f^2a^3 \approx \text{const} = \lambda^3 N/2\pi^2, \tag{4.35}$$

which has a simple physical sense: the total number of scalar particles in the Universe volume $2\pi^2a^3$, where radius is measured in units of λ, is constant with time. Within this approximation the amplitude squared $f^2(t)$ is proportional to the particles number density. So, in this approximation from the quantum viewpoint there is now creation of particles in the Universe.

In order to find the properties of matter content of the Universe in this approximation let us put our approximate solution (4.35) into the scalar field stress-energy tensor (4.28). Then, we get:

$$T_1^1 = T_2^2 = T_3^3 \approx 0;\ T_0^0 \approx Nmc^2/2\pi^2a^3. \tag{4.36}$$

If we interprete space conpoments T_1^1, T_2^2, T_3^3 as a pressure with the negative sign, i.e. $T_1^1 = T_2^2 = T_3^3 = -p$ and T_0^0 as an energy density, i.e. $T_0^0 = \varepsilon$, then in this approximation the conformal scalar field is equivalent to a dust-like matter.

Let us find out the concrete conditions of the nonrelativistic approximation. We had to use for that the well known solution of Einstein-Hilbert equations for dust in a closed model [7]:

$$a(\eta) = a_0\ (1-\cos\eta); \tag{4.37}$$
$$ct = a_0\ (\eta - \sin\eta), \tag{4.38}$$

where $a_0 = 2GM/3\pi c^2$. Due to (4.37) the scale factor changes with time from zero to its maximal value which may be expressed via mass of a particle and Planck values of mass m_L and length L:

$$2a_0 = 2\ (2/3\pi)(Nm/m_L)\ L. \tag{4.39}$$

For the condition of nonrelativistic approximation (4.32) to be valid, it is necessary, as it follows from (4.37), that an inequality

$$2a_0c^2/a^3 \ll \omega^2 \tag{4.40}$$

holds. Using also (4.38) we come to the following restriction on the radius of the Universe:

$$a \gg \lambda, \tag{4.41}$$

and on the total number of particles in the Universe:

$$N \gg (m_L/m)^2. \tag{4.42}$$

It is very interesting that for the elementary particles characteristic masses $m \approx 10^{-24}g$ and length $\lambda \approx 10^{-13}cm$ we get for the macro parameters (scale factor and total number of particles) $a \gg 10^{-13}cm$ and $N \gg 10^{40}$ correspondingly.

¿From (4.41) it follows that in the nonrelativistic limit for a conformal scalar particle the radius of the Universe should be much larger than the Compton wavelength of particles. The same conclusion remains in this approximation for the self-consistent system of minimally coupled (without $R/6$ term) scalar field as this term is small compared to a particle mass for $a \gg \lambda$. Really, $m \approx \lambda^{-1}$, $R \approx a^{-2}$, so $m^2 \gg R$ when (4.41) holds. So, both approaches (with conformal and minimally coupled scalar fields) are identical in the nonrelativistic approximation.

Ultra relativistic approximation (or exact solution for $m = 0$). Let us now investigate equation (4.31) in the time interval t when the scalar factor $a \ll \lambda$ and is a rapidly changing function of t ($|\dot{a}|/a \gg \omega$) with respect to the Compton frequency of the particle. If the initial moment $t = 0$ corresponds to collapse, then this interval corresponds to small times.

Within this approximation equation(4.31) transforms to

$$\ddot{\psi} + 3(\dot{a}/a)\dot{\psi} + \psi(\ddot{a}a + \dot{a}^2)/a^2 = 0 \tag{4.43}$$

It's integral is:

$$a(t)\psi(t) = A\int d\tau/a(\tau) + B\ \ (A, B = \text{const}). \tag{4.44}$$

In the same approximation components of the energy-momentum tensor are

$$\varepsilon = T_0^0 \approx (m^2c/2\hbar)[\dot{\psi}\dot{\psi}^* + (\dot{a}^2/a^2)\psi\psi^*]; \tag{4.45}$$
$$T_1^1 = T_2^2 = T_3^3 \approx \frac{m^2c}{2\hbar}\left[-\frac{1}{3}\left(\dot{\psi}\dot{\psi}^* + \frac{\dot{a}^2}{a^2}\psi\psi^*\right)\right]. \tag{4.46}$$

Introducing once more pressure p and energy density ε, we come from (4.45)-(4.46) to the equation of state $p = \varepsilon/3$ which corresponds to the case of ultrarelativistic gas (radiation). Let us use the well-known solution of Einstein-Hilbert equations for this case [7]:

$$a(\eta) = a_1 \sin\eta; \quad ct = \int_0^\eta a(\eta)d\eta = a_1(1-\cos\eta). \tag{4.47}$$

Then, from (4.44) and (4.47) it follows that

$$\psi(t) = At/ca(t) + B/a(t). \tag{4.48}$$

The total energy of the scalar field $2\pi^2 a^3 T_0^0$ in the whole volume of the universe in this ultrarelativistic approximation due to (4.47),(4.48) and (4.45),(4.46) is:

$$2\pi^2 a^3 T_0^0 \approx \text{const}/a, \tag{4.49}$$

i.e. it is diminishing with a scale factor (time). It is connected with the fact that we neglected mass here, so, the dependence of $\varepsilon(a)$ should be as for the radiation filed, *i.e.* $\varepsilon \sim a^{-4}$ and it is just what we have in (4.49). For the case of the minimally coupled scalar field (without $R/6$ term) this dependence is different, namely: $2\pi^2 a^3 T_0^0 \approx \text{const}/a^3$. So, $R/6$ term is essential both for the solution and T_0^0 dependence (*i.e.* changes the equation of state) *. We stress here, that our approximate solution for the ultrarelativistic case is the exact solution for a massless conformal scalar field.

Einstein-Hilbert and PCT Equations in the Hamilton Form. Let us investigate the system of equations (4.29) -(4.31) in more detail. Equation (4.29) is of the second order and equation (4.30) is of the first order with respect to a scale factor time derivative.

It is easy to see that the first order Einstein equation (4.30) is the first integral of the system (4.29)-(4.31) which transforms exactly to (4.30) if the constant of integration is taken as zero. So, we may not take into account the Einstein equation of the first order which makes easy the transition to the canonical formalism.

The system of equations (4.28),(4.29) and (4.31) allows representations both in the form of Lagrange and Hamilton. Transition from the Lagrange formalism to the Hamilton one is done according to usual classical mechanics prescription. Taking the scale factor a as a generalized gravitational field coordinate and real and imaginary parts of the scalar field $q_A (A = 1, 2)$

$$q_1 = \text{Re}\psi; \quad q_2 = \text{Im}\psi$$

as generalized coordinates of a matter content, we obtain the following Lagrangian density:

$$\mathcal{L} = \frac{m^2 c}{2\hbar}\left[\sum_A \left(\dot{q}_A^2 - \omega^2 q_A^2 - \frac{\ddot{a}a + \dot{a}^2 + c^2}{a^2} q_A^2\right)\right] + \frac{3(\ddot{a}a + \dot{a}^2 + c^2)}{\varkappa a^2}. \tag{4.50}$$

Now, using a standard procedure we get the Hamiltonian in the form:

$$H \equiv \dot{a}p + \sum_A \dot{q}_A p_A - \mathcal{L} = -\frac{3\pi c^4}{4G}a + \sum_A \frac{M}{2\lambda}\left(\omega^2 + \frac{c^2}{a^2}\right) q_A^2 + \\ + \sum_A \frac{\lambda}{2M} p_A^2 - \frac{G}{3\pi c^2 a^3}\left(ap - \sum_A q_A p_A\right)^2, \tag{4.51}$$

where $M = 2\pi^2 a^3 m$ is a scalar field mass, p is a generalized gravitational field momentum and p_A are generalized momenta of the scalar field.

*For the minimally coupled scalar field the equation of state in this approximation is $\varepsilon = p$ (see farther).

Hamiltonian equations

$$\left.\begin{array}{rl} \dot{a} = \partial H/\partial p; & \dot{p} = -\partial H/\partial a; \\ \dot{q}_A = \partial H/\partial p_A; & \dot{p}_A = -\partial H/\partial q_A \end{array}\right\} \tag{4.52}$$

are equivalent to the system of equations (4.28),(4.29) and (4.31).

As this Hamiltonian doesn't depend explicitly on time, the Hamiltonian equations allow the existence of an energy integral:

$$H = H_{\rm g} + H_m = E = \text{const} \tag{4.53}$$

Taking into account that

$$H_m = 2\pi^2 a^3 T_0^0, \tag{4.54}$$

we see that the energy integral (4.53) coincides with (4.30) if one puts $E = 0$. So, the total Hamiltonian of gravitational and scalar fields is null:

$$H = 0, \tag{4.55}$$

as it is in a general case for the gravitational field and any matter content[2]. Equation (4.55) is called the equation of dynamical, or Hamiltonian constraint.

Quantum Treatment of scalar and gravitational fields system in a closed isotropic model [17]. Quantum Hamiltonian constraint. Let us reduce the Hamiltonian (4.51) to the canonical form using the following canonical transformations.

$$\left.\begin{array}{l} \Pi_A = q_A p_A; \quad \Pi = (a/L)Lp - q_1 p_1 - q_2 p_2; \\ Q_A = (a/L)q_A = x_A; \quad Q = (a/L)^{3/2} = x. \end{array}\right\} \tag{4.56}$$

Let coordinate operators Q, Q_A and their corresponding momenta operators Π, Π_A obey the standard commutation relations:

$$\left.\begin{array}{l} [Q,\Pi] = i\hbar; \quad [Q_A,\Pi_B] = i\hbar\delta_{AB}; \\ [Q,\Pi_A] = [Q_A,\Pi] = [Q,Q_A] = [\Pi,\Pi_A] = 0, \quad A,B = 1,2. \end{array}\right\} \tag{4.57}$$

In the representation when generalized coordinate operators Q, Q_A are diagonal, the momenta operators Π, Π_A have a differential operator form:

$$\Pi = -i\hbar\partial/\partial Q, \quad \Pi_A = -i\hbar\partial/\partial Q_A.$$

They act on a state function $\Phi = \Phi(t, Q, Q_A)$ of gravitational and scalar fields, which has the meaning of a quantum-mechanical probability amplitude.

Let us write down the Schrödinger equation for Φ:

$$i\hbar\partial\Phi/\partial t = H\Phi.$$

As the Hamiltonian doesn't depend on time, substituting

$$\Phi(t, Q, Q_A) = \exp[(iE/\hbar)t]\Phi(Q, Q_A),$$

we shall obtain the time-independent Schrödinger equation

$$H\Phi = E\Phi.$$

Then, as we already mentioned, we put $E=0$, as it follows from the Einsein equations, and come to the equation of the quantum Hamiltonian constraint

$$H\Phi = 0.$$

The noncomuting operators Π and Q, Π_A and Q_A we order in such a way, as it was suggested by DeWitt, that corresponding differential terms appear as one-dimensional Laplace-Beltrami operators. As a result, the quantum Hamiltonian constraint takes the form:

$$H(x,x_A)\Phi = m_L c^2 \left[-\frac{3\pi}{4}x^{2/3} + \frac{3}{4\pi}\Delta + \pi^2 v^2 x^{2/3}(v^2 + x^{4/3})(x_1^2 + x_2^2) - \right.$$
$$\left. -\frac{v^{-2}x^{-2/3}}{4\pi^2}(\Delta_1 + \Delta_2)\right]\Phi = 0, \quad v = m/m_L. \tag{4.58}$$

Let us rewrite it in such a manner:

$$D\Phi = (4\pi/3)\mathcal{H}_m\Phi, \tag{4.59}$$

where

$$D = -\Delta + \pi^2 x^{2/3},$$

Δ is the Laplace operator and

$$\mathcal{H}_m = \sum_{A=1,2}\left[-\frac{v^{-2}x^{-2/3}}{4\pi^2}\Delta_A + \pi^2 v^2 x^{2/3}(v^2 + x^{-4/3})\right].$$

Equation (4.59) is the second order hyperbolic equation in partial derivatives. We are interested in its solution which leads to the normalized state function of gravitational and scalar fields Φ:

$$\int_0^\infty dx \int\int_{-\infty}^\infty dx_1 dx_2 |\Phi(x,x_A)|^2 < \infty.$$

¿From this condition it follows that the state function $\Phi(x,x_A)$ is diminishing rather fast at infinity for $x\to\infty$ and also for $x_A\to\infty$.

Variables in the equation (4.59) are not separable. This makes difficult obtaining an exact solution, so, we restrict ourselves to approximate solutions. As in the classical case we investigate two limits: nonrelativistic and ultra relativistic ones. In this limits variables in (4.59) become separable.

Nonrelativistic approximation. In this limit x-variable takes large values (with respect to L due to (4.56)). Then, equation (4.59) may be solved by the method known in quantum mechanics as the method of rapid and slow subsystems. Here, the rapid subsystem is a scalar field and the slower one is the gravitational field.

Let us consider the scalar field Hamiltonian $\mathcal{H}_m$ at a fixed value of x-variable. Denote by $\Phi_N^{(m)}(x,x_A)$ and $\varepsilon_N^{(m)}$ its eigen function and eigen values correspondingly by:

$$\mathcal{H}_m\Phi_N^{(m)} = \varepsilon_N^{(m)}\Phi_N^{(m)},$$

where $\varepsilon_N^{(m)} = v(N_1+N_2+1); N=\{N_1,N_2\}$.

The solution of (4.59) we shall seek in the form:

$$\Phi(x,x_A) = \sum_N \Phi_N^{(g)}(x)\Phi_N^{(m)}(x,x_A). \tag{4.60}$$

Substituting (4.60) into (4.59), we obtain for the coefficients of (4.60) the following system of equations:

$$(D-\varepsilon_N^{(m)})\Phi_N^{(g)}=-\sum_{N'}(\Phi_N^{(m)}/[D,\Phi_{N'}^{(m)}]\Phi_{N'}^{(g)})_x, \tag{4.61}$$

where $(\Phi_1/\Phi_2)_x$ is a scalar product in the functions space of x_A; $[D,\Phi_{N'}^{(m)}]$ is a commutator.

R.h.s. of (4.61) are proportional to terms depending only on first and second derivatives of $\Phi_{N'}^{(m)}$ over x.These derivatives lead to terms proportional to x^{-1} and x^{-2}. If x takes rather large values (with respect to L) then r.h.s. of (4.61) may be taken as zero. After that we come already to independent equations

$$D\Phi_N^{(g)}=\varepsilon_N^{(m)}\Phi_N^{(g)}.$$

Such kind of equations were studied by De Witt in the case of a dust-like matter in a closed Friedman Universe. Formally, they describe a quantum mechanical motion in a potential well limited by an infinite wall at $x=0$ and the potential proportional to $x^{2/3}$ for $x>0$.

Quasiclassical eigen values in this case are:

$$\varepsilon_n^{(g)}=(4/3)\pi\sqrt{6(n+3/4)},\quad n=0,1,2,...$$

Eigen functions $\Phi_n^{(g)}$ are exponentially damped behind the turning point x_n, though they are non null at any a. The radius a_n corresponding to the turning point is $a_n/L=x^{2/3}=\varepsilon_n^{(g)}/\pi^2$. We point out that for a minimally coupled scalar field (without $R/6$) in a nonrelativistic quantum case we obtain the same results [9]. The condition of the nonrelativistic quantum approximation leads to:

$$N^2/x^2<<\varepsilon_N^{(m)};\quad L^3N^3<<\varepsilon_N^{(m)}a^3.$$

¿From these inequalities the same limits for a and the number of scalar particles follow as for the classical nonrelativistic case (4.41) and (4.42).But the only difference is that in principle the ψ-function is not null at any a, though extremely small after the turning point. In this sense the closed model conception in a quantum case is relative.

Ultrarelativistic approximation. In this limit $x\to 0$.Then eq.(4.61) has the form:

$$-x^{2/3}\frac{\partial^2\Phi}{\partial x^2}=\frac{4\pi}{3}\sum_A\left(-\frac{v^2}{4\pi^2}\frac{\partial^2}{\partial x_A^2}+\pi^2v^2x_A^2\right)\Phi. \tag{4.62}$$

and variables are separable:

$$d^2U_A/dx_A^2+(B_A-4\pi^4v^4x_A^2)U_A=0,\quad A=1,2; \tag{4.63}$$

$$d^2U/dx^2+(v^{-2}/3\pi)(B_1+B_2)x^{-2/3}U=0; \tag{4.64}$$

$$\Phi=U(x)\;U_1(x_1)\;U_2(x_2). \tag{4.65}$$

As one may see, (4.63) is a one dimensional Schrödinger equation for the oscillator. So, its solution is well known:

$$U_{A,n_A}(x_A)=\frac{\exp(-\pi^2v^2x_A^2)H_{n_A}(\sqrt{2}\pi v x_A)}{\sqrt{2^{n_A}n_A!\sqrt{\pi}/\sqrt{2}\pi v}};$$

$$B_{A,n_A}=4\pi^2v^2(n_A+1/2),$$

where H_{n_A} is the Hankel function. The solution of (4.64) is

$$U(x)=\sqrt{x}K_{i3/4}\left(\left[\frac{4\pi v^{-2}}{2}(n_1+n_2+1)\right]^{4/9}x^{2/3}\right),$$

where K is the MacDonald function. Let us analyse this solution. For $x \to 0$ we come to $K_{i3/4}(...) \sim x^{-i3/4}$, so, $|K_{i3/4}|^2 \approx 1$ and then

$$|\Phi(x, x_A)|^2 \sim x. \tag{4.66}$$

It means that the probability of the collapse of the Universe to a point (or, more strictly, of a state with a null value of a scale factor) is zero. We stress that this result has a quantum origin as $x = (a/L)^{3/2} \sim (a/\hbar^{1/2})^{3/2}$.

For $x \to 0$, that means $a \gg L$, we find that

$$|\Phi(x, x_A)|^2 \sim x^{1/3} \exp(-2\alpha x^{2/3}), \tag{4.67}$$

with $\alpha = \text{const}$. So, the square of the state function is exponentially small outside the region $a > L \sim 10^{-33} cm$ and the system is concentrated at the Planck scale within this approximation. Due to (4.66) and (4.67) the state function $\Phi(x, x_A)$ is normalizable with respect to x- variable.

Let us denote a scalar field energy as E_m. In this ultrarelativistic approximation one may define E_m at a fixed x as an eigen value of the Hamiltonian $\mathcal{H}_m$ through

$$\mathcal{H}_m \Phi = E_m \Phi.$$

It leads to

$$E_{n_1,n_2} = \frac{4\pi}{3} \frac{v^{-2}}{4\pi^2} \frac{n_1 + n_2 + 1}{x^{2/3}} m_L c^2.$$

As $x^{2/3} \sim a$ this formula coincides qualitatively with the classical result (4.49) in the same approximation.

Though in a classical context a and x are well determined values, then in a quantum case they are not. Mean value $\langle E_{m,n} \rangle_x$ of the scalar field energy over the probability of a variable x distribution for fixed values of x_A is:

$$\langle E_{m,n} \rangle = \int_0^\infty dx E_{m,n} |\Phi(x, x_A)|^2 = \text{const} \int_0^\infty |\Phi|^2 dx / x^{2/3}. \tag{4.68}$$

This integral converges for both $x \to \infty$ (see (4.67)) and for $x = 0$, so this energy is finite.

Ultrarelativistic quantum approximation for minimally coupled scalar field (without $R/6$ term)[9]. In this case quantum equation of the Hamiltonian constraint has the same form as (4.59) but D and $\mathcal{H}_m$ are a bit different:

$$D = -\partial^2/\partial x^2 + \pi^2 x^{2/3}; \tag{4.69}$$

$$\mathcal{H}_m \equiv \frac{H_m}{m_L c^2} = \sum_A \left(-\frac{1}{2\mu} \frac{\partial^2}{\partial x_A^2} + \frac{1}{2} \mu v^2 x_A^2 \right), \tag{4.70}$$

where μ and v are dimensionless effective mass and frequency correspondingly:

$$\mu = 2\pi^2 v^2 x^2; \quad v = m/m_L. \tag{4.71}$$

Potential energy $(1/2)\mu v^2 x_A^2$ of the scalar field effective oscillator $(A = 1, 2)$ in its Hamiltonian $\mathcal{H}_m$, proportional to a small quantity x^2, is a slowly varying monotonous function of x_A. Far from turning points the scalar field energy is not essential and we may neglect it. After that equation (4.59) takes the form:

$$x^2 \left(-\frac{\partial^2}{\partial x^2} + \pi^2 x^{2/3} \right) \Phi = -\frac{4\pi}{3} \sum_A \frac{1}{2} \frac{1}{2\pi^2 v^2} \frac{\partial^2}{\partial x_A^2} \Phi. \tag{4.72}$$

In equation (4.72) variables are separable. Its solution decreasing for $x \to \infty$ up to the normalizing constant is:

$$\Phi(x_1, x_2, x) = \sqrt{x} K_{i\tau}\left(\frac{3\pi}{4} x^{4/3}\right) \exp[\pm i(\alpha_1 x_1 + \alpha_2 x_2)], \tag{4.73}$$

where α_1, α_2 are integration constants, $K_{i\tau}$ is the MacDonald function of imaginary index $i\tau$:

$$\tau = \frac{3}{8}\sqrt{\frac{\alpha_1^2 + \alpha_2^2}{\alpha_0^2} - 1}, \quad \alpha_0 = \left(\frac{3\pi}{4}\right)^{1/2} \frac{m}{m_L}.$$

Let us analyse the physical meaning of equation (4.73). When $x \to 0$ the square $|\Phi(x, x_A)|^2$ of the state function tends to zero as

$$|\Phi(x, x_1, x_2)|^2 \sim \begin{cases} 1 - \sqrt{1 - \frac{\alpha_1^2+\alpha_2^2}{\alpha_0^2}}, & \alpha_1^2 + \alpha_2^2 < \alpha_0^2; \\ x, & \alpha_1^2 + \alpha_2^2 > \alpha_0^2. \end{cases} \tag{4.74}$$

As in the case of the conformal scalar field (with $R/6$) the probability of the collapsed state of the Universe is zero. This result also has a quantum origin as α_0^2 is inversely proportional to the Planck constant, $\alpha_0^2 \sim \hbar^{-1}$ and tends to infinity for $\hbar \to 0$.

For $x \to \infty$ the square of the state vector is exponentially decreasing

$$|\Phi(x, x_A)|^2 \sim x^{-1/3} \exp[-(3\pi/2)x^{4/3}].$$

So, this state function is normalizable with respect to x. What role do parameters α_1 and α_2 play? To find it let us define a scalar field energy E_m:

$$\mathcal{H}_m \Phi = E_m \Phi; \quad E_m = \frac{3}{16\pi} \frac{\alpha_1^2 + \alpha_2^2}{\alpha_0^2} \frac{1}{x^2} m_L c^2.$$

As $x^2 \sim a^3$, E_m is qualitatively similar to its classical analogue in the ultrarelativistic case. The mean value $\langle E_m \rangle_x$ of the scalar field energy over x distribution for fixed values of x_1 and x_2 is:

$$\langle E_m \rangle_{x_A} = \int_0^\infty E_m |\Phi(x, x_A)|^2 dx = \frac{3}{16\pi} \frac{\alpha_1^2 + \alpha_2^2}{\alpha_0^2} m_L c^2 \times \\ \times \int_0^\infty |\Phi(x, x_A)|^2 \frac{dx}{x^2}.$$

Taking into account (4.74) for $\alpha_1^2 + \alpha_2^2 > \alpha_0^2$ we get that this integral is logarithmically divergent at the lower limit $x = 0$.

So, we saw that solutions of the quantum equation (4.59) in nonrelativistic and relativistic limits are different. In the nonrelativistic limit the state function $\Phi(a, q_A)$ is essentially non zero in the region of a of the order of the Compton wavelength λ, $a \lesssim \lambda$. The total energy of the scalar field (both conformal and minimal) in the Universe is (m_L/m) times larger than the Planck energy $m_L c^2 : E_m \sim (m_L/m) m_L c^2$. In the ultrarelativistic limit the square of the state function $\Psi(a, q_A)$ is substantially nonzero only for a of the order of the Planck length L, $a \lesssim L$. The total energy of the scalar field E_m in this case is proportional to the Planck energy $m_L c^2$.

4.5 Quantization of a Friedman Universe matched to the Kruskal space-time[12]

Let us return to the metric of the closed Friedman universe filled with dust. It is a particular solution of Einstein equations found by Tolman (1934). The Tolman solution describes a gravitational field of a spherical dust distribution in a comoving reference frame. Let us consider such particular Tolman solution when dust fills homogeneously a part of a sphere $0 \le \chi \le \chi_0$ and use a spherical coordinate system. In this case the space-time metric inside the sphere coincides with the Friedman cosmological model and outside — with the Kruskal metric having the form:

$$ds^2 = c^2 d\tau^2 - \exp\lambda d\chi^2 - r^2 d^2\Omega. \tag{4.75}$$

Here $\lambda = \lambda(\chi, \tau)$ and $r = r(\chi, \tau)$ are functions of χ and τ. If we introduce new variables ρ and T according to

$$\left.\begin{array}{l} \rho = (\chi - \chi_1)/\sin^3\chi_0; \\ T = (\rho^2+1)^{-3/2}\tau/\sin^3\chi_0; \\ \chi_1 = \chi_0 - \sin^2\chi_0\cos\chi_0, \end{array}\right\} \tag{4.76}$$

then functions $\exp\lambda$ and r take the form:

$$\left.\begin{array}{r} \exp\lambda = 4(\rho^2+1)[a(T) - (3/2)T da(T)/dT]^2; \\ r = (\rho^2+1)a(T)\sin^3\chi_0. \end{array}\right\} \tag{4.77}$$

¿From matching conditions it follows that at the surface of a material ball variable T and function r take values τ and $r = r_0(\tau) = a(\tau)\sin\chi_0$ correspondingly.

Matched Friedman and Kruskal metrics describe the phenomenon of a gravitational closure of a collapsing material ball, filled with a dust-like matter. It is interesting to transform these metrics from the comoving frame with coordinates $\chi, \theta, \varphi, \tau$ to the spherical ones with coordinates r, θ, φ, t. After this transformation we see that time of the infinitely remote observer for the point at the surface of a material ball when its radius $r_0(\tau)$ tends to the gravitational one

$$r_g = a_m \sin^3\chi_0,$$

asymptotically tends to

$$t \sim (r_g/c)\ln[(r_{0m} - r_g)/(r_0 - r_g)].$$

Here $a_m = 4GM/3\pi c^2$ is the maximal value of the scale factor $a(\tau)$; $r_{0m} = a_m \sin\chi_0$ is a maximal value of the radius $r_0(\tau)$ of the material ball.

Brightness I of the radiation from a material ball surface, which is experiencing a gravitational closure, tends asymptotically [14] to $[(r_0 - r_g)/(r_{0m} - r_g)]^4$.

Relations (4.77) show that metrical coefficients of matched Friedman (4.13) and Kruskal (4.75) metrics are expressed via a scale factor $a(\tau)$ of the Friedman cosmological model satisfying equation (4.14). So, it seems that quantization of these matched metrics may be reduced † to the quantization of the Friedman model, done by De Witt[2].

This idea was used in [12] for quasiclassical quantization of matched Friedman and Kruskal metrics.

Instead of stationary equation (4.15) for the quasiclassical quantization of Friedman cosmological model nonstationary Schrödinger equation is used:

$$i\hbar\frac{\partial\Psi(q,\tau)}{\partial\tau} = -\frac{\hbar^2}{2m_L}\frac{\partial^2\Psi(q,\tau)}{\partial q^2} + V(q)\Psi(q,\tau). \tag{4.78}$$

†It is not evident that it is so, as it is not clear how formulas (4.77) would be in a quantum case.

Besides, here the coordinate is $q = L(a/L)^{3/2}$ and the potential is

$$V(q) = (9/8)m_L c^2 (q/L)^{2/3}.$$

The Schrödinger equation (4.78) describes a motion of a material point with mass m_L in a potential well field $V(q)$ with the Hamiltonian

$$H = p^2/2m_L + V(q). \tag{4.79}$$

For this Schrödinger equation (4.78) the problem is set with a given wave function $\Psi(q,0) = \Psi_0(q)$ at initial moment $\tau = 0$. In a quasiclassical approximation mean quantum mechanical value $\langle\hat{\mathcal{L}}(\hat{p},\hat{q})\rangle$ of some operator function $\hat{\mathcal{L}}(\hat{p},\hat{q})$ of coordinate $\hat{q}$ and momentum $\hat{p} = -i\hbar\partial/\partial q$ operators is calculated using the approximate formula [8]:

$$\langle\hat{\mathcal{L}}(\hat{q},\hat{p})\rangle \approx \int \mathcal{L}\{q(q_0,p_0,\tau); p(q_0,p_0,\tau)\} f_0(q_0,p_0)dq_0dp_0. \tag{4.80}$$

Here $q = q(q_0,p_0,\tau)$, $p = p(q_0,p_0,\tau)$ is a classical trajectory of a dynamical system in a phase space q,p, starting at an initial moment $\tau = 0$ from the point q_0,p_0; $f_0(q_0,p_0)$ is an initial value of a density matrix of this dynamical system.

Let us take an initial wave function $\Psi_0(q)$ in the form of a Gaussian wave packet:

$$\Psi_0(q) = (1/\sqrt{\sqrt{\pi}\sigma})\exp[-(q-\langle q\rangle)^2/2\sigma^2], \tag{4.81}$$

corresponding to the state of maximal expansion of a material ball under classical treatment of its evolution, when

$$\chi_0 = \pi/2 - \varepsilon, \quad \varepsilon \ll 1. \tag{4.82}$$

In this case the classical value of the scale factor $a(\tau)$ only slightly differs from its maximal value a_m up to the moment of a gravitational closure of the ball.

We may define a dispersion δ of the wave packet (4.81) from the condition

$$\langle K\rangle = \hbar\omega, \tag{4.83}$$

where K is a kinetic energy of a material point with the Hamiltonian H equal to (4.79); ω is its frequency with energy $\langle H\rangle$ in a potential well limited by a curve $V(q)$ and infinitely high wall at $q = 0$. From (4.83) it follows that

$$\sigma \sim \left(\frac{\langle a_m\rangle}{L}\right)^{1/2} L.$$

Using a density matrix $f_0(q_0,p_0)$ corresponding to a wave packet (4.81) and (formula (4.80)), we get the estimates:

$$\left.\begin{aligned} \langle t\rangle &\sim \left(\frac{\langle r_g\rangle}{c}\right)\ln\left(\frac{\varepsilon\langle a_m\rangle}{L}\right); \\ \langle I\rangle &\sim \left(\frac{\varepsilon\langle a_m\rangle}{L}\right)^{-4}. \end{aligned}\right\} \tag{4.84}$$

¿From them one may see that, due to the quasiclassical quantization of matched Friedman and Kruskal metrics, the mean time of a gravitational closure of a material ball by an infinitely remoted observer clocks and the mean observed by him brightness of radiation from the ball surface are estimated by finite values contrary to the classical result.

One should make some comment at this point. It is related to a validity of the quasiclassical approximation to the Schrödinger equation (4.78) in the vicinity of a turning point of the classical trajectory (its reflection from potential $V(q)$). From this condition we obtain the following restriction on a small parameter in equality (4.82):

$$\varepsilon^2 \left(\frac{\langle a_m \rangle}{L} \right)^2 \gg 1. \tag{4.85}$$

A comparison of estimates (4.84) and (4.85) shows that mean time of closure $\langle t \rangle$ and mean brightness $\langle I \rangle$ tend to logarithmical infinity and zero correspondingly for $\varepsilon \to 0$.

4.6 On Cosmological Constant in Quantum Cosmology [13]

Let us describe the quantization of a dust-like closed isotropic cosmological model with a cosmological constant by the method similar to [2]. The system of units is chosen so that $c = \hbar = 16\pi G = 1$. The corresponding Lagrangian is :

$$L = L_g + L_m = 12\pi^2 \alpha R - 12\pi^2 \alpha^{-1} R \, (R_{,0})^2 - 4\pi^2 \alpha \Lambda R^3 + L_m,$$

where R is a radius (scalar factor) of the Universe, Λ is a cosmological constant and $-\alpha^2 = g_{00}$. L_m is a matter Lagrangian (of N noninteracting particles with mass m). Transforming according to standard rules to the Hamiltonian and using the Hamiltonian constraint [2] we come to the condition:

$$\mathcal{H}_g + \mathcal{H}_m = 0, \tag{4.86}$$

where $\mathcal{H}_g = -\Pi^2/48R - 12\pi^2 R + 4\pi^2 \Lambda R^3$; $\mathcal{H}_m = Nm$; Π is a momentum conjugate to R. In a quantum case (4.86) becomes the condition on a state vector Ψ.

$$(\mathcal{H}_g + \mathcal{H}_m) \; \Psi = 0.$$

All the information about our system is contained in this quantum constraint. Using operator ordering such that the differential operator be the one-dimensional Laplace-Betrami operator, we get in R-representation the following equation:

$$\left(\frac{1}{48\pi^2} R^{-1/4} \frac{\partial}{\partial R} R^{-1/2} \frac{\partial}{\partial R} R^{-1/4} - 12\pi^2 R + 4\pi^2 \Lambda R^3 + Nm \right) \Psi = 0.$$

Here m is the mass operator. State function Ψ doesn't depend on α due to constraints. Supposing that Ψ is an eigenfunction of the m operator, so that we treat it here as a c-number and using the transformation $\chi = R^{3/2}$, $\Phi = R^{-1/4} \; \Psi$, $\Lambda R^2 = \lambda$, we come to the equation:

$$-(3/64\pi^2) \; \ddot{\Phi} + (12\pi^2 \chi^{2/3} - 4\pi^2 \lambda \chi^2) \; \Phi = Nm\Phi,$$

which differs from the DeWitt equation for the dust by the form of the potential. We put the boundary condition of DeWitt $\Phi(0) = 0$, or $\Psi(0) = 0$. So, we obtain the Schroedinger equation for the particle with mass $32\pi^2/3$ moving with energy Nm in the one-dimensional potential

$$V(\chi) = \begin{cases} \infty, & \chi < 0, \\ 12\pi^2 \chi^{2/3} - 4\pi^2 \lambda \chi^2, & \chi \geq 0. \end{cases}$$

There are three possible cases for values of the cosmological constant: $\Lambda < 0$, $\Lambda = 0$ and $\Lambda > 0$. $\Lambda = 0$ is treated in [2] in the WKB approximation.

It is obtained there that the total energy

$$Nm = [48\pi^2(n+3/4)]^{1/2}, n = 0,\ 1,\ 2,\ \ldots$$

For each Nm there is a maximal radius of the Universe R_{max} after which Ψ exponentially decreases.

For $\Lambda < 0$ we also have a limited value for the Universe radius and, in particular, for small Nm, i.e. $R_{max} \ll (-3/\Lambda)^{1/2}$ the same expression (4.86) is obtained as in the case of $\Lambda = 0$. For large Nm, i.e. $R_{max} \gg (-3/\Lambda)^{1/2}$, as it is seen from the expression for the potential,

$$Nm \sim (n+3/4)\ \sqrt{-\lambda},$$

similar to the energy of the harmonic oscillator. The motion of the equivalent particle will also be finite; $0 \leq R \leq R_{max}$. So, for $\Lambda < 0$ there are no principal differences with the case $\Lambda = 0$. Besides, for $\Lambda < 0$ results correspond to the classical case.

For $\Lambda > 0$ V has the form of a potential barrier which at large χ do not differ from the parabolic one. For $Nm \geq 8\pi^2\lambda^{-1/2} = V_{max}$ the spectrum is continuous and the effective motion is infinite in one direction. For $N^m < V_{max}$ the spectrum is also continuous. But if we choose for the solution the radiation condition, i.e. the solution after the barrier is taken in the form of an outgoing wave, then this solution choose from the continuous spectrum separate levels which are not stationary as in cases $\Lambda < 0$ and $\Lambda = 0$. In order to find these, so called, quasistationary levels we use the well known in a quantum mechanics procedure [7], after which we obtain the equation

$$\sin\left(\int_0^{\chi_1} p d\chi + \frac{\pi}{4}\right)\exp\left(\int_{\chi_1}^{\chi_2}|p|d\chi\right) + \cos\left\{\int_0^{\chi_1} p d\chi + \frac{\pi}{4}\right\}\times$$
$$\times\exp\left\{i\frac{\pi}{2} - \int_{\chi_1}^{\chi_2}|p|d\chi\right\} = 0, \tag{4.87}$$

where $|p| = \sqrt{(64\pi^2/3)[V(\chi) - Nm]}$; χ_1 and χ_2 are turning points.

Let us solve this equation approximately supposing that

$$\int_{\chi_1}^{\chi_2}|p|d\chi \gg 1,\ i.e.\ Nm/8\pi^2\lambda^{-1/2} \ll 1. \tag{4.88}$$

Energies far less than V_{max} correspond to this case. As a result, we get that

$$Nm = [48\pi^2(n+3/4)]^{1/2}.$$

Writing the total energy as $Nm = E = E_0 + i\Gamma$, where E_0 is the energy of quasistationary levels and $1/\Gamma = \tau$ is a lifetime of this quasistationary levels with the energy E_0, we calculate the penetration coefficient

$$D = \exp\{-2\int_{\chi_1}^{\chi_2}|p|d\chi\},$$

which in this approximation has the form:

$$D = \exp\left\{-18\pi^3\lambda^{-1}\left[1 - \frac{(4+3\sqrt{2})}{\sqrt{3}}\frac{E_0\lambda^{1/2}}{36\pi^2}\right]\right\}.$$

In the same approximation for the life time τ we have from (4.87) the following equation:

$$\mathrm{sh}(E_0\Gamma/24\pi) = D,$$

or, at small $(E_0\Gamma)$ approximately $E_0\Gamma/24\pi \approx D$. Then, from here we get

$$\tau = \frac{E_0}{24\pi}\exp\left\{18\pi^3\lambda^{-1}\left[1-\left(\frac{4+3\sqrt{2}}{\sqrt{3}}\right)\frac{E_0\lambda^{1/2}}{36\pi^2}\right]\right\}.$$

Let us try to compare these results with the classical ones. When the condition (4.88) is realized there are models of two types - oscillating of the first kind and monotonous of the second kind. In quantum approach there is a nonnull probability of a transition from one kind of a model to another, though rather small at values of E_0 and Λ in the modern epoch [14]. The same problem may be solved in other than (4.88) limit when $E \approx V_{max} = 8\pi^2\lambda^{-1/2}$. Then, the barrier becomes nearly parabolic and the expression for the penetration coefficient D and, therefore, for the above barrier reflection coefficient, $R = 1 - D$, is known [7]. In our case it has the form:

$$D = \{1+\exp[-2\pi\lambda^{-1/2}(E_0 - 8\pi^2\lambda^{-1/2})]\}^{-1}$$

or, as $|E_0 - 8\pi^2\lambda^{-1/2}| \ll 8\pi^2\lambda^{-1/2}$,

$$D = \left\{1+\exp\left[-16\pi^3\lambda^{-1}\left(\frac{E_0 - 8\pi^2\lambda^{-1/2}}{8\pi^2\lambda^{-1/2}}\right)\right]\right\}^{-1}.$$

As it is seen from these formulas, $D = 1/2$, i.e. it is not small for $E = V_{max} = 8\pi^2\lambda^{-1/2}$. In the classical approach three type of models correspond to this variant: two asymptotic models and the static Einstein universe. In the quantum approach the static model is absent but the system is described by asymptotic model with equal probability.

So, the quantization of the closed isotropic model with Λ-term leads to new interesting results, in particular to the finite lifetime of the system, to the appearance of the Universe after the barrier and question even the fact of division of models into closed and open ones. One should stress that these purely quantum effects appear when $\Lambda > 0$ and this is connected, probably, to the approach interpreting the Λ-term as a vacuum energy density [15,16].

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Chapter 5

Self-Consistent Treatment of Quantum Gravitational Effects

5.1. Cosmological Consequencies of Spontaneously Broken Gauge Symmetry in Isotropic Model [21,13,32]

Models with spontaneous symmetry breakdown are very popular in modern quantum theory. In particular, they are investigated in external gravitational fields [1]. We shall be interested mainly in their cosmological effects in the early stages of the evolution as quantum vacuum properties must be essential in strong gravitational fields. These effects will be treated in the framework of the self-consistent conformal scalar and Einstein gravitational field.

A theory of gravitation with a conformal quantized scalar field without self-interaction was suggested in ref. [2], where static and weakly nonstatic cosmological solutions were found. Nonstatic solutions for a massless scalar field were found in ref. [3]. For results on localized systems including the electromagnetic field see refs. [4,5].

Here we shall demonstrate that the quantum properties of the vacuum lead to cardinal cosmological consequences, i.e. to avoidance of a singularity. Moreover, particle masses vary from the Planck mass at the initial moment up to the graviton mass in the present stage. In the following we treat 3 sources for the Einstein equations: 1) a massless scalar field, 2) the same with radiation and incoherent matter and 3) a massive scalar field with radiation and dust.

5.1.1. Let the Lagrangian be $L = L_g + L_\varphi$, $c = \hbar = 1$, where $L_g = R/2k$,

$$L_\varphi = g^{\alpha\beta}\nabla_\alpha\varphi^*\nabla_\beta\varphi - \frac{1}{6}R\varphi^*\varphi - \frac{1}{6}\lambda(\varphi^*\varphi)^2. \tag{5.1}$$

To this Lagrangian correspond Einstein equations with the scalar field energy-momentum tensor, suggested in ref. [6] and the scalar field equation

$$\Box\varphi + \frac{1}{6}R\varphi + \frac{1}{3}\lambda\varphi^*\varphi^2 = 0. \tag{5.2}$$

Eq. (5.2) is conformal-invariant. The Einstein Lagrangian (as well as the mass) violate the conformal invariance of the whole system. The gravitational field is considered to be classical, but the φ-field quantized [2, 1], so in the right-hand side of the Einstein equations we have the vacuum expectation value of the energy-momentum tensor [2].

The Lagrangian (5.1) is invariant under the gauge transformations of the scalar field: $\varphi \to \varphi\,\mathrm{e}^{i\alpha}$, $\varphi^* \to \varphi^*\,\mathrm{e}^{-i\alpha}$. If we neglect vacuum fluctuations, one may show [1] that for a massless scalar field in the metric of the open isotropic model,

$$ds^2 = a^2(\eta)[d\eta^2 - d\chi^2 - \mathrm{sh}^2\,\chi(d\theta^2 + \sin^2\theta d\varphi^2)],$$

there is a spontaneous breakdown of gauge symmetry. For the vacuum expectation value, defined in the initial moment, there are stable solutions of eq. (5.2),

$$\ddot{f} - f + f^3 = 0, \quad < O|\varphi|O >= \sqrt{3/\lambda} f(\eta)/a(\eta),$$

where $f = \pm 1$, to which a negative energy integral corresponds. The trivial solution $f = 0$ with zero energy is unstable. According to ref. [1] the energy and pressure densities of a massless scalar field in a nonsymmetric vacuum state are:

$$\epsilon(\eta) =< O|T_0^0|O >= -3/2\lambda a^4(\eta), \quad p(\eta) =< 0|T_i^i|0 >= -1/2\lambda a^4(\eta) \tag{5.3}$$

(no summation over i), i.e. $\epsilon = 3p$ as it should be for a massless conformal field. The general form of eq. (5.3) may also be obtained from the conservation law [7]. Let us find out what type of geometry generates effects of spontaneous symmetry breakdown in theory (5.1). We shall use the integral of the $\binom{0}{0}$-component of the Einstein equations (see ref. [7]):

$$\eta = \pm \int da/(\frac{1}{3}k\epsilon_{\mathrm{tot}} a^4 + a^2)^{1/2}.$$

After transforming to the synchronous time $dt = ad\eta$, taking into account eq. (5.3), we come to

$$a = (L^2 + t^2)^{1/2}, \tag{5.4}$$

where $L = (k/2\lambda)^{1/2} \sim 10^{-33}$ cm ($\lambda \sim 1$). So we obtained that quantum vacuum effects, i.e. spontaneous breakdown of gauge symmetry, lead to avoidance of a cosmological singularity as it is energetically preferable during the whole evolution of the model and, thus, should be realized from the beginning. It is natural to define the initial vacuum state at the moment of maximal compression, when $a = L$. A solution of type (5.4) was used in ref. [8] to analyze cosmological models with a variable number of particles.

In ref. [1] the mass spectrum of particles was also found:

$$m_1^2 = 3/a^2, \quad m_2^2 = 1/a^2. \tag{5.5}$$

According to eq. (5.4), in the moment of maximal compression they correspond to the Planck masses:

$$m_1^2 = 3/L^2 = 6\lambda/k \sim m_L^2, \quad m_2^2 = 2\lambda/k \sim m_L^2 \quad (\lambda \sim 1), \tag{5.6}$$

which to the present epoch $t \sim 10^{27}$ cm vary up to the graviton masses:

$$m_1 \sim m_2 \sim m_L[1 + (t/L)^2]^{-1/2} \quad \sim m_L \times 10^{-60} \sim 10^{-65} g. \tag{5.7}$$

So, these relic scalar particles if ever found could serve as detectors of the quantum vacuum properties at the initial stages of the evolution of the Universe, $t = m_L^{-1} \sim 10^{-33}$ cm.

It is also shown in ref. [1] that addition of an electromagnetic field interacting with a scalar field causes the appearance of the mass of the vector particle: $m_V^2 = 6\,\mathrm{e}2/\lambda a^2$, i.e. in e^2 times less, than for the scalar case. So, in our scheme the photon masses should vary from the electromagnetic Planck (maximon) masses at $t \sim 10^{-33}$ cm up to $m_V \sim 10^{-66}$ g nowadays as

$$m_V^2 \sim m_L^2\alpha/[1 + (t/L)^2], \tag{5.8}$$

where α is the fine structure constant. Nonzero photon and graviton masses in an isotropic world were first obtained by Staniukovich [9] when analyzing electrodynamics in Riemann space-time.

The treatment of the massive field in ref. [1] is based on the assumption that the scale factor $a(\eta) = a_1\eta^p$ and that the solution for a massless field can be used as initial condition for the massive case when $\eta \to 0$. We have shown that for the massless case the solution is not singular, so, this solution cannot be as initial condition when $m \neq 0$ (the limits $m \to 0$ and $\eta \to 0$ are not equivalent). So, we do not consider all the calculations and conclusions of ref. [1] for a massive scalar field to be correct. This problem should be treated with different initial conditions.

5.1.2. Let there also be incoherent matter and radiation without direct interaction with each other and the scalar field. Such a model is a good approximation at large and small times (densities). In this case, from $\binom{0}{0}$-component of the Einstein equations follows:

$$d\eta = da/(\frac{1}{3}k\epsilon_{\rm tot}a^4 + a^2)^{1/2}, \tag{5.9}$$

with $\epsilon_{\rm tot} = \epsilon_s + \epsilon_r + \epsilon_d$ the total energy density. Using thermodynamic arguments, one may get for the energy densities of radiation and dust:

$$\epsilon_r = \epsilon_{r0}a_0^4/a^4, \quad \epsilon_d = \epsilon_{d0}a_0^3/a^3, \tag{5.10}$$

where ϵ_0, a_0 are the present values of the energy density and the radius. Substituting eqs. (5.10) and (5.3) into eq. (5.6) and integrating, we obtain:

$$a = -b + (b^2 + c)^{1/2}\,\mathrm{ch}\,\eta, \quad b = \epsilon_{d0}a_0^3k/6, \quad c = k/(2\lambda) - \epsilon_{r0}a_0^4k/3, \tag{5.11}$$

with

$$a > a_{\rm min} = -b + (b^2 + c)^{1/2}, \tag{5.12}$$

$$\lambda < 3/2\epsilon_{r0}a_0^4 \sim 10^{-118}. \tag{5.13}$$

So, if there is a quantized massless scalar field in the Universe, then radiation and matter do not lead to a singular state. Using eq. (5.13), i.e. $\lambda \sim 10^{-118}$, we come to the following value of the minimal scale factor:

$$a_{\rm min} \sim (\epsilon_{r0}/\epsilon_{d0})a_0(3/2\lambda\epsilon_{r0}a_0^4 - 1). \tag{5.14}$$

The mass spectrum of physical particles will again be given by eq. (5.5), that is in the moment of maximal compression $m_1 \sim m_2 \sim a_{\rm min}^{-1}$. From eq. (5.11) $a \sim t$ when $t \gg a_{\rm min}$ and we get for the evolution of the masses: $m_1 \sim m_2 \sim t^{-1}$. For the present epoch, $t \sim 10^{27}$ cm: $m_1 \sim m_2 \sim 10^{-65}$ g.

5.1.3. In the massive case ($m < m_L$) eq. (5.2) in the metric of the open isotropic model is
$\ddot{f} + (m^2a^2 - 1)f + f^3 = 0$.

It is easy to show that for $\eta \ll 1$ its solution will be

$$f = (1 - m^2a_m^2)^{1/2}\,\mathrm{Cn}(ma_m\eta, 1/2ma_m),$$

where $1 - m^2a_m^2 > 0$. Expanding in series in η, one may find up to $ma_m\eta$:

$$f(0) = (1 - m^2a_m^2)^{1/2}, \dot{f}(0) = 0.$$

The energy density of the massive scalar field [1] in this approximation is:

$$\epsilon(\eta) = (3f^2/\lambda a^4)(m^2a^2 - 1 + f^2/2 + \dot{f}^2/f^2) \approx -3(1 - m^2a_m^2)/2\lambda a_m^4 < 0, \tag{5.15}$$

i.e. in the massive case a spontaneous breakdown of gauge symmetry is also realised at small times as the energy at the extremal value of f is less than one at the trivial solution $\epsilon(f) = \epsilon(0) = 0$. Considering as in section 5.1.2 a cosmological model of the open type filled with radiation and dust we obtain eqs. (5.11), (5.12) with

$$b \to b' = b/d, \quad c \to c' = c/d, \quad \eta \to \eta' = \sqrt{d\eta}, \quad d = (l + m^2 k/\lambda), \quad \eta' \ll 1,$$

where a_m (when $ma_m \ll 1$) coincides with (5.14) and in the synchronous time $a \approx t$, $t \gg a_m$. As is seen from eq. (5.15), at large times $\epsilon > 0$, so, at some t_0 the gauge symmetry must be restored. ¿From dimensional considerations this moment should be $t_0 \sim m^{-1}$. For $m = a^{-1}$ see eq. (5.15), there is no spontaneous symmetry breakdown from the beginning.

5.2. Spontaneously Broken Conformal Symmetry in a Curved Space-Time

There is presently a great interest in models of unified teories of the various physical interactions. Popular also are models with spontaneous breaking of gauge symmetry, of the type of the Weinberg-Salam model [10], which unify the weak and electromagnetic interactions, as well as models which also incorporate the strong interactions. An attractive feature of such models is their renormalizability by virtue of the high symmetry of the initial Lagrangian. The quantum properties of the vacuum, which lead to the appearance of the particle masses, play an important role in these models. It is known that in theories of gravitation a nonzero vacuum is a source of a gravitational field, just as matter is. Since all real processes take place in the gravitational field of the metagalaxy, gravitational interactions must also be used for the construction of unified theories [9]. The spontaneous breaking of gauge symmetry in a given external gravitational field has been considered in a number of works (see Ref. 11, for example and section 5.1). It is known that any problem in an external gravitational field is approximate. In the spirit of Einstein's theory of gravitation, a self-consistent analysis with allowance for the equations of the gravitational field is necessary.

Here, we will consider within the framework of a self-consistent scheme a spontaneous breaking of conformal symmetry.

5.2.1. The basis is a conformally invariant generalization of a Lagrangian of the Weinberg-Salam type with the gauge group [12] $U(1)$. Suppose, that only the scalar field possesses a nonzero vacuum average. Then, in the tree approximation the effective vacuum Lagrangian possessing conformal symmetry has the form

$$S_\varphi = \int d^4x\sqrt{-g}L_\varphi = \int d^4x\sqrt{-g}\left\{\frac{1}{2}g^{\alpha\beta}\Phi^*_\alpha\Phi_\beta - \frac{R}{12}\Phi^*\Phi - \frac{\lambda}{12}(\Phi^*\Phi)^2\right\},$$
$$\Phi_\alpha = \frac{\partial\Phi}{\partial x^\alpha}. \tag{5.16}$$

Changing next to the real scalar field $\Phi = e i\alpha\varphi$ and varying the action with respect to $g^{\mu\nu}$ and φ, we arrive at the following system of equations:

$$\frac{\varphi^2}{6}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right) = \varphi_\alpha\varphi_\beta - \frac{1}{2}g_{\alpha\beta}\left(g^{\mu\nu}\varphi_\mu\varphi_\nu - \frac{\lambda}{6}\varphi^4\right) - \frac{1}{6}(\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box)\varphi^2, \tag{5.17}$$

$$\varphi_\alpha = \frac{\partial\varphi}{\partial x^\alpha}, \quad \nabla_\alpha = \frac{\partial}{\partial x^\alpha}, \quad \Box = \nabla^\alpha\nabla_\alpha, \quad \Box\varphi + \frac{R}{6}\varphi + \frac{\lambda}{3}\varphi^3 = 0. \tag{5.18}$$

It is well known that Eq. (5.18) is a consequence of (5.17), i.e., it is not an independent dynamical field in the case of "conformal" gravitation. Actually, the trace of (2) is

$$\varphi(\Box\varphi + \frac{R}{6}\varphi + \frac{\lambda}{3}\varphi^3) = 0,$$

which gives (5.18) when $\varphi \neq 0$ (spontaneous breaking of the symmetry). By virtue of the conformal invariance of (5.16), and hence of (5.17) and (5.18), and using the conformal transformation

$$g'_{\mu\nu}(x) = \exp[-\rho(x)]g_{\mu\nu}(x), \quad \varphi'(x) = \exp(\rho/2)\varphi(x) = \chi = \text{const},$$

it is straightforward to reduce (5.l6) to the form

$$S = \frac{\chi^2}{12}\int d^4x\sqrt{-g'}(R' + 2\lambda\chi^2). \tag{5.19}$$

Thus, upon spontaneous breaking of the conformal symmetry, as mentioned in Ref. 12, we arrive at the fact that the Einstein gravitational constant is expressed in terms of the vacuum average $k = 6/\chi^2$ of the scalar field, and the analog of the cosmological term $\lambda\chi^2/2$ appears in the Lagrangian. But, we wish to emphasize that something else is no less important: the appearance of the Einstein term $R/2k$ in the Lagrangian. This fact can be interpreted as the appearance of a new macroscopic characteristic – the curvature of space-time – as a result of spontaneous breaking of conformal symmetry. In other words, Einstein gravitation can be considered as a classical manifestation of the quantum properties of a vacuum. In this connection it is not ruled out that the attempts, as yet unsuccessful, to quantize Einstein gravitation have a profound basis. It is very likely that a gravitational field may be essentially classical in nature and should not be quantized. Therefore, we investigated the effect of the spontaneous breaking of the gauge symmetry within the framework of a self-consistent scheme with allowance for Einstein classical equations of gravitation in section 5.1. For this we added the Lagrangian $L_g = R/2k$ of the gravitational field, obtained in (5.19), to the Lagrangian of (5.16). We assumed that the gravitational field was classical but φ was quantized. We suggested just such an approach within the framework of the Einstein theory of gravitation with a conformal quantum scalar field without self-interaction in Ref. 2, where we found static and weakly nonstatic cosmological solutions, and in Ref. 3, where we found nonstatic solutions.

5.2.2. Continuing the topic of sect.5.1., suppose the masses appear only through spontaneous breaking of the gauge symmetry of the massless scalar field. Substituting the expressions (5.8) for the effective boson masses into the formula corresponding to the massive case [1],

$$\epsilon(\eta) \equiv < 0|T_0^0|0 > = \frac{3f^2}{\lambda a^4}\left(m^2a^2 - 1 + \frac{f^2}{2} + \frac{\dot{f}^2}{f^2}\right).$$

we find that

$$\epsilon(\eta) = \frac{3f^2}{\lambda a^4}\left(k_i + \frac{f^2}{2} + \frac{\dot{f}^2}{f^2}\right) > 0. \quad k_i = 2:0,$$

while f is determined from the equation

$$\ddot{f} + (1 + k_i)f + f^3 = 0. \tag{5.20}$$

Thus, in the massive case the minimum energy density $\varepsilon = 0$ is realized for the trivial solution $f = 0$. For other solutions ε is always positive. Therefore, in the given massive case there is no spontaneous breaking of the gauge symmetry during the entire evolution of the model. Moreover, there is also no particle creation in this case, since the dependence on a drops out of (5.9), i.e., the effective frequency does not depend on time. With other masses particle creation should occur. But here we should emphasize the following: First, the problem of particle creation in an external gravitational field is an approximate problem; strictly it would be desirable to solve it within the framework of self-consistent schemes in some modification of the Einstein equations, and then, both the metric and the creation processes would be consistent.

Because of the extreme difficulty of the given problem it is necessary to solve it in a given field even in the first stages, but here one must bear in mind that this field must be understood as a real gravitational field. It cannot be transformed, e.g., to the flat case, as is done in the case of a conformally invariant scalar field, since we thereby remove the gravitational field itself. Moreover, returning conceptually to a self-consistent scheme, we see that the gravitational equations would also be changed in this transformation, since neither the Einstein equations nor the majority of their modifications are conformally invariant, so that the very metric in which the analysis began would also change.

Secondly, an analysis of a metric in which the scale factor equals zero at the starting time is, at least, non-physical. There are many reasons why the quantum properties of the vacuum must prevent a singular state in cosmology. In the present section we showed that within the framework of the self-consistent problem of the interaction of a scalar and a gravitational field allowance for the effect of spontaneous breaking of the gauge symmetry actually leads to a nonsingular solution. Therefore, as the "real" metrics one must take those in which the expansion proceeds from a nonzero value (usually the Planck scale). Then, the results are essentially altered. For example, if we analyze particle creation in our metric (5.4) by the method presented in Ref. 15 then at small times we find that the particle-number density is $n \approx m^3\eta^4/2^7\pi$, and not $m^3/24\pi^4$ as calculated in Ref. 15 on the basis of a singular metric, so that in the early, vacuum stage the creation proceeds extremely slowly. It is also small in the present epoch [15,16], so that the main creation must occur at times $\tau \approx m^{-1}$ (Ref. 15).

In connection with the foregoing, we can offer the following senario for the Universe: 1) spontaneous breaking of the conformal symmetry of the vacuum Higgs field and the appearance of Einstein gravitation as a classical manifestation of this effect; 2) spontaneous breaking of the gauge symmetry and the start of evolution from the nonsingular state [13] as a quantum effect of the vacuum; 3) the appearance of all masses as an effect of creation in the evolving gravitational field [17], in agreement with the Dirac hypothesis of Large Numbers. An analog of points 1) and 2) for a closed model was first considered in Ref. 13.

5.3. Cosmological Models with Broken Gauge Symmetry and Vacuum Polarization [53]

In recent years, a number of scenarios of the evolution of the universe have been presented [19–26], in which attempts were made to solve the fundamental problems of cosmology: the development of the universe as a result of quantum fluctuation [20,26], explanation of isotropicity, flatness, barion asymmetry, etc. Many of these problems were solved by introduction of the de Sitter stage. It should be noted that the majority of such scenarios were based on a closed model. However, observations still indicate in favor of flat or an open models: the limit of barion density for cosmic nucleosynthesis of light elements is as follows: $\Omega \leq 0,12$ (where Ω is the ratio of the present density of matter to the critical density) for a Hubble constant $H \geq 65$ km/s Mpc, while limitations on the dynamics of clusters and superclusters of galaxies given in [28] indicate that $0,3 \leq \Omega \leq 0,7$. Similar results ($\Omega < 1$) were reached in [29]. In connection with this, it is reasonable still to study scenarios in which an open model is used.

In 1978, using such a model, one of the first scenarios was proposed [21], including the following stages: 1) spontaneous breaking of the conformal symmetry of Weinberg-Salam type theories with a gravitational background, assuming that only the conformal Higgs and gravitational fields have nonzero vacuum mean values. This effect leads (as noted in [21]) to a gravitation theory with the Einstein Lagrangian and a cosmological constant of the order of the Planck constant; 2) spontaneous breaking of gauge symmetry of the conformal Higgs field in the gravitational field [31], leading to a nonsingular cosmological solution [21]; 3) particle generation in the nonsteady state gravitational field and explanation of the observed masses of matter in this process [17,19]; and, finally, 4) the possibility of explaining variations in the effective gravitational coupling within the framework of the model used in [2]. This model is initially "cold," but then becomes "hot," due to particle generation in the phase transition.

5.3.1. Model. We assume the validity of the semiclassical approach to gravitation, i.e., the Einstein equations $G_\mu^\nu = -8\pi G < T_\mu^\nu >$ with quantum mean value of the energy-momentum tensor of physical fields. We will consider a vacuum-dominated isotropic universe with metric

$$ds^2 = a^2(\eta)[d\eta^2 - dr^2/(1-\chi r^2) - r^2(d\theta^2 + \sin^2\theta d\Phi^2)], \tag{5.21}$$

where $k = 0, \pm 1$, so that the scalar curvature and Einstein tensor are given by

$$R = 6a^{-3}(\ddot{a} + \chi a); \quad G_0^0 = -3a^{-4}(\dot{a}^2 + \chi a^2). \tag{5.22}$$

Let the following quantities appear in the tensor $< T >$:

1) the effective cosmological constant Λ, the result of quantum gravitation effects ($\Lambda \sim G^{-1}$) or (and) the big-bang theory ($\Lambda \sim 10^{-8} G^{-1}$), or torsion effects [30].

2) The contribution of vacuum polarization of any free massless conformal-invariant fields which, as is well known, has the form [31]

$$8\pi T_\mu^{\nu(\mathrm{pol})} = N_1^{(3)} H_\mu^\nu + N_0^{(1)} H_\mu^\nu - N_2 \delta_{\chi,-1} I_\mu^\nu, \tag{5.23}$$

where the tensor ${}^{(1)}H_\mu^\nu$, including higher derivatives of $g_{\mu\nu}$, creates variation in $g^{\mu\nu}$ of the term R^2 in the Lagrangian, and, consequently, can be eliminated by final renormalization; in connection with this we take $N_0 = 0$. For the conservative tensors ${}^{(3)}H_\mu^\nu$ and I_μ^ν it will suffice to describe the time components:

$${}^{(3)}H_0^0 = 3a^{-8}(\dot{a}^2 + \chi a^2)^2; \quad I_0^0 = a^{-4}. \tag{5.24}$$

The constants N_1 and N_2 depend on the number and spins of the fields [31]:

$$N_1 = \frac{1}{360\pi}\left(N^{(0)} + \frac{11}{2}N^{(1/2)} + 62N^{(1)} + \ldots\right), \tag{5.25}$$

$$N_2 = \frac{1}{360\pi}\left(6N^{(0)} + \frac{51}{2}N^{(1/2)} + 61N^{(1)} + \ldots\right), \tag{5.26}$$

where $N^{(s)}$ is the number of different fields with spin s, while the particle and antiparticle fields are considered to be different.

3) The vacuum mean value of the complex conformal scalar field with Lagrangian

$$L = \overset{*}{\varphi}{}^{,\alpha}\varphi_{,\alpha} - \frac{1}{6}R\overset{*}{\varphi}\varphi - \frac{\lambda}{6}(\overset{*}{\varphi}\varphi)^2. \tag{5.27}$$

The field φ possesses $U(1)$-symmetry. We will consider conditions under which the breaking of this symmetry occurs, characterized by a nonzero value of $\varphi_0 =< 0|\varphi|0 >$. In view of the symmetry of the problem, we take $\varphi_0 = \varphi_0(\eta)$. Then, from the classical equation of the field φ for $f \equiv a\varphi_0$ we obtain the Duffing equation

$$\ddot{f} + \chi f + \lambda f^3/3 = 0, \tag{5.28}$$

which has a stable nontrivial solution only at $k = -1$ (to which case we will limit ourselves in the future):

$$f = \sqrt{3/\lambda}; \quad \varphi_0 = a^{-1}\sqrt{3/\lambda}. \tag{5.29}$$

In this state the energy-momentum tensor of the field φ has the form

$$T_\mu^\nu(\varphi_0) = -(2\lambda a^4)^{-1}\,\mathrm{diag}(3,-1,-1,-1) = -I_\mu^\nu/6\lambda. \tag{5.30}$$

5.3.2. Model Dynamics. The evolution of $a(\eta)$ is defined by the time component of the Einstein equations, which has the form

$$3a^{-4}(\dot{a}^2 - a^2) = \Lambda + 3GN_1 a^{-8}(\dot{a}^2 - a^2)^2 - 3GN_3 a^{-4}, \tag{5.31}$$

where $3N_3 = N_2 + 12\pi/\lambda$. It follows from Eq. (5.31) that

$$\frac{\dot{a}^2}{a^2} = 1 + \frac{a^2}{2GN_1}\left[1 \pm \sqrt{1 + 4Gn_1\left(\frac{GN_3}{a^4} - \frac{\Lambda}{3}\right)}\right]. \tag{5.32}$$

We choose the minus sign before the radical, as required for a regular minimum of $a(\eta)$. Obviously,

$$a^2_{\rm min} = \frac{3}{2\Lambda}\left[-1 + \sqrt{1 + \frac{4G\Lambda}{3}(N_3 - N_1)}\right]; \tag{5.33}$$

for $G\Lambda \ll 1$ we have $a^2_{\rm min} = G(N_3 - N_1)$. According to Eq. (5.33), the regular minimum exists at $N_3 > N_1$. This condition can be satisfied for any set of physical fields by choice of a sufficiently low self-action constant λ.

Equation (5.32) can be integrated in quadratures, although in the general case it is not possible to present the result in a closed form, and it is more convenient to deal with Eq. (5.32) itself. For large a, the behavior of the model depends significantly on the sign of the constant

$$A = 1 - 4N_1 G\Lambda/3. \tag{5.34}$$

For $A > 0$, neglecting the term with a^{-4} (i.e., assuming $a^2 \gg 2G\sqrt{N_1 N_3/A^{1/2}}$) and transforming to world time t with the expression $dt = a(\eta)d\eta$, we obtain the asymptotic de Sitter behavior:

$$a^2(\eta) = a^2(t) \approx \frac{1}{s}(\,{\rm e}\sqrt{2st} - 1), \quad s = \frac{1 - \sqrt{A}}{GN_1}. \tag{5.35}$$

The asymptotic Hubble parameter $H = a^{-1}da/dt = \sqrt{s/2}$ for small $\Lambda (G\Lambda \ll 1)$ is equal to $\sqrt{\Lambda/3}$. At $\Lambda = 0$, instead of the de Sitter, we obtain the Milne asymptote $a(t) = t$. In the special case $N_1 = N_2 = 0$, $N_3 = 4\pi/\lambda$, this is the model considered in [19,21,32]: $a(t) = (4\pi G/\lambda + t^2)^{1/2}$ (consideration of only the field φ with spontaneous breaking of symmetry).

At $A = 0$, Eq. (5.32) gives

$$a = a_{\rm min}\cosh(t/\sqrt{2GN_1}). \tag{5.36}$$

i.e., qualitatively, the solution behaves as at $A > 0$.

For $A < 0$ there exists an $a = a_{\rm max} = [4G^2 N_1 N_3/(-A)]^{-1/4}$, at which the root in Eq. (5.32) vanishes; further expansion is impossible. Although the rate of expansion at the moment when $a_{\rm max}$ is reached is finite $[(da/dt)^2 = 1 + a^2_{\rm max}/2GN_1]$, the curvature R is infinite, i.e., expansion ends in a singularity. In this case, of the material sources of gravitation the vacuum polarization Eq. (5.23) is singular, or, more precisely, the tensor ${}^{(3)}H^\nu_\mu$.

Thus, with the given set of material fields, i.e., at fixed N_1, the universe expands without limit only if $\Lambda \leq \Lambda_{\rm max} = 3/(4GN_1)$. The conditions of our solution then describe a smooth transition from de Sitter compression to a regular bounce, and further to de Sitter inflation.

The characteristic values of the model parameters are of the Planck order, defined by the constant G. Thus, the constant N_1 for big-bang and supersymmetry models currently popular is close to unity in order of magnitude. If we assume, for example, that $N_1 = 1$ and $N_3 = 10$, then $G\Lambda_{\rm max} = 3/4$; for $G\Lambda = 1/2$ we obtain $a_{\rm min} = 2,22\sqrt{G}$; $A = 1/3$; $H\sqrt{G} = 0,46$.

5.3.3. The Problem of Consideration of Deviation of the Field φ. Until now we have considered only the classical field $\varphi = \varphi_0$, corresponding to the broken symmetry, and have ignored the possible contribution to vacuum polarization of deviations $\chi = \varphi - \varphi_0$, which are of a quantum character. Neglect of the latter is probably justifiable if the number of fields is large. However, for a precise treatment of the problem, it is necessary to define the vacuum energy-momentum tensor corresponding to the field χ.

It is simple to show that for $\chi_1 = \sqrt{2}Re\chi$ and $\chi_2 = \sqrt{2}Im\chi$ in the linear approximation the equations

$$(\Box + R/6 + 3/a^2)\chi_1 = 0 : \quad (\Box + R/6 + 1/a^2)\chi_2 = 0. \tag{5.37}$$

are valid, describing fields with variable masses $\sqrt{3}/a$ and $1/a$. Such masses do not lead to particle creation [32] (separation into positive- and negative-frequency solutions is done just as for massless fields). However, the question of vacuum polarization is complex, since even the classical energy-momentum tensor is nonconservative in the approximation quadratic in χ.

In fact, we will consider a more general Lagrangian for φ:

$$L = \overset{*}{\varphi}{}^{,\alpha}\varphi_{,\alpha} - \xi R\Phi - V(\Phi), \quad \Phi \equiv \overset{*}{\varphi}\varphi, \quad \xi = \text{const}, \tag{5.38}$$

not fixing the function $V(\Phi)$. The full energy-momentum tensor has the form

$$T^\nu_\mu = \overset{*}{\varphi}{}^{,\nu}\varphi_{,\mu} + \overset{*}{\varphi}{}_{,\mu}\varphi^{,\nu} - \delta^\nu_\mu L - 2\xi(R^\nu_\mu + \nabla^\nu\nabla_\mu - \delta^\nu_\mu\Box)\overset{*}{\varphi}\varphi \tag{5.39}$$

and is conservative ($\nabla_\alpha T^\alpha_\mu = 0$) with account of the field equation

$$(\Box + \xi R + V')\varphi = 0, V' \equiv dV/d\Phi. \tag{5.40}$$

We now expand φ as a background component F [which satisfies Eq. (5.40)] and a perturbation $\chi \equiv \varphi - F$, and write the equation for the field χ to the accuracy of $O(\chi^2)$:

$$(\Box + \xi R + V')\chi + V''F^2(\chi^* + \chi) + V''F\chi(2\chi^* + \chi) + \frac{1}{2}V'''F^2(\chi^* + \chi)^2 = 0, \tag{5.41}$$

where the derivatives of V are taken at $\Phi = F^2$. Representing the energy-momentum tensor in the form $T(F) = O(1)$, $T^{\rm lin}(F,\chi) = O(\chi)$, $T^{\rm quad}(F,\chi) = O(\chi^2)$ and the residue $o(\chi^2)$, it is simple to show that:

1) the tensor $T^{\rm lin}$ is conservative due to Eq. (5.41) [to the accuracy of $O(\chi)$];

2) the tensor $T^{\rm quad}$, taken separately, is not conservative:

$$\nabla_\alpha T^\alpha_\mu \quad = V'_{,\mu}\overset{*}{\chi}\chi + \left[\frac{1}{2}F^2V''_{,\mu} + V''FF_{,\mu}\right](\overset{*}{\chi} + \chi)^2 + o(\chi^2).$$

Nonconservativeness develops in any nonlinear theory as soon as $\varphi_{\rm background} \equiv F \neq const$.

3) the sum $T^{\rm lin} + T^{\rm quad}$ is conservative to the accuracy of $O(\chi^2)$, if $\nabla T^{\rm lin}$ is represented by Eq. (5.41) to quadratic accuracy. Thus, a coherent treatment of the field χ must necessarily include its self-interaction, and, obviously, the corresponding particle creation.

If in a first approximation we include in the full tensor $< T >$ only $< T^{\rm lin} >$, then the vacuum models described in Sec.5.3.3 do not change, since the vacuum mean values of $< 0|T^{\rm lin}|0 >= 0$. Such models are coherent, in that the initial vacuum state of the field remains a vacuum state and there is no particle creation further on.

5.3.4. Conclusions

The nonsingular model considered is symmetric with respect to time, although it is natural to assume that the state with $a = a_{\min}$ developed not upon classical compression, but due to some processes [33,34], whose description requires either quantum cosmological methods [19] (penetration through a barrier due to tunneling), or concepts of gravitational field generation due to spontaneous breaking of conformal symmetry [21,41] or induction.

A significant feature of the scenario proposed is the idea of evolution from a purely vacuum (cold) state. The scenario includes two vacuum stages: 1) the beginning of expansion from the nonsingular state, produced by the effect of spontaneous breaking of symmetry of the Higgs field; and, 2) the stage of inflation due to the effective Λ-term which, as is well known, permits solution of a number of fundamental cosmological problems.

Further evolution with transition to a Friedman regime may occur, for example, because of particle generation and growth of gravitational and scalar perturbations, as has been considered by many authors [19, 35-38], with heating of the model occurring by one or the other method, so that it becomes "hot."

5.4. A Scalar Field with Self-interaction Leads to the Absence of a Singularity in Cosmology [54]

5.4.1. *Introduction.* The elimination of a singular state in cosmological models is one of the most actual problems in modern cosmology (see ref. [41], where examples of nonsingular solutions are reviewed). Here we present some other nonsingular solutions obtained through computer calculations for the system of gravitational and Higgs massive fields for both the isotropic and the anisotropic case. A Higgs field is an essential part of modern unified models; its vacuum average is nonzero, so this field must be important at early stages of the universe.

5.4.2. A nonsingular open isotropic model with the conformal Higgs scalar field. We start a self-consistent treatment of scalar and gravitational fields from the lagrangian $L = L_g + L_{\psi_1}$, where $L_g = R/2k$, $c = \hbar = 1$, k is the gravitational constant,

$$L_{\psi_1} = g^{\alpha\beta}\nabla_\alpha\psi_1^*\nabla_\beta\psi_1 - (m_\psi^2 + R/6)\psi_1^*\psi_1 - \frac{1}{6}\lambda_\psi(\psi_1^*\psi_1)^2. \tag{5.42}$$

The sign of the curvature tensor is the same as in ref. [6]. We use an open isotropic metric

$$dS_1^2 = a^2(\eta)[d\eta^2 - (1+r^2)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi_1^2)], \tag{5.43}$$

where the proper time t_1 and the conformal time η are connected as $dt_1 = a d\eta$. The scalar field equation, obtained from (5.42) is

$$\Box\psi_1 + (m_\psi^2 + R/6)\psi_1 + \frac{1}{3}\lambda_\psi(\psi_1^*\psi_1^2) = 0. \tag{5.44}$$

The gravitational field is treated classically, the ψ_1 field is quantised, so on the right-hand side of the Einstein equations we have vacuum averages of the energy-momentum tensor. These equations with the so-called modified tensor $T^\nu_{(s)\mu}$, corresponding to our lagrangian [2] are:

$$R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R = -k < 0|T^\nu_{(s)\mu}|0 > . \tag{5.45}$$

If we neglect vacuum fluctuations and take into account that the metric is homogeneous, then one obtains

$$< 0|\psi_1^*(x,\eta)|0 > \approx < 0|\psi_1(x,\eta)|O > = < 0|\psi_1(0,\eta)|0 > \equiv (3/\lambda_\psi)^{1/2}\psi(\eta)/a(\eta), \tag{5.46}$$

where the vacuum average phase is zero, for simpicity Using (5.43)–(5.46) we get the following equation for th scalar field:

$$\psi'' + (m_\psi^2 a^2 - 1)\psi + \psi^3 = 0, \tag{5.47}$$

and the $\binom{0}{0}$ component of the Einstein equations:

$$(a')^2 - a^2 = (k/3)a^4 \epsilon_\psi(\eta), \tag{5.48}$$

with the scalar-field energy density [11]:

$$\epsilon_\psi(\eta) = (3\psi^2/\lambda_\psi a^4)[m_\psi^2 a^2 - 1 + \frac{1}{2}\psi^2 + (\psi')^2/\psi^2]. \tag{5.49}$$

Earlier we studied this system analytically for both the massless case [21,32] and the case with a cosmological constant [40] and obtained nonsingular solutions. Further the system (5.47) was treated numerically for both the massless and the massive case with the Λ term and a conclusion was obtained about the possible existence of nonsingular solutions for a number of values of the parameter $n_\psi = m_\psi^2 k/\lambda_\psi$. Here we present new solutions also obtained by computer calculations for the case $\Lambda = 0$ and $n_\psi = 10^{-2}$, 10^{-1}, 10^0, 10^1, 9×10^1. The examples are given in fig. 5.1.

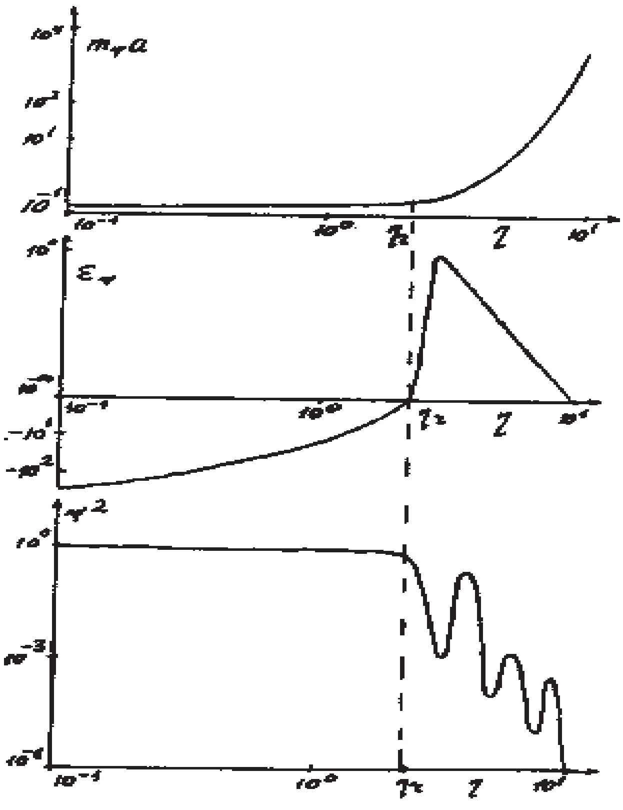

Fig. 5.1

The moment of symmetry breaking η_v at the stage of contraction to the state of maximal compression and the moment of symmetry restoration η_r at the expansion stage are found. Evidently $\eta_v = -\eta_r$ because of the T-invariance of the Einstein equations. We also found that changes of the particle number density

ψ^2, i.e. a creation or annihilation of ψ particles took place at $m_\psi a \approx 10^{-4}$, that is at a scale factor values larger than the Compton one.

5.4.3. Massive Higgs fields lead to nonsingular anisotropic models. So, in section 5.4.2 we defined the moment η_v, when spontaneous symmetry breaking occurred. The vacuum energy density became negative. According to Bogolubov's ideas [42] we suppose that starting from this moment vacuum domains and consequently domain walls appeared. These walls are considered as a new ξ_1 field, described by the lagrangian

$$L_{\xi_1} = g^{\alpha\beta}\nabla_\alpha\xi_1^*\nabla_\beta\xi_1 - (m_\xi^2 + R/6)\xi_1^*\xi_1 - \frac{1}{6}\lambda_\xi(\xi_1^*\xi_1)^2, \qquad (5.50)$$

where $m_\xi^2 > 0$ and $\lambda_\xi < 0$. The total lagrangian is now

$$L = L_g + L_{\psi_1} + L_{\xi_1}. \qquad (5.51)$$

It is natural to believe that the wall self-interaction is weaker than for ψ particles, i.e. $|\lambda_\xi| < |\lambda_\psi|$. Zei'dovich et al. [43] showed that the masses of the walls are larger than the ψ-particle masses. We also suppose that in our scheme $t = m_\psi/m_\xi \lesssim 1$. Interactions between walls and particles are neglected.

So, we consider the evolution of the universe filled with two nonlinear scalar fields. Earlier Starobinsky using our [21,32] solution showed [44] that in the open isotropic universe filled with a conformal Higgs field with spontaneous symmetry breaking anisotropic disturbances of a Bianchi type may arise and this leads to a cosmological singularity. These disturbances start when the scalar-field energy density becomes negative and this energy defines the evolution of the universe. Here we show that a vacuum domain structure leads to a nonsingular anisotropic cosmology at certain values of the wall parameters. Starting from the moment η_v we analyse the system in a Bianchi type metric:

$$dS_1^2 = dt_1^2 - a^2(t_1)dx^2 - b^2(t_1)[S^2(t_1)dy^2 + S^{-2}(t_1)dz^2]\,\mathrm{e}2x. \qquad (5.52)$$

For simplicity we suppose that $a = b$ [45]. Let

$$dt_1 = a(\eta)d\eta, \quad t = m_\psi/m\xi, \quad n_\psi = m_\psi^2 k/\lambda_\psi, \qquad (5.53)$$

$$n_\xi = m_\xi^2 k/\lambda_\xi, \quad A^2 = m_\psi m_\xi a^2, \quad ' = d/d\eta, \qquad (5.54)$$

$$\psi_1 = (3/\lambda_\psi)^{1/2}\psi(\eta)/a(\eta), \quad \xi_1 = (3/|\lambda_\xi|)^{1/2}\psi(\eta)/a(\eta). \qquad (5.55)$$

Using (5.52) and a trace of Einstein equations we get the following equations for the scalar fields and the scale factor:

$$\psi'' + (A^2 t - A''/A + n_\xi\xi^2)\psi + (n_\psi + 1)\psi^3 = 0, \qquad (5.56)$$

$$\xi'' + (A^2/t - A''/A + n_\psi\psi^2)\xi + (n_\xi - 1)\xi^3 = 0. \qquad (5.57)$$

The integration of the combination of the $\binom{2}{2}-\binom{3}{3}$ Einstein equations leads to:

$$(S'/S)^2 = 3C^2/(A^2 - n_\psi t^{-1}\psi^2 - n_\xi t\xi^2)^2 \equiv 3C^2/Q^2. \qquad (5.58)$$

where C^2 is a constant. The dimensionless energy is

$$\epsilon_{\psi\xi}(\eta)\lambda_\psi/m_\psi^2 m_\xi^2 \equiv E = 3A^{-4}\Big\{-[C^2/Q^2 + 1] + \psi^2 + w\xi^2 + (\psi')^2$$
$$+w(\xi')^2 + (\psi^4 + w\xi^4)/2 + A^2 t\psi^2 + A^2(n_\xi/t^3 n_\psi)\xi^2\Big\}, \quad w \equiv n_\xi/t^2 n\psi, \qquad (5.59)$$

where $\epsilon_{\psi\xi}(\eta)$ is the energy density of two scalar fields. We solve the system (5.53) and the $\binom{0}{0}$ component of the Einstein equations numerically with the following values of the parameters:

$$n_\psi = 10^{-1}, \quad n_\xi = -1, \quad t = 10^p(109 - q),$$
$$p = -2, \ldots, -7, \quad q = 10, 11, \ldots, 20, 60, 100. \qquad (5.60)$$

Initial values for ψ and A at $\eta = \eta_v$ we take from section 5.4.2. As to the initial values of ξ and ξ' we use the following considerations. It seems natural to suppose that the wall density ξ^2 at η_v is less than that of particles ψ^2. Besides, the walls arose practically at once, i.e. $d(\xi^2)/d\eta|_{\eta=\eta_v} = 0$. So, we choose that at $\eta = \eta_v$

$$\xi = |\psi/(101-l)| \times 10; \quad \xi' = 0, \quad l = 1, 2, \ldots, 10, 50, 90. \tag{5.61}$$

The constant C^2 is defined from the following. In the contraction of the universe there was one ψ field and its energy density ϵ_ψ at $\eta = \eta_v - 0$ was zero. At the moment $\eta = \eta_v + 0$ already two fields exist. So, C^2 may be defined from the conservation-of-energy condition at given values of the ξ-field. In other words, because of the "conservation law of energy" (in the sense that the energy of the ψ field is transformed into the energy of ψ and ξ fields and the "anisotropic" energy after the moment η_v) an arbitrary choice of domain wall parameters fixed an initial anisotropy "energy".

The solutions are represented in figs. 5.2, 5.3.

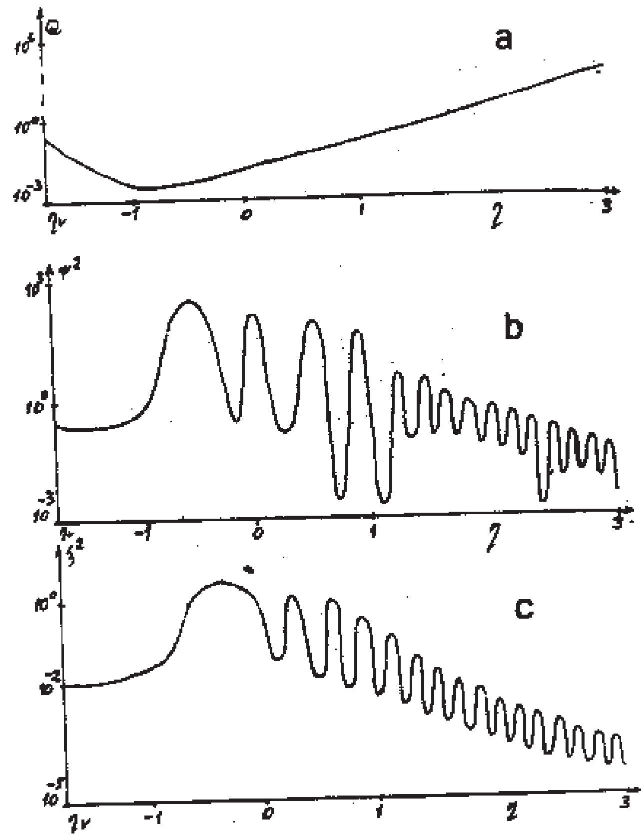

Fig. 5.2

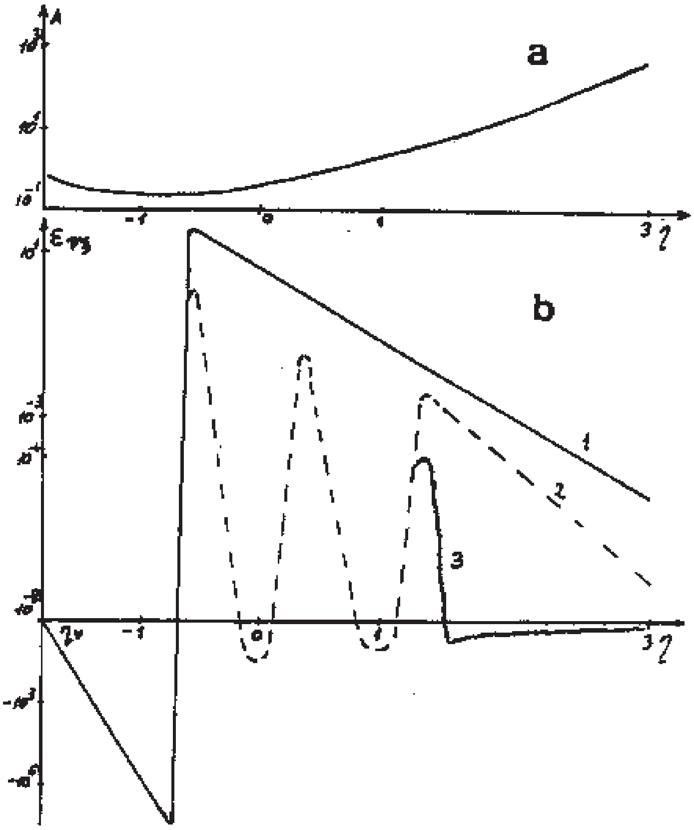

Fig. 5.3

For $t = 0,99 - 0,96$ nonsingular solutions are obtained. From fig. 5.2a and eq. (5.54) we see that the square of the anisotropy rate of growth first increases and then tends to zero as A^{-4}. This means that in our nonsingular models anisotropic disturbances do not grow infinitely in a limited time interval. From figs. 5.2a–2c ones sees that ψ particles and domain wall densities decrease the expansion of the universe. In fig. 5.3b we give three curves which differ essentially at large A or η. This is caused by a different choice of initial values of t and ξ, so depending on this choice the evolution of the system will be such that the energy density may change sign several times and becomes positive (curve 2) or negative (curve 3). If we interpolate variable Λ term then we may say that during the evolution of the universe this Λ term may change several times and its final value may be both positive and negative.

Our model is really nonsingular, because in the proper time t_1 the behavior of the scale factor qualitatively the same as in the conformal time η. From fig. 5.3a one may see that near the minimum of the scale factor A_m

$$A = A_m 1 + [A"(\eta_m)/2A_m](\eta - \eta_m)^2,\ A''(\eta_m) > 0\ .$$

By integrating we obtain in the proper time t_1

$$A = A_m 1 + [A"(\eta_m)/2A_m] m_\psi m_\xi (t_1 - t_{1m})^2, \quad |\eta - \eta_m| < 0,27, \quad -l < \eta_m < -0,05$$

depending on field parameters. In the expansion stage we have for $(\eta_m + 0,27) < \eta < 0,3$:

$$A\alpha(t_1 - t_{1m})^\tau, \quad 1,08 < \tau < 2$$

and for $\eta > 0,3$:

$A\alpha(t_1 - t_{1m})^{1,08}$.

In the small contraction stage from η_v to η_m:

$A \approx -\tau(t_1 - t_{1m}), \quad 0,3 < \tau < 0,57.$

Thus, in the proper time we observe a decrease of the scale factor from the symmetry breaking point up to $\min A$ and then an increase of A. We stress that for $\eta > 0,3$: $A \approx 0,86t_1^{1,08}$, the Hubble constant $H \approx 1,08t_1^{0,92}$, which is close to the linear-expansion law used in ref. [9] and is satisfied by modern observation data. The particle number density ψ^2 oscillates (fig. 2b). Its average over the interval $\eta > 0.3$ is

$\bar{\psi}^2\alpha t_1^{-l,16}$.

Then the total number of particles increases as $N = \bar{\psi}^2 A^3 \alpha t_1^{2,123}$ which is close to Dirac's "large numbers" law. Nonsingular solutions with other nonlinear fields may be found in ref. [46].

5.5. Influence of Higgs Particles on General Cosmological Models with Λ [40]

At the present time, the problem of a singular state in cosmology is of great interest in connection with various attempts to take into account quantum effects in the early stages of evolution of the Universe [9]. In such a case, the conditions under which the well-known singularity theorems [10] hold may be violated, and one then obtains nonsingular cosmological solutions even if the gravitational field is treated at the classical level. The most interesting quantum effects that can in principle prevent a singular state are particle production in a nonstationary gravitational field and the effects of spontaneous symmetry breaking of the physical fields which fill the Universe. This last effect is very interesting, since it is related to the quantum properties of the vacuum.

We shall be interested in the spontaneous breaking of the gauge symmetry of a conformally invariant scalar field and its influence on the metric of homogeneous and isotropic models. In [13] , we considered the case of an open isotropic model. Here, we treat all models in the framework of a unified scheme with allowance for the radiation field and the cosmological constant.

The Lagrangian for a system of a self-interacting scalar field and radiation has the form

$$L = \sqrt{-g}[g^{\mu\nu} \cdot \frac{\partial\varphi^*}{\partial x^\mu}\frac{\partial\varphi}{\partial x_\nu} - (m^2 + R6)\varphi^*\varphi - \lambda(\varphi^*\varphi)^2 6] - L_0 \tag{5.62}$$

We use the Robertson-Walker metric, in which the cosmological proper time τ and the variable η are connected by $dr = ad\eta$:

$$ds^2 = a^2(\eta)[d\eta^2 - dr^2(1 - kr^2) - r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{5.63}$$

and the sign of the curvature tensor is such that for any A_σ: $(\nabla_\mu\nabla_\nu - \Delta_\nu\Delta_\mu)A_\sigma = R^\alpha_{\mu\nu\sigma}A_\alpha$; $c = \hbar = 1$. We assume that the gravitational field satisfies the equations

$$R^\nu_\mu - \delta^\nu_\mu R2 + \delta^\nu_\mu\Lambda = \chi(T^\nu_{(s)\mu} + T^\nu_{(u)\mu}), \tag{5.64}$$

where $T^\nu_{(s)\mu})$ and $T^\nu_{(u)\mu}$ are the energy-momentum tensors of the scalar field and the radiation. $T^\nu_{(s)\mu}$ has the form

$$T^\nu_{(s)\mu} = \nabla_\mu\varphi^*\nabla^\nu\varphi + \nabla^\nu\varphi^*\nabla_\mu\varphi - \delta^\nu_\mu[\nabla^\alpha\varphi^*\nabla_\alpha\varphi - (m^2 + R/6)\varphi^*\varphi]$$
$$-(R^\nu_\mu + \nabla_\mu\nabla^\nu - \delta^\nu_\mu\Box)\varphi^*\varphi/3 + \lambda\delta^\nu_\mu(\varphi^*\varphi)^2/6. \tag{5.65}$$

We restrict ourselves to a matter tensor in the form of radiation, since we are interested in effects at small values of the scale factor, when radiation dominates over the matter. Moreover, it is shown in [47] that the contemporary temperature of the microwave background will be close to the observed value only under the condition that the equation of state in the early stages in the evolution of the Universe had the form $\varepsilon = 3p$. If the radiation and the scalar field do not interact with one another, then the $\binom{0}{0}$ component of Eq. (5.64) is

$$(da/d\eta)^2 - (2\Lambda/3) \cdot a^4 4 + ka^2/2 = -h^2/2, \quad (5.66)$$

$$h^2 = -\chi a^4(\varepsilon_u + \varepsilon_s)/3 = \text{const} > 0, \quad (5.67)$$

in which ε_u and ε_s are the energy densities of the radiation and the scalar field; as can be seen from (5.67), they are assumed to depend on a in accordance with the law

$$\varepsilon_u = \text{const}/a_4, \quad \varepsilon_s = \tilde{\text{const}} a^4, \quad (5.68)$$

which [48] holds for ε_u and a massless scalar field, since $T^{\nu}_{(s)\nu} = 0$. Denoting

$$\Pi = ka^2/2 - (2\Lambda/3)a^4/4, \quad (5.69)$$

we make a qualitative investigation [49,50] of the equation for the scale factor, assuming $h^2 > 0$, which in accordance with [13] holds for spontaneous symmetry breaking in the open model. In what follows, we shall show that this case is also realized in the other models in the massive cases at short times. From (5.66) and (5.69) we obtain the necessary condition of evolution of the scale factor for any k:

$$|\dot{a}_\eta| \equiv |da/d\eta)| = \sqrt{2(-h^2/2 - \Pi)} \geq 0, \quad (5.70)$$

i.e., $\Pi \leq -h^2/2$. We consider open models, i.e., $k = +1$. For this case, the phase plane of Eq. (5.66) is shown in Fig. 5.4.

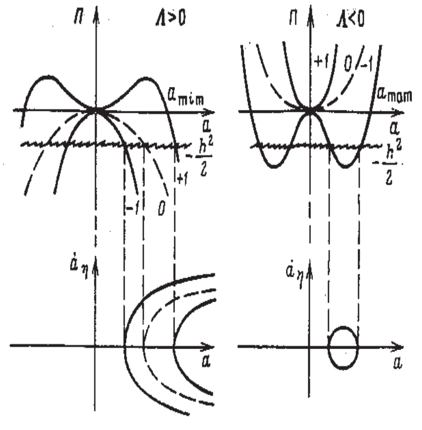

Fig. 5.4. – a) $\Lambda > 0$, b) $\Lambda < 0$. At the top, the potential energy (5.69); $a_{\rm mim} = \sqrt{3/\Lambda}$, $a_{\rm mam} = \sqrt{6/|\Lambda|}$; the numbers of the curves indicate the type of space, $k = -1, 0, 1$. At the bottom, the phase plane of Eq. (5.66).

The graphical construction shows us that for $\Lambda > 0$ (Fig. 5.4a) the scalar field, which gives a negative energy density such that (5.67) is satisfied, ensures a nonsingular cosmology; $\min a$ is determined from the condition

$$\Pi = -h^2/2, \quad (5.71)$$

and minimorum $\min a \equiv a_{\rm mim}\sqrt{3/\Lambda}$ from the condition

$$\Pi = -h^2/2 = 0. \quad (5.72)$$

If $\Lambda < 0$ (Fig. 5.4b), then in such a closed model the presence of a scalar field satisfying the condition (5.67) is impossible. Study of the quasi-Euclidean model with $k = 0$ determines the phase plane shown by the broken curve in Fig. 5.4. As in the case of the closed model, in this case, for $\Lambda > 0$, a nonsingular cosmology is realized, and $\min a$ is also determined in accordance with (5.71); for $\Lambda < 0$, the condition (5.70) is not satisfied, which means that a scalar field realizing the condition (5.67) cannot exist in such a Euclidean model. The investigation of the open model with $k = -1$ (see Fig. 5.4) for $\Lambda > 0$ gives a nonsingular cosmology; as in the two preceding cases, $\min a$ is also determined by (5.71). For $\Lambda < 0$, there is an interesting open cosmology with oscillating scale factor. In this case, the condition (5.71) gives both $\min a$ and $\max a$, and the condition (5.72) both $a_{\rm mim} = 0$ (trivial case) and maximorum $\max a \equiv a_{\rm mam}$.

Thus, we have shown that if the scalar field satisfies the condition (5.67) nonslngular models of four kinds are possible. Going over to the cosmological time τ in Eq. (5.66) and integrating, we obtain the following dependences of the scale factor for these cosmological models:

a) $\Lambda > 0$; $k = l, 0, -l$:

$$a = \sqrt{3/2\Lambda}(|k| + 4\Lambda h^2/3)^{1/2}\,\mathrm{ch}(\sqrt{4\Lambda/3}\tau) + k^{1/2}; \tag{5.73}$$

b) $\Lambda = 0$; $k = -1$, [13]: $a = \sqrt{h^2 + \tau^2}$. It is obvious that tor $\Lambda = 0$ and $k = 1$ and 0 there are no cosmological models. At the same time, $a_{\rm min} = \sqrt{3/2\Lambda}(|k| + 4\Lambda h^2/3)^{1/2} + k^{1/2} > 0$ for all models with $\Lambda > 0$ and $\Lambda = 0$:

$\Lambda < 0; \quad k = -1;$

$$a = \sqrt{3/2|\Lambda|}1 - (1 - 4|\Lambda|h^2/3)^{1/2}\cos(\sqrt{4|\Lambda|/3}\tau)^{1/2}. \tag{5.74}$$

We verify the fulfillment of the condition (5.67), for which we calculate h^2, using the equation for the field φ, which is obtained by varying the action integral for (5.62) with respect to φ and averaging over the vacuum neglecting as in [51,52] the vacuum fluctuations:

$$\Box\upsilon + (m^2 + R/6)\upsilon + \lambda\upsilon^3/3 = 0, \tag{5.75}$$

where $\Box$ is the covariant d'Alembertian, and $\upsilon \equiv <0|\varphi|0>$. The substitution $\upsilon = \sqrt{3/k}f(\eta)a(\eta)$ reduces (5.75) to an equation for the function $f(\eta)$:

$$\ddot{f} + (m^2a^2 + k)f + f^3 = 0. \tag{5.76}$$

Suppose that the cosmological model tilled with the scalar field is nonsingular (our solutions (5.73) and (5.74)), i.e., $a \approx \min a \equiv a_m$ tor small η. In this case, the first integral (5.76) is

$$\dot{f}/2 + \Pi_m = f_m, \tag{5.77}$$

where

$$\Pi_m = (m^2a_m^2 + k)f^2/2 + f^4/4, \tag{5.78}$$

$$f_m = \mathrm{const}. \tag{5.79}$$

The qualitative investigation of (5.77)–(5.79) shows that if

$$m^2 < -k/a_m^2, \tag{5.80}$$

then Eq. (5.77) will have stable nonzero solutions, and this, in its turn, indicates the presence of nonzero vacuum expectation values of the scalar field. As in [1], using (5.65) , we obtain the energy density which the field φ possesses in the asymmetric vacuum state:

$$\varepsilon(\eta) \equiv <0|T_0^0|0> = (3f^2/\lambda a^4)(m^2A^2 + k + f^2/2 + \dot{f}^2/f^2). \tag{5.81}$$

Using (16) for $a \approx a_m$, we obtain

$$f_m = (\lambda/6)a_m^4 \varepsilon(\eta). \tag{5.82}$$

We see that in the massive case (5.82) does indeed reduce to (5.68) at short times. For $m = 0$, (5.68) is always valid. Under the condition $a \approx a_m$, the solution to Eq. (5.76) to first order in η is

$$f(0) = \sqrt{|m^2 a_m^2 + k|}, \quad \dot{f}(0) = 0. \tag{5.83}$$

In this case, the energy density can be determined from (5.81) as

$$\varepsilon \approx -3(m^2 a_m^2 + k)^2 / 2\lambda a_m^4, \quad \eta \ll 1. \tag{5.84}$$

Using (5.67), we can now readily determine h^2, which becomes greater than zero at

$$\lambda < 3(m^2 a_m^2 + k)^2 / 2\varepsilon_{0u} a_0^4, \quad a_0 \sim 20^{28} cm, \quad \varepsilon_{0u} \sim 10^{-31} g/cm^3. \tag{5.85}$$

Thus, we have shown that the presence of the scalar field (5.62) with the self-interaction (5.85) eliminates the singularity in the closed and quasi-Euclidean models if the square of the unrenormalized mass is negative and satisfies (5.81). In the open model, we obtain such a result if $m^2 < 1/a_m^2$. We emphasize that the massless case is admissible only in the open model [13] . It should be noted that in the closed model we found that $a_m = \sqrt{3/\Lambda} \sim 10^{28}$ cm always. Therefore, in the present case the assumption of a dominant role of radiation is incorrect on account of the large initial value of the scale factor, so that it is also necessary to take into account matter, which will be done in the following paper. The assumption of radiation dominance at short times in the open and Euclidean models is correct, since λ is a free parameter of the theory: an appropriate choice of it sufficiently close to the value of (5.85) can make the scale factor of the initial nonsingular state arbitrarily small.

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