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QUANTIZATION OF BIANCHI MODELS IN $N = 1$ SUPERGRAVITY WITH A COSMOLOGICAL CONSTANT

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We study the quantization of some cosmological models within the theory of N=1 supergravity with a positive cosmological constant. We find, by imposing the supersymmetry and Lorentz constraints, that there are *no* physical states in the models we have considered. For the k=1 Friedmann-Robertson-Walker model, where the fermionic degrees of freedom of the gravitino field are very restricted, we have found two bosonic quantum physical states, namely the wormhole and the Hartle-Hawking state. From the point of view of perturbation theory, it seems that the gravitational and gravitino modes that are allowed to be excited in a supersymmetric Bianchi-IX model contribute in such a way to forbid any physical solutions of the quantum constraints. This suggests that in a complete perturbation expansion we would have to conclude that the *full* theory of N=1 supergravity with a non-zero cosmological constant should have *no* physical states.

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I. Introduction

Recently a number of quantum cosmological models have been studied in which the action is that of supergravity, with possible additional coupling to supermatter [1-11,14,15,18,26-29,31-34,38,41]. In addition, a review on this rather fascinating subject is under preparation [12]. It is sufficient, in finding a physical state, to solve the Lorentz and supersymmetry constraints of the theory [13,14]. Because of the anti-commutation relations $\left[S_A,\ \widetilde{S}_{A'}\right]_+ \sim \mathcal{H}_{AA'}$, the supersymmetry constraints $S_A\Psi = 0$, $\overline{S}_{A'}\Psi = 0$ on a physical wave function Ψ imply the Hamiltonian constraint $\mathcal{H}_{AA'}\Psi = 0$ [13,14].

In the case of the Bianchi-I model in $N = 1$ supergravity with no cosmological constant ($\Lambda = 0$) [8,43], the quantum states are in the bosonic and filled fermionic sectors and are of the form $\exp(-\frac{1}{2}h^{-\frac{1}{2}})$, where $h = \det h_{ij}$ is the determinant of the three-metric. In the case of Bianchi IX with $\Lambda = 0$, there are two states, of the form $\exp(\pm I/\hbar)$ where I is a certain Euclidean action, one in the empty and one in the filled fermionic sector [9,15]. When the usual choice of spinors constant in the standard basis is made for the gravitino field, the bosonic state $\exp(-I/\hbar)$ is the wormhole state [9,16]. With a different choice, one obtains the Hartle–Hawking state [15,17]. Similar states were found for $N = 1$ supergravity in the more general Bianchi models of class A [10]. [Supersymmetry (as well as other considerations) forbids mini-superspace models of class B.]

It is of interest to extend these results, by studying more general locally supersymmetric actions, initially in Bianchi models. Possibly the simplest such generalization is the addition of a cosmological constant in $N = 1$ supergravity [19]. On the one hand, the appearance of a cosmological constant term in some supergravity models is a consequence of the coupling to matter. In particular, when one gauges internal SO(2) or SO(3) symmetries in N=2, N=3 *extended* supergravities [20] (coupling the spin-1 fields in a electromagnetic way to the gravitino) one needs at the same time a cosmological constant and a mass-like term for the gravitino in the action [20–24]. Such interesting connection between spin-1 fields and a cosmological constant has led to the suggestion that the electromagnetism might be due to a De Sitter space-time curvature within a supergravity context [20]. Furthermore, the action for our model presented in Eq. (2.1) can be obtained from the O(2)-gauge extended supergravity model when one eliminates from the lagrangian the spin $(\frac{3}{2}$,1)-multiplet while keeping a non-zero gauge coupling constant [25]. But, on the other hand, our model can also be derived as an extension of pure N=1 supergravity [19]. Further, little has been written about extensions of pure supergravity to include R^2 terms, etc. The main idea in this extension is based on the fact that as the presence of a non zero

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cosmological constant induces a constant curvature of space-time independent of matter, the symmetry properties of such spaces will be in correspondence with the De Sitter group rather than the Poincaré group. The new action is determined by the prescription that quantities such as the covariant derivative and curvature terms which are characteristic of the Poincaré group should be replaced in the *field equations* by new ones which are charactristic of the De Sitter group. Following this procedure, one concludes that even in N=1 supergravity one needs a mass-like term for the spin-$\frac{3}{2}$ field if one adds a non-zero cosmological constant even though there are no spin-1 fields.

Using the triad ADM canonical formulation we shall see that there are no physical quantum states in the cases of Bianchi type-I and IX (diagonal) models [26,27]. The calculations are described in Sec. II and include some new corrections in the Bianchi type-I case, allowing to obtain the right bosonic and filled fermionic parts of the wave function [43]. We also treat briefly in Sec. III the spherical $k = +1$ Friedmann model, and find that there is a two-parameter family of solutions of the quantum constraints with a Λ-term. Nevertheless, as will be seen, the Bianchi-IX model provides a better guide to the generic result, since more spin-$\frac{3}{2}$ modes are available to be excited in the Bianchi-IX model, while the form of the fermionic fields needed for supersymmetry in the $k = +1$ Friedmann model is very restrictive [6]. In Sec. IV we briefly comment on how other different approaches from the one presented in the previous sections allows one to extract some similar results [4,5,28,29], [30-34]. Sec. V contains the Conclusion.

II. Quantum States for the Bianchi Models with a Λ-Term

Using two-component spinors [6,14], the action [19] is

$$\begin{aligned} S \;=\; & \int d^4x\big[\left(2\kappa^2\right)^{-1}(\det e)\;\left(R-3g^2\right) \\ & +\; \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}^{A'}{}_{\mu}e_{AA'\nu}D_\rho\psi^A{}_\sigma + H.c.\right) \\ & -\; \frac{1}{2}g(\det e)\;\left(\psi^A{}_\mu e_{AB'}{}^{\mu}e_B{}^{B'\nu}\psi^B{}_\nu + H.c.\right)\big] \end{aligned} \quad (2.1)$$

Here the tetrad is $e^a{}_\mu$ or equivalently $e^{AA'}{}_\mu$. The gravitino field $\left(\psi^A{}_\mu, \overline{\psi}^{A'}{}_\mu\right)$ is an odd (anti-commuting) Grassmann quantity. The scalar curvature R and the covariant derivative D_ρ include torsion. We define $\kappa^2 = 8\pi$. Here g is a constant, and the cosmological constant is $\Lambda = \frac{3}{2}g^2$.

There are two possible approaches to the quantization of this model. One possibility is to substitute the Bianchi Ansatz, e.g.,

$$\begin{aligned} ds^2 = & \left(N^jN_j - N^2\right)dt^2 \\ & +2N_idt\;dx^i + h_{ij}dx^idx^j\;, \end{aligned} \quad (2.2)$$

for the geometry $e^{AA'}{}_\mu$, where all metric components are functions of time, and gravitino field $\left(\psi^A{}_\mu, \overline{\psi}^{A'}{}_\mu\right)$ into the action (2.1). The components $\psi^A{}_\mu e^{BB'\mu}$ and $\overline{\psi}^{A'}{}_\mu e^{BB'\mu}$ are required to be spatially constant with respect to the standard triad [35] on the Bianchi hypersurfaces. One finds that, in order for the form of the Ansatz to be left invariant by one-dimensional local supersymmetry transformations, possibly corrected by coordinate and Lorentz transformations [6], one must study the general non-diagonal Bianchi model [35]. The reduced action could then be computed, leading to the Hamiltonian standard form

$$\begin{aligned} H \;=\; & \widetilde{N}\mathcal{H} + \rho_AS^A + \widetilde{S}^{A'}\tilde{\rho}_{A'} \\ & +\; M_{AB}J^{AB} + \widetilde{M}_{A'B'}\widetilde{J}^{A'B'}. \end{aligned} \quad (2.3)$$

Hence $\mathcal{H}$ is the generator of local time translations, S^A and $\widetilde{S}^{A'}$ are the generators of local supersymmetry transformations, and J^{AB} and $\widetilde{J}^{A'B'}$ are the generators of local Lorentz transformations; they are formed from the basic dynamical variables $\left(e^{AA'}{}_i, \psi^A{}_i, \widetilde{\psi}^{A'}{}_i\right)$. [At this point it is natural in the classical theory to free $\widetilde{\psi}^{A'}{}_i$ from being the hermitian conjugate of $\psi^A{}_i$.] The quantities $\widetilde{N}, \rho_A, \tilde{\rho}_{A'}, M_{AB}$ and $\widetilde{M}_{A'B'}$ are Lagrange multipliers. The supersymmetry and Lorentz constraints are imposed on physical wave functions but they would be complicated because of the number of parameters needed to describe the off-diagonal model.

The other alternative, taken here, is to apply the supersymmetry constraints of the general theory at a Bianchi geometry [9]. This is valid since the supersymmetry constraints are of first order in bosonic derivatives, and give expressions such as $\delta\Psi/\delta h_{im}(x)$ in terms of known quantities and Ψ. These equations can be evaluated in the case of, e.g., a diagonal Bianchi-IX geometry, parametrized by three radii A, B, C. One multiplies (e.g.) by $\delta h_{im}(x) = \partial h_{im}/\partial A$ and integrates $\int d^3x(\quad)$ to obtain an equation for $\partial\Psi/\partial A$ in terms of known quantities. The need to consider off-diagonal metrics is thereby avoided.

In general, it is only necessary to solve the quantum constraints

$$S^A\Psi = 0,\quad \bar{S}^{A'}\Psi = 0,\quad J^{AB}\Psi = 0,\quad \bar{J}^{A'B'}\Psi = 0, \quad (2.4)$$

for a physical state Ψ, since the anti-commutator of S^A and $\bar{S}^{A'}$ includes $\mathcal{H}^{AA'}$, so that Eq. (2.4) implies also $\mathcal{H}^{AA'}\Psi = 0$. The wave function can (e.g.) be taken as $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ or $\widetilde{\Psi}\left(e^{AA'}{}_i, \widetilde{\psi}^{A'}{}_i\right)$. These representations are related by a fermionic Fourier transform [5, 9].

Classically, the supersymmetry constraints are

$$\begin{aligned} S_A \;=\; & gh^{\frac{1}{2}}e_A{}^{A'i}n_{BA'}\psi^B{}_i + \epsilon^{ijk}e_{AB'i}\;{}^{3s}D_j\overline{\psi}^{B'}{}_k \\ & -\; \frac{i}{2}\kappa^2p_{AA'}{}^i\widetilde{\psi}^{A'}{}_i, \end{aligned} \quad (2.5)$$

$$\overline{S}_{A'} = gh^{\frac{1}{2}} e^A{}_{A'}{}^i n_{AB'} \overline{\psi}^{B'}{}_i + \epsilon^{ijk} e_{AA'i}\, {}^{3s}D_j \psi^A{}_k + \frac{i}{2}\kappa^2 \psi^A{}_i p_{AA'}{}^i \ . \quad (2.6)$$

Here $n^{AA'}$ is the spinor version of the unit future-pointing normal n^μ to the constant t surface. It is a function of the $e^{AA'}{}_i$, defined by

$$n^{AA'} e_{AA'i} = 0 \ , \quad n^{AA'} n_{AA'} = 1 \ . \quad (2.7)$$

In Eq. (2.5),(2.6), $p_{AA'}{}^i$ is the momentum conjugate to $e^{AA'}{}_i$. The expression ${}^{3s}D_j$ denotes the three-dimensional covariant derivative without torsion. Since the components of $\psi^A{}_k$ are taken to be constant in the Bianchi basis, one can replace ${}^{3s}D_j\psi^A{}_k$ by $\omega^A{}_{Bj}\psi^B{}_k$, where $\omega^A{}_{Bj}$ gives the torsion-free connection [14].

Quantum-mechanically, in the representation $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$, one has [9,14]

$$p_{AA'}{}^i \longrightarrow -i\hbar \frac{\delta}{\delta e^{AA'}{}_i} + \tfrac{1}{2}\epsilon^{ijk}\psi_{Aj}\bar{\psi}_{A'k} \ , \quad (2.8)$$

where

$$\bar{\psi}^{A'}{}_i \longrightarrow -i\hbar D^{AA'}{}_{ji} h^{\frac{1}{2}} \frac{\partial}{\partial \psi^A{}_j} \ , \quad (2.9)$$

where $\partial/\partial\psi^A{}_j$ denotes left differentiation [13], and

$$D^{AA'}{}_{ji} = -2ih^{-\frac{1}{2}} e^{AB'}{}_i e_{BB'j} n^{BA'} \ . \quad (2.10)$$

We have made the replacement $\delta\Psi/\delta\psi^B{}_j \longrightarrow h^{\frac{1}{2}}\partial\Psi/\partial\psi^B{}_j$. This replacement is important when considering space-time manifolds whose spatial sections are compact. The $h^{\frac{1}{2}}$ factor ensures that each term has the correct weight in the equations, namely when one takes a variation of a (compact) Bianchi geometry, multiplying by $\delta/\delta h_{ij}$ and integrating over the three-geometry (see Eq. (2.21),(2.23), (2.24)). One can check, e.g., that this replacement gives the correct supersymmetry constraints in the $k = +1$ Friedmann model, where the model was quantized using the alternative approach via a supersymmetric Ansatz [6].

The corresponding quantum constraints read, with the help of [14],

$$\overline{S}_{A'}\Psi = -i\hbar g h^{\frac{1}{2}} e^A{}_{A'}{}^i n_{AB'} D^{BB'}{}_{ji}\left(h^{\frac{1}{2}}\frac{\partial\Psi}{\partial\psi^B{}_j}\right) + \epsilon^{ijk} e_{AA'i}\omega^A{}_{Bj}\psi^B{}_k\Psi - \frac{1}{2}\hbar\kappa^2\psi^A{}_i \frac{\delta\Psi}{\delta e^{AA'}{}_i} = 0, \quad (2.11)$$

$$S_A\Psi = gh^{\frac{1}{2}} e_A{}^{A'i} n_{BA'}\psi^B{}_i\Psi - i\hbar\omega_A{}^B{}_i\left(h^{\frac{1}{2}}\frac{\partial\Psi}{\partial\psi^B{}_i}\right) + \frac{i}{2}\hbar^2\kappa^2 D^{BA'}{}_{ji}\left(h^{\frac{1}{2}}\frac{\partial}{\partial\psi^B{}_j}\right)\frac{\delta\Psi}{\delta e^{AA'}{}_i} = 0. \quad (2.12)$$

The constraints $J^{AB}\Psi = 0$, $\bar{J}^{A'B'}\Psi = 0$ imply that $\Psi\left(e^{AA'}{}_i, \psi^A{}_i\right)$ is a Lorentz-invariant function. One solves them by taking expressions in which all spinor indices have been contracted together. As described in [5], it is reasonable also to consider only wave functions Ψ which are spatial scalars, where all spatial indices $i, j, \ldots$ have also been contracted together. To specify this, note the decomposition [11] of $\psi^A{}_{BB'} = e_{BB'}{}^i\psi^A{}_i$:

$$\psi_{ABB'} = -2n^C{}_{B'}\gamma_{ABC} + \frac{2}{3}\left(\beta_A n_{BB'} + \beta_B n_{AB'}\right) - 2\epsilon_{AB} n^C{}_{B'}\beta_C, \quad (2.13)$$

where $\gamma_{ABC} = \gamma_{(ABC)}$ is totally symmetric and ϵ_{AB} is the alternating spinor. The general Lorentz-invariant wave function is a polynomial of sixth degree in Grassmann variables:

$$\begin{aligned}\Psi\left(e^{AA'}{}_i,\ \psi^A{}_i\right) =& \Psi_0\left(h_{ij}\right) + \left(\beta_A\beta^A\right)\Psi_{21}\left(h_{ij}\right) \\ &+ \left(\gamma_{ABC}\gamma^{ABC}\right)\Psi_{22}\left(h_{ij}\right) \\ &+ \left(\beta_A\beta^A\right)\left(\gamma_{BCD}\gamma^{BCD}\right)\Psi_{41}\left(h_{ij}\right) \\ &+ \left(\gamma_{ABC}\gamma^{ABC}\right)^2\Psi_{42}\left(h_{ij}\right) \\ &+ \left(\beta_A\beta^A\right)\left(\gamma_{BCD}\gamma^{BCD}\right)^2\Psi_6\left(h_{ij}\right). \end{aligned} \quad (2.14)$$

Any other Lorentz-invariant fermionic polynomials can be written in terms of these. Note that, for example, the term $\left(\beta^A\gamma_{ABC}\right)^2 = \beta^A\gamma_{ABC}\beta^D\gamma_D{}^{BC}$ can be rewritten, using the anti-commutation of the β's and γ's, as

$$\begin{aligned}&\text{const}\cdot\beta^E\beta_E\epsilon^{AD}\gamma_{ABC}\gamma_D{}^{BC} \\ &= \text{const}\cdot\left(\beta_E\beta^E\right)\left(\gamma_{ABC}\gamma^{ABC}\right). \end{aligned} \quad (2.15)$$

Similarly, any quartic in γ_{ABC} can be rewritten as a multiple of $\left(\gamma_{ABC}\gamma^{ABC}\right)^2$. Since there are only four independent components of $\gamma_{ABC} = \gamma_{(ABC)}$, only one independent quartic can be made from γ_{ABC}, and it is sufficient to check that $\left(\gamma_{ABC}\gamma^{ABC}\right)^2$ is non-zero. Now $\gamma_{ABC}\gamma^{ABC} = 2\gamma_{000}\gamma_{111} - 6\gamma_{100}\gamma_{011}$. Hence $\left(\gamma_{ABC}\gamma^{ABC}\right)^2$ includes a non-zero quartic term const. $\gamma_{000}\gamma_{100}\gamma_{110}\gamma_{111}$. Unlike the case of $N = 1$ supergravity [5], here the nonzero g (or Λ) implies that there is coupling between different fermionic levels.

Quantization of the diagonal Bianchi type-IX model

We now proceed to solve the supersymmetry and Lorentz constraints for the case of a diagonal Bianchi-IX [29], whose three-metric is given in terms of the three radii A, B, C by

$$h_{ij} = A^2E^1{}_iE^1{}_j + B^2E^2{}_iE^2{}_j + C^2E^3{}_iE^3{}_j \ , \quad (2.16)$$

where $E^1{}_i, E^2{}_i, E^3{}_i$ are a basis of unit left-invariant one-forms on the three-sphere [35]. In the calculation, we shall repeatedly need the expression:

$$\omega_{ABi}n^A{}_{B'}e^{BB'j} = \frac{i}{4}\left(\frac{C}{AB}+\frac{B}{CA}-\frac{A}{BC}\right)E^1{}_iE^{1j} + \frac{i}{4}\left(\frac{A}{BC}+\frac{C}{AB}-\frac{B}{CA}\right)E^2{}_iE^{2j} + \frac{i}{4}\left(\frac{B}{CA}+\frac{A}{BC}-\frac{C}{AB}\right)E^3{}_iE^{3j}. \quad (2.17)$$

This can be derived from the expressions for $\omega^{AB}{}_i$ given in [9,13].

First consider the $\overline{S}_{A'}\Psi = 0$ constraint at the level ψ^1 in powers of fermions. One obtains

$$\frac{3}{16}\hbar g h^{\frac{1}{2}} e_{BA'}{}^i \psi^B{}_i \Psi_{21} + \epsilon^{jki} e_{AA'j}\omega^A{}_{Bk}\psi^B{}_i\Psi_0 + \hbar\kappa^2 e_{BA'j}\psi^B{}_i\frac{\delta\Psi_0}{\delta h_{ij}} = 0. \quad (2.18)$$

Since this holds for all $\psi^B{}_i$, one can conclude

$$\frac{3}{16}\hbar g h^{\frac{1}{2}} e_{BA'}{}^i \Psi_{21} + \epsilon^{jki} e_{AA'j}\omega^A{}_{Bk}\Psi_0 + \hbar\kappa^2 e_{BA'j}\frac{\delta\Psi_0}{\delta h_{ij}} = 0. \quad (2.19)$$

Now multiply this equation by $e^{BA'm}$, giving

$$-\frac{3}{16}\hbar g h^{im} h^{\frac{1}{2}}\Psi_{21} + \epsilon^{jki} e_{AA'j}e^{BA'm}\omega^A{}_{Bk}\Psi_0 - \hbar\kappa^2\frac{\delta\Psi_0}{\delta h_{im}} = 0. \quad (2.20)$$

The second term can be simplified using [6]

$$e_{AA'j}e^{BA'}{}_m = -\frac{1}{2}h_{jm}\epsilon_A{}^B + i\epsilon_{jmn}h^{\frac{1}{2}}n_{AA'}e^{BA'n}. \quad (2.21)$$

One then notes, as above, that by taking a variation among the Bianchi-IX metrics, such as

$$\delta h_{ij} = \frac{\partial h_{ij}}{\partial A} = 2AE^1{}_iE^1{}_j\ , \quad (2.22)$$

multiplying by $\delta\Psi_0/\delta h_{ij}$ and integrating over the three-geometry, one obtains $\partial\Psi_0/\partial A$. Putting this information together one obtains the constraint

$$\hbar\kappa^2\frac{\partial\Psi_0}{\partial A} + 16\pi^2 A\Psi_0 + 6\pi^2\hbar gBC\Psi_{21} = 0 \quad (2.23)$$

and two others given by cyclic permutation of ABC.

Next we consider the $S_A\Psi = 0$ constraint at order ψ^1. One uses the relations

$$\partial\left(\beta_A\beta^A\right)/\partial\psi^B{}_i = -n_A{}^{B'}e_{BB'}{}^i\beta^A$$

and

$$\partial\left(\gamma_{ADC}\gamma^{ADC}\right)/\partial\psi^B{}_i = -2\gamma_{BDC}\ n^{CC'}e^D{}_{C'}{}^i$$

and writes out β^A and γ_{BDC} in terms of $e^{EE'}{}_j$ and $\psi^E{}_j$. Proceeding by analogy with the previous calculation, one again 'divides out' by $\psi^B{}_j$. One replaces the free spinor indices AB by the spatial index n on multiplying by $n^A{}_{D'}e^{BD'n}$, then multiplying by different choices $\delta h_{im} = \partial h_{im}/\partial A$ etc. and integrating over the manifold, one finds the constraints

$$\frac{1}{16}\hbar^2\kappa^2A^{-1}\left(A\frac{\partial\Psi_{21}}{\partial A}+B\frac{\partial\Psi_{21}}{\partial B}+C\frac{\partial\Psi_{21}}{\partial C}\right) - \frac{\hbar\kappa^2}{3}\left[3\frac{\partial\Psi_{22}}{\partial A}-A^{-1}\left(A\frac{\partial\Psi_{22}}{\partial A}+B\frac{\partial\Psi_{22}}{\partial B}+C\frac{\partial\Psi_{22}}{\partial C}\right)\right] - 16\pi^2 gBC\Psi_0 - \pi^2\hbar BC\left(\frac{A}{BC}+\frac{B}{CA}+\frac{C}{AB}\right)\Psi_{21} + \frac{1}{3}\left(16\pi^2\right)\hbar BC\left(\frac{2A}{BC}-\frac{B}{CA}-\frac{C}{AB}\right)\Psi_{22} = 0. \quad (2.24)$$

and two more equations given by cyclic permutation of ABC.

Now consider the $\overline{S}_{A'}\Psi = 0$ constraint at order ψ^3. It will turn out that we need go no further than this. The constraint can be written as

$$\frac{1}{2}\hbar g h^{\frac{1}{2}} e^B{}_{A'j}n_C{}^{B'}e_{BB'}{}^j\beta_C\left(\gamma_{DEF}\gamma^{DEF}\right)\Psi_{41} + \epsilon^{ijk}e_{AA'i}\omega^A{}_{Bj}\psi^B{}_k \times\left[\left(\beta_C\beta^C\right)\Psi_{21}+\left(\gamma_{CDE}\gamma^{CDE}\right)\Psi_{22}\right] - \frac{1}{2}\hbar^2\kappa^2\psi^A{}_i\left[\left(\beta_C\beta^C\right)\frac{\delta\Psi_{21}}{\delta e^{AA'}{}_i} + \left(\gamma_{CDE}\gamma^{CDE}\right)\frac{\delta\Psi_{22}}{\delta e^{AA'}{}_i}\right] = 0. \quad (2.25)$$

The terms $\psi^B{}_k$ and $\psi^A{}_i$ in the last two lines can be rewritten in terms of β_A and γ_{FGH}, using Eq. (2.13). Then one can set separately to zero the coefficient of $\beta^C\left(\gamma_{DEF}\gamma^{DEF}\right)$, the symmetrized coefficient of $\gamma_{DEF}\left(\beta_C\beta^C\right)$ and the symmetrized coefficient of $\gamma_{FGH}\left(\gamma_{CDE}\ \gamma^{CDE}\right)$. These three equations give

$$\frac{3}{4}\hbar g h^{\frac{1}{2}}n^C{}_{A'}\Psi_{41} - \frac{8}{3}\epsilon^{ijk}e_{AA'i}\omega^A{}_{Bj}n^B{}_{C'}e^{CC'}{}_k\Psi_{22} + \frac{4}{3}\hbar\kappa^2 n^A{}_{B'}e^{CB'}{}_i\frac{\delta\Psi_{22}}{\delta e^{AA'}{}_i} = 0, \quad (2.26)$$

$$2\epsilon^{ijk}e_{AA'i}\omega^A{}_{Bj}n^D{}_{B'}e^{CB'}{}_k\Psi_{21} - \hbar\kappa^2 n^D{}_{B'}e^{CB'}{}_i\frac{\delta\Psi_{21}}{\delta e^{BA'}{}_i} + (BCD\to CDB) + (BCD\to DBC) = 0 \quad (2.27)$$

and Eq. (2.27) with Ψ_{21} replaced by Ψ_{22}. Contracting Eq. (2.26) with $n_C{}^{A'}$ and integrating over the three-surface gives

$$\frac{3}{4}\left(16\pi^2\right)\hbar gABC\Psi_{41} + \frac{2}{3}\left(16\pi^2\right)\ \left(A^2+B^2+C^2\right)\Psi_{22}$$

$$+\frac{2}{3}\hbar\kappa^2\left(A\frac{\partial\Psi_{22}}{\partial A}+B\frac{\partial\Psi_{22}}{\partial B}+C\frac{\partial\Psi_{22}}{\partial C}\right)=0\ . \quad (2.28)$$

Contracting Eq.(2.27) with $e^{BA'\ell}n_{CC'}e_D{}^{C'N}$, multiplying by $\delta h_{\ell n}=\partial h_{\ell n}/\partial A$ and integrating gives

$$3\hbar\kappa^2\frac{\partial\Psi_{21}}{\partial A}-\hbar\kappa^2A^{-1}\left(A\frac{\partial\Psi_{21}}{\partial A}+B\frac{\partial\Psi_{21}}{\partial B}+C\frac{\partial\Psi_{21}}{\partial C}\right)$$

$$-16\pi^2BC\left(\frac{C}{AB}+\frac{B}{CA}-2\frac{A}{BC}\right)\Psi_{21}=0\ , \quad (2.29)$$

and two more equations given by permuting ABC cyclically. The equation (2.29) also holds with Ψ_{21} replaced by Ψ_{22}.

There is a duality between wave functions $\Psi\left(e^{AA'}{}_i,\psi^A{}_i\right)$ and wave functions $\widetilde{\Psi}\left(e^{AA'}{}_i,\ \widetilde{\psi}^{A'}{}_i\right)$, given by a fermionic Fourier transform [14]. The S_A and $\overline{S}_{A'}$ operators interchange rôles under this transformation, and the rôles of Ψ_0 and Ψ_6, Ψ_{21} and Ψ_{42}, and Ψ_{22} and Ψ_{41} are interchanged. We shall proceed by showing that Ψ_{22}, Ψ_{21} and Ψ_0 must vanish for $g\neq 0$ (or $\Lambda\neq 0$), and hence by the duality the entire wave function must be zero.

Consider first the equation (2.29) and its permutations for Ψ_{21} and Ψ_{22}. One can check that these are equivalent to

$$\hbar\kappa^2\left(A\frac{\partial\Psi_{21}}{\partial A}-B\frac{\partial\Psi_{21}}{\partial B}\right)=16\pi^2\left(B^2-A^2\right)\Psi_{21} \quad (2.30)$$

and cyclic permutations. One can then integrate Eq. (2.30) along a characteristic $AB=$ const., $C=$ const., using the parametric description $A=w_1e^{\tau}$, $B=w_2e^{-\tau}$, to obtain

$$\Psi_{21}=h_1(AB,C)\exp\left[-\frac{8\pi^2}{\hbar\kappa^2}\ \left(A^2+B^2\right)\right]. \quad (2.31)$$

Replacing A,B for B,C in Eq. (2.30) gives the solution

$$\Psi_{21}=h_2(BC,A)\exp\left[-\frac{8\pi^2}{\hbar\kappa^2}\ \left(B^2+C^2\right)\right]. \quad (2.33)$$

Eqs. (2.31) and (2.33) are only consistent if Ψ_{21} has the form

$$\Psi_{21}=F(ABC)\exp\left[-\frac{8\pi^2}{\hbar\kappa^2}\ \left(A^2+B^2+C^2\right)\right]. \quad (2.34)$$

Similarly

$$\Psi_{22}=G(ABC)\exp\left[-\frac{8\pi^2}{\hbar\kappa^2}\ \left(A^2+B^2+C^2\right)\right]. \quad (2.35)$$

Substituting Eqs. (2.34),(2.35) into Eq. (2.24), one obtains

$$16\pi^2g\Psi_0=-2\pi^2\hbar(ABC)^{-1}\left(A^2+B^2+C^2\right)(\exp)F$$
$$+\frac{3}{16}\hbar^2\kappa^2(\exp)F'$$
$$+\frac{2}{3}(16\pi^2)\hbar(ABC)^{-1}(2A^2-B^2-C^2)(\exp)G \quad (2.36)$$

and cyclically, where

$$\exp=\exp\left[-\frac{8\pi^2}{\hbar^2\kappa^2}\ \left(A^2+B^2+C^2\right)\right]\ . \quad (2.37)$$

Now Ψ_0 should be invariant under permutations of A,B,C. Hence $G=0$. I.e.

$$\Psi_{22}=0\ . \quad (2.38)$$

The equation (2.36) and its cyclic permutations, with $\Psi_{22}=0$, must be solved consistently with Eq. (2.23) and its cyclic permutations. Eliminating Ψ_0, one finds

$$\frac{3\hbar^3\kappa^4}{16\,(16\pi^2g)}F''-\frac{\hbar^2\kappa^2}{8g}\ \frac{\left(A^2+B^2+C^2\right)}{ABC}F'$$
$$+6\pi^2\hbar gF-\frac{\hbar^2\kappa^2}{4g}\ \frac{1}{B^2C^2}F$$
$$+\frac{\hbar^2\kappa^2}{8g}\ \frac{\left(A^2+B^2+C^2\right)}{(ABC)^2}F=0 \quad (2.39)$$

and cyclic permutations. Since $F=F(ABC)$ is invariant under permutations, the $(BC)^{-2}F$ term and its permutations imply $F=0$. Thus

$$\Psi_{21}=0. \quad (2.40)$$

Hence, using Eq.(2.36),

$$\Psi_0=0.$$

Then we can argue using the duality mentioned earlier, to conclude that

$$\Psi_{41}=\Psi_{42}=\Psi_6=0\ . \quad (2.41)$$

Hence there are no physical quantum states obeying the constraint equations in the diagonal Bianchi-IX model. This result will be discussed further in Sec.V.

Quantization of the Bianchi type-I model

The same conclusion can be reached for the case of a (non-diagonal) Bianchi type I model. Following Ref. [26], one can use the averaged ordering [5], with

$$p_{AA'}{}^i\bar{\psi}^{A'}{}_i\ \longrightarrow\ \tfrac{1}{2}\left(\bar{\psi}^{A'}{}_ip_{AA'}{}^i+p_{AA'}{}^i\bar{\psi}^{A'}{}_i\right),$$
$$\psi^A{}_ip_{AA'}{}^i\ \longrightarrow\ \tfrac{1}{2}\left(\psi^A{}_ip_{AA'}{}^i+p_{AA'}{}^i\psi^A{}_i\right). \quad (2.42)$$

With this ordering, there is a certain symmetry between the operators S_A and $\bar{S}_{A'}$ as viewed in the two representations $\Psi\left(e^{AA'}{}_i,\psi^A{}_i\right)$ and $\widetilde{\Psi}\left(e^{AA'}{}_i,\widetilde{\psi}^{A'}{}_i\right)$, provided one changes $g\rightarrow -g$. However, the final result is not an artefact of the symmetric factor ordering (2.42) used here. One can repeat the calculations using a general factor ordering

$$p_{AA'}{}^{i}\bar{\psi}^{A'}{}_{i} \longrightarrow \left(\tfrac{1}{2}+s\right)\bar{\psi}^{A'}{}_{i}p_{AA'}{}^{i}+\left(\tfrac{1}{2}-s\right)p_{AA'}{}^{i}\bar{\psi}^{A'}{}_{i}\ ,$$
$$\psi^{A}{}_{i}p_{AA'}{}^{i} \longrightarrow \left(\tfrac{1}{2}-s\right)\psi^{A}{}_{i}p_{AA'}{}^{i}+\left(\tfrac{1}{2}+s\right)p_{AA'}{}^{i}\psi^{A}{}_{i}\ , \tag{2.43}$$

to reach the same conclusion.

Let us begin with the constraint $\bar{S}_{A'}\Psi = 0$ at order ψ^1. This gives [26,43]

$$\left(-\frac{3}{4}\hbar g\Psi_{21}+\frac{4}{3}\hbar\kappa^2 h^{\frac{1}{2}}h_{ij}\frac{\partial\Psi_0}{\partial h_{ij}}-\hbar\kappa^2\Psi_0\right)n_{AA'}\beta^A$$
$$-\tfrac{1}{2}e_{BB'i}n_C{}^{B'}e_{AA'j}\gamma^{ABC}\frac{\partial\Psi_0}{\partial h_{ij}}=0, \tag{2.44}$$

for all β^A and γ^{ABC}. Take the symmetrized coefficient of $\gamma^{ABC} = \gamma^{(ABC)}$ and contract it with $e^{AA'}{}_k n^{BC'} e^C{}_{C'\ell}$ to get

$$\left(3h_{ik}h_{j\ell}-h_{k\ell}h_{ij}\right)\frac{\partial\Psi_0}{\partial h_{ij}}=0\ . \tag{2.45}$$

Since the $\frac{\partial\ h}{\partial\ h_{ij}} = h\ h^{ij}$, the general solution of (2.45) may be taken in the form

$$\Psi_0 = A\ f(u),\ \ u\equiv B\ h^m, \tag{2.46}$$

where A, B, m are constants. Taking now the β^A part of Eq.(2.44) and using Eq. (2.46) we get

$$-\frac{3}{4}\hbar g\Psi_{21}+4\hbar\kappa^2 Af^{'}Bmh^{m+\frac{1}{2}}-\hbar\kappa^2 Af=0, \tag{2.47}$$

where $(')$ denotes a derivative with respect to the u variable. In the case $g = 0$ ($\Lambda = 0$) one would get a bosonic as well as a filled fermionic sector of the form $\exp(-\frac{1}{2}h^{-\frac{1}{2}})$. The other intermediate sectors will be absent [8,43]. However, in the case of $g \neq 0$ one gets as solutions

$$\Psi_{21}=\frac{4}{3}g^{-1}A\kappa^2(4f^{'}Bmh^{m+\frac{1}{2}}-f)\ . \tag{2.48}$$

The constraint $S_A\Psi = 0$ at order ψ^1 can be shown to yield

$$2gh^{\frac{1}{2}}\Psi_0\beta_A-\frac{1}{4}\hbar^2\kappa^2 h^{-\frac{1}{2}}h_{ij}\frac{\partial\Psi_{21}}{\partial h_{ij}}\beta_A$$
$$-\frac{8}{3}\hbar^2\kappa^2 h^{-\frac{1}{2}}\Psi_{22}\beta_A-\frac{9}{8}\hbar^2\kappa^2\Psi_{21}\beta_A$$
$$-\ 4e_{AA'k}D^{BA'}{}_{ji}n_E{}^{C'}e_{DC'}{}^{j}\epsilon_{BC}h^{m+\frac{1}{2}}\frac{\partial\Psi_{22}}{\partial h_{ik}}\gamma^{CDE} \tag{2.49}$$

for all β^A and γ^{CDE}. From the γ^{CDE} part of Eq.(2.49) one gets an equation for Ψ_{22} like Eq.(2.45) and then the solutions Ψ_{22} are of the form of Eq.(2.48). Now let us consider the $\bar{S}_{A'}\Psi = 0$ at order ψ^3. This reads

$$\hbar ge^{B}{}_{A'j}e_{BC'}{}^{j}n_C{}^{C'}\Psi_{41}\beta^C\gamma_{DEF}\gamma^{DEF}$$
$$-\ \hbar\kappa^2 h^{\frac{1}{2}}\frac{8}{3}n^B{}_{C'}e^{EC'}{}_k e_{BA'j}\frac{\partial\Psi_{22}}{\partial h_{jk}}\beta_E\gamma_{ACD}\gamma^{ACD}$$
$$+\ 2\hbar\kappa^2 h^{\frac{1}{2}}n^E{}_{C'}e^{FC'}_{\ \ k}\,e_{BA'j}$$
$$\times\left(\frac{\partial\Psi_{21}}{\partial h_{jk}}\beta^A\beta_A\gamma^B{}_{FE}+\frac{\partial\Psi_{22}}{\partial h_{jk}}\gamma^B{}_{FE}\gamma_{ACD}\gamma^{ACD}\right)$$
$$-\ \tfrac{1}{2}\hbar\kappa^2\Big(\tfrac{1}{2}n_{AA'}\gamma^{ADP}\gamma_{FDP}\beta^F$$
$$-\frac{4}{3}n^P{}_{A'}\beta^D\gamma_{FDP}\beta^F\Big)\Psi_{21}$$
$$-\ \frac{1}{2}\hbar\kappa^2\Psi_{22}\begin{bmatrix}\frac{8}{3}n_{CA'}\beta_B\gamma^B{}_{DA}\gamma^{ACD}\\ -2n^E{}_{A'}\beta_A\gamma_{BDA}\gamma^{ABD}\\ +4n^E{}_{A'}\gamma^B{}_{DA}\gamma^{ACD}\gamma_{CBE}\end{bmatrix}=0 \tag{2.50}$$

for all β^A and γ^{BCD}. Taking the part

$$-4\hbar\kappa^2 h^{\frac{1}{2}}n^E{}_{C'}e^{FC'}_{\ \ k}\,e_{BA'j}\frac{\partial\Psi_{22}}{\partial h_{jk}}\gamma^B{}_{FE}\gamma_{ACD}\gamma^{ACD}$$

$$-2\hbar\kappa^2 n^E{}_{A'}\gamma^B{}_{DA}\gamma^{ACD}\gamma_{CBE}\Psi_{22}=0$$

and substituting the solution for Ψ_{22} of the form of Eq.(2.46), we get that the term in $\gamma^B{}_{FE}\gamma_{ACD}\gamma^{ACD}$ is zero and so $\Psi_{22} = 0$. Using this result back into Eq. (2.49) together with Eq. (2.46) and (2.49) we obtain the following equation for f:

$$2gAhf+\frac{3}{4}\hbar\kappa^4 g^{-1}Af-2\hbar\kappa^4 g^{-1}ABmh^{m+\frac{1}{2}}f^{'}$$
$$-\ 4\hbar\kappa^4 g^{-1}ABm^2h^m f^{'}-4\hbar\kappa^4 AB^2m^2h^{2m+\frac{1}{2}}f''=0. \tag{2.51}$$

Expanding $f, f^{'}$ and $f^{''}$ in a power series with respect to the u-variable and substituting back into Eq.(2.51) one concludes that for the constraint to be satisfied at any moment of time one needs that $A = 0$, this to say that $\Psi_0 = \Psi_{21} = 0$. Using the duality mentioned earlier, we again argue to conclude that $\Psi_{41} = \Psi_{42} = \Psi_6 = 0$. Hence, there are *no* physical states in the Bianchi - I case either.

III. Quantization of the $k = +1$ FRW Model with a Λ-Term

The $k = +1$ Friedmann model without a Λ term has been discussed in [2,6]. There are two linearly independent physical quantum states. One is bosonic and corresponds to the wormhole state [16], the other is at quadratic order in fermions. The Hartle–Hawking state [17] is also found [15]. In the Friedmann model with Λ term, the coupling between the different fermionic levels 'mixes up' this pattern [4].

In the Friedmann model, the wave function has the form [6]

$$\Psi = \Psi_0(A) + \left(\beta_C \beta^C\right) \Psi_2(A) \ . \tag{3.1}$$

As part of the Ansatz of [6], one requires $\psi^A{}_i = e^{AA'}{}_i \widetilde{\psi}_{A'}$ and $\widetilde{\psi}^A{}_i = e^{AA'}{}_i \psi_A$; this is in order that the form of the one-dimensional Ansatz should be preserved under one-dimensional local supersymmetry, suitably modified by local coordinate and Lorentz transformations. Thus the gravitino field is truncated to spin $\frac{1}{2}$. Note that $\beta^A = \frac{3}{4} n^{AA'} \widetilde{\psi}_{A'}$.

One then proceeds as in Sec. II to derive the consequences of the $\overline{S}_{A'}\Psi = 0$ and $S_A \Psi = 0$ constraints at level ψ^1, by writing down the general expression for a constraint and then evaluating it at a Friedmann geometry. Note that it is not equivalent to set $A = B = C$ in Eqs. (2.23) and (2.24); the coefficients in the constraint equations are different. One then obtains

$$\hbar \kappa^2 \frac{d\Psi_0}{dA} + 48\pi^2 A \Psi_0 + 18\pi^2 \hbar g A^2 \Psi_2 = 0 \tag{3.2}$$

and

$$\hbar^2 \kappa^2 \frac{d\Psi_2}{dA} - 48\pi^2 \hbar A \Psi_2 - 256\pi^2 g A^2 \Psi_0 = 0 \ . \tag{3.3}$$

These give second-order equations, for example

$$A \frac{d^2\Psi_0}{dA^2} - 2\frac{d\Psi_0}{dA} + \left[-\frac{48\pi^2}{\hbar\kappa^2} A - \frac{(48)^2 \pi^4}{\hbar^2 \kappa^4} A^3 + \frac{9 \times 512 \pi^4 g^2}{\hbar^2 \kappa^4} A^5 \right] \Psi_0 = 0. \tag{3.4}$$

This has a regular singular point at $A = 0$, with indices $\lambda = 0$ and 3. There are two independent solutions, of the form

$$\begin{aligned} \Psi_0 &= a_0 + a_2 A^2 + a_4 A^4 + \ldots \ , \\ \Psi_0 &= A^3 \left(b_0 + b_2 A^2 + b_4 A^4 + \ldots\right) \ , \end{aligned} \tag{3.5}$$

convergent for all A. They obey complicated recurrence relations, where (e.g.) a_6 is related to a_4, a_2 and a_0.

One can look for asymptotic solutions of the type $\Psi_0 \sim \left(B_0 + \hbar B_1 + \hbar^2 B_2 + \ldots\right) \exp\left(-I/\hbar\right)$, and find

$$I = \pm \frac{\pi^2}{g^2} \left(1 - 2g^2 A^2\right)^{\frac{3}{2}} \ , \tag{3.6}$$

for $2g^2 A^2 < 1$. The minus sign in I corresponds to taking the action of the classical Riemannian solution filling in smoothly inside the three-sphere, namely a portion of the four-sphere S^4 of constant positive curvature. This gives the Hartle–Hawking state [17]. For $A^2 > \left(1/2g^2\right)$, the Riemannian solution joins onto the Lorentzian solution [36]

$$\Psi \sim \cos\left\{ \hbar^{-1} \left[\frac{\pi^2 \left(2g^2 A^2 - 1\right)^{\frac{3}{2}}}{g^2} - \frac{\pi}{4} \right] \right\} \ , \tag{3.7}$$

which describes the de Sitter space-time.

IV. Other Approaches: $\sigma-$Model Supersymmetric Extension and Ashtekar Canonical Quantization

In this section we describe briefly other approaches to study the quantization of cosmological models with supersymmetry, which allows one to extract similar conclusions.

The $\sigma-$model supersymmetric extension in quantum cosmology has been developed by R. Graham and collaborators [4,5,28,29]. As is well known, the geometrodynamics of the Bianchi models reduce, formally, to the Hamiltonian dynamics of a particle with coordinates $q^{\tilde{\mu}}$ in a three or two dimensional potential $V(q^{\tilde{\mu}})$ [35]. In this approach, quantum models are constructed by coupling additional fermionic degrees of freedom to the purely gravitational ones (the minisuperspace vielbein) in such a way that the coupled system acquires a larger symmetry, namely local supersymmetry. The supersymmetric extension of a particle motion in a potential well is treated by supersymmetric quantum mechanics [28]. The case of dynamics on a curved manifold with metric $ds^2 = \tilde{g}_{\tilde{\mu}\tilde{\nu}} dq^{\tilde{\mu}} dq^{\tilde{\nu}}$, where $\tilde{g}_{\tilde{\mu}\tilde{\nu}}$ is the minisuperspace metric, has been studied in the N=2 supersymmetric model $\sigma-$model [28]. Supersymmetry than requires that the potential $V(q^{\tilde{\mu}})$ must be derivable from an underlying superpotential $\Phi(q^{\tilde{\mu}})$ as

$$V(q^{\tilde{\mu}}) = \tfrac{1}{2} \tilde{g}_{\tilde{\mu}\tilde{\nu}} \frac{\partial \Phi}{\partial q^{\tilde{\mu}}} \frac{\partial \Phi}{\partial q^{\tilde{\nu}}} \ . \tag{4.1}$$

Such conditions were verified for the Bianchi type I, II, VII, VIII, IX, Kantowski-Sachs, Taub and Taub-Nut models.

It is important to note that this supersymmetry extension of Hamiltonian dynamics of a particle only leads to a N=2 supersymmetry. By contrast, from (1+3) dimensional N=1 supergravity a dimensional reduction allows one to obtain a (1+0)-dimensional theory with N=4 supersymmetry [28,29]. (The extension of R. Graham's approach to N=4 is non-trivial but such an extension has been provided for the case of a Bianchi type-IX without matter [29]). One may consider the N=2 supersymmetry as a *sub*symmetry of the larger N=4 supersymmetry obtained from supergravity and an attempt to clarify that connection explicitly would be interesting. In particular [28], while in the N=2 supersymmetry extension it is possible to construct solutions in all fermionic sectors, this is different from N=4 supersymmetry minisuperspace models, in which an additional internal rotational symmetry inherited from the Lorentz invariance of supergravity rules out all states except those in the empty and filled fermionic sectors. It is also curious to mention, even if not yet clear, that if the minisuperspace metric is

Wick-rotated (i.e., the scale factor is complexified) this leads to the same restrictions of physical states as the requirement of all the symmetries included in the N=4 supersymmetry models.

The application of the $\sigma-$model supersymmetry extension programme to a general non-diagonal Bianchi type-IX model with a cosmological constant term is given in Refs.[5, 28]. The complete Hamiltonian and the classical constraints of the model are then derived. The system is quantized *à la* Dirac, replacing brackets by commutators or anti-commutators and the canonical momenta by appropriate derivatives with respect to the canonical coordinates. However, this supersymmetry framework was then applied to the particular case of a closed Friedmann-Robertson-Walker case. The wave function of the Universe can be written as

$$|\Psi\rangle = (\Psi_0 + \Psi_a c_a^\dagger + \Psi_{a\Lambda} c_\Lambda^\dagger c_a^\dagger + \Psi_\Lambda c_\Lambda^\dagger)|0\rangle \ , \quad (4.2)$$

where $c_a^\dagger, c_\Lambda^\dagger$ stand for a set of creation operators that replace the fermionic partners in a Fock state representation; $|0\rangle$.

From the N=2 supersymmetry constraints R. Graham obtains that the admissable solution is of the form

$$\Psi_0 = \Psi_{a\Lambda} = 0 \ , \quad (4.3)$$

$$\Psi_{a(\Lambda)}^{WBK} = \left\{(-)1 + \left[1 - \frac{2\Lambda}{9\pi^2}6\pi a^2\right]^{-1/2}\right\}^{1/2} \times \exp\left(-\tfrac{1}{2}\int_0^{6\pi a^2} du\sqrt{1-\frac{2\Lambda}{9\pi^2}u}\right), \quad (4.4)$$

which in the limit $a \to 0$ gives $\Psi_a \to 0$; $\Psi_\Lambda \to \exp(-\Phi)$ where Φ is the superpotential previously described and $\Phi = 3\pi a^2$. For $6\pi a^2 > [2\Lambda/(9\pi^2)]^{-1}$ the exponent in (4.4) is oscillatory. These results point to the wave function as being in the the initial state found by Vilenkin [37], apart from the appearance of additional Grassmann variables.

Now, let us make some brief comments about canonical quantization of supergravity using Ashtekar variables. Recently Ashtekar has presented a new formulation of Einstein gravity. One of the remarkable features of this formalism is that the constraints of gravity are simple polynomials of the canonical variables. So the constraints are more manageable than the ones expressed in the metric representation. One may hope that this feature persists in the canonical quantization of N = 1 supergravity. This turns out to be true: the supersymmetry constraints are again simple polynomials of the canonical variables.

The Ashtekar formalism of the N = 1 supergravity has been formulated by Jacobson [31]. Capovilla and Guven [38] have successfully carried out the quantization of all the Bianchi type A models using this formalism. They obtained similar results to D'Eath [9]. Recently Capovilla and Obregon [33], Sano and Shiraishi [32] have also studied the quantization of the N = 1 supergravity with a cosmological constant. In [32] they found a semi-classical solution in the full theory. It has the form of exponential of the N =1 supersymmetric extension of the Chern-Simons functional. They applied this semi-classical wavefuction to the FRW universe using WKB approximation. The general line element is

$$ds^2 = -d\tau^2 + \frac{\sigma}{8}d^2\Omega \ , \quad (4.5)$$

They have considered four cases: τ : real, $\sigma < 0$; τ : imaginary, $\sigma > 0$; τ : real, $\sigma > 0$; τ : imaginary, $\sigma < 0$. They try to find the classical solution of σ under these 4 cases. In the first case, the universe has the form of an Euclidean hyperbola. In case 2, the universe is a 4-sphere. In case 3, it is an open universe. In the last case, there is no solution.

The reality condition on the gravitino is the Majorana condition. Their solution does not satisfy this condition in general. To obtain the real solution, they must transform the solutions by the transformations corresponding to the symmetries in the theory. They did not solve this problem in [32] and will consider it on another occasion.

In [33], the quantization of the class A Bianchi Models was studied. In the following, we describe briefly the work of Capovilla and Obregón.

The canonical variables are

$$\tilde{\sigma}^i_{AB}, A_i{}^{AB}, \psi_{iA}, \tilde{\pi}^{iA} \ , \quad (4.6)$$

where $A_i{}^{AB}$ is a self-dual connection, $\tilde{\sigma}^i_{AB}$ is a densitized SU(2) soldering form where

$$\tilde{\sigma}^i_{AB} = ih^{\frac{1}{2}} n^{AA'} e^{iB}{}_{A'} \ , \quad (4.7)$$

and $\det(q)q^{ij} = \tilde{\sigma}^{iAB}\tilde{\sigma}^j{}_{AB}$. Here $\tilde{\pi}^{iA}$ is the momentum conjugate to ψ_{iA}.

In class A Bianchi Models, a triad of basis vectors satisfies

$$[X_a, X_b]^i = C_{ab}{}^c X_c^i \ , \quad (4.8)$$

where $C_{ab}{}^d = \epsilon_{abc}M^{cd}$ denote the structure constant and $M^{ab} = M^{ba}$.

The Jacobson phase space variables may be expanded with respect to the triad vectors and their duals χ_i^a

$$A_{iA}{}^B = A_{aA}{}^B \chi_i^a \ , \quad (4.9a)$$

$$\tilde{\sigma}^{iAB} = \det(\chi)\sigma^{aAB} X_a^i \ , \quad (4.9b)$$

$$\psi_i{}^A = \psi_a{}^A \chi_i^a \ , \quad (4.9c)$$

$$\tilde{\pi}^{iA} = \det(\chi)\pi^{aA} X_a^i \ , \quad (4.9d)$$

The fundamental Poisson brackets are given by

$$\{\sigma^{aAB}, A_{bCD}\} = \frac{i}{\sqrt{2}}\delta_a^b \delta_{(C}{}^A \delta_{D)}{}^B \ , \quad (4.10)$$

$$\{\pi^{aA}, \psi_{bB}\} = \frac{i}{\sqrt{2}}\delta_b^a \delta_B{}^A. \quad (4.11)$$

Because of the use of the self-dual connection as a field variable, we need to impose the *reality condition* [39] for the construction of the inner product of the physical wave function. However, we will not need it in here because one finds that the only physical wave function is the trivial one. In quantum theory we get from the triad representation

$$A_a{}^{AB} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\sigma^a{}_{AB}}, \tag{4.12}$$

$$\pi^a{}_A = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\psi_a{}^A}. \tag{4.13}$$

Using this ansatz, one obtains

$$S^A = \frac{1}{2}\frac{\partial^2}{\partial\sigma^a_{AB}\partial\psi_{aA}{}^B} + i2\sqrt{2}m\left(\sigma^a\psi_a\right)^A, \tag{4.14}$$

$$\begin{aligned}\bar{S}^A &= -\frac{1}{2}\epsilon_{abd}M^{dc}(\sigma^a\sigma^b\psi_c)^A \\ &+ \frac{1}{\sqrt{2}}\left(\sigma^{[a}\sigma^{b]}\frac{\partial\psi_b}{\partial\sigma^a}\right)^A - i2mh^{-\frac{1}{2}}\left(\sigma_a\frac{\partial}{\partial psi_{aA}}\right)^A\end{aligned} \tag{4.15}$$

where $\Lambda = -4m^2$.

The last terms in $S^A, \bar{S}^A$ are the contributions from the cosmological constant. The supersymmetry constraints here are simpler than those in the metric formulation. Using the same decompositon of the gravitino field and the same ansatz of the wavefunction, one again obtains a set of equations. The only solution which satisfies this set of equations is the trivial solution: there are no physical states. The simple polynomial form of the constraints suggests that the study of the full theory with a cosmological constant term might be easier in this formalism.

V. Conclusion

The results hereby present as well as the ones in Ref. [26,27,43] and the ones of Capovilla and Obregon [33] indicate that for Bianchi models in $N = 1$ Supergravity with a Λ-term there are **no** physical quantum states. The physical states found in Sec. III for the $k = +1$ Friedmann model, where the degrees of freedom carried by the gravitino field are β_A, disappear when the further fermionic degrees of freedom γ_{ABC} of the Bianchi-IX model are included.

One could also study this from the point of view of perturbation theory about the $k = +1$ Friedmann model. As well as the usual gravitational harmonics [40], gravitino harmonics can be used [41]. For example, the Bianchi-IX model with radii A, B, C close together describes a particular type of 'gravitational wave' distortion of the Friedmann model; similarly for the γ_{ABC} of the Bianchi-IX model, which describes a particular 'gravitino wave' distortion. Quite generally, in perturbation theory [40,42] one expects to find a wave function which is a product of the background wave function $\Psi(A)$ times an infinite product of wave functions ψ_n (perturbations) where n labels the harmonics. And one further expects that the perturbation wave function corresponding to the Bianchi-IX modes must be zero, by a perturbative version of the argument of Sec. II. [It will be interesting to investigate this.] Hence the complete perturbative wave function should be zero; then physical states would be forbidden for a generic model of the gravitational and gravitino fields with Λ-term. This suggests that the full theory of $N = 1$ supergravity with a non-zero Λ-term should have *no* physical states.

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