

MULTIDIMENSIONAL INTEGRABLE COSMOLOGICAL MODELS WITH POSITIVE EXTERNAL SPACE CURVATURE

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Multidimensional cosmological models with n ($n > 1$) spaces of constant curvature are discussed classically in the context of canonical quantization. These models are integrable in the case of Ricci-flat internal spaces. For positive curvature in the external space we find exact solutions modelling dynamical as well as static internal spaces.

1. Introduction

The interest in the Kaluza-Klein idea of geometric unification of interactions actually initiates a lot of work. On the one hand, a consistent formulation of supersymmetry or superstring theory is possible only in a space-time with more than four dimensions. In field theory higher dimensions act on four-dimensional space-time through the generation of particle masses [1 - 3], or the influence on the effective constants of nature (like the constant of gravitation) [4-7]. Higher dimensions can generate observable remnants like cosmic rays of ultrahigh energy [8]. They are also used for a geometric interpretation of internal quantum numbers (like electric charge) [2].

On the other hand, multidimensional models are successfully considered in cosmology. It was shown that such higher dimensional stages may have significant influence on the evolution of our external space. Obviously, the internal dimensions are not observable at the actual state of the universe. Therefore, all multidimensional cosmological models (MCM) have to describe the compactification of the internal dimensions by the present time [1 - 3], [9 - 19]. As a consequence, all MCM can be divided into two different classes. The first class consists of models where from the very beginning the internal dimensions are assumed to be static and of the scale of the Planck length $L_{\text{Pl}} \sim 10^{-33}$ cm. The other class consists of models where the internal dimensions undergo a dynamical evolution like the external space-time; during the evolution of the universe the internal spaces

contract for several orders with respect to the external one. In both classes we find purely gravitational models as well as models with different types of matter (scalar field, electromagnetic field, Yang-Mills field, perfect fluid, one-loop quantum corrections etc.).

Of special interest are exact solutions ([5, 14, 18], [20 - 30]) because they can be used for a detailed study of the evolution of our space, of the compactification of the internal spaces and of the behaviour of matter fields.

One of the most natural MCM generalizing the Friedmann-Robertson-Walker (FRW) universe is given by a toy model with the topology $R \times M_1 \times \dots \times M_n$ where M_i ($i = 1, \dots, n$) denote spaces of constant curvature. One of these spaces, say M_1 , describes the external space but all the others are internal spaces. This model was considered especially in papers [31 - 41].

In [31] a gauge covariant Wheeler-DeWitt equation (WDW) for this model was given. As shown in [34 - 38], the classical and quantum field equations can be integrated for this model if one of the spaces M_i ($i = 1, \dots, n$) is non-Ricci-flat. This integrability takes place also for a MCM with only Ricci-flat spaces and a non-vanishing cosmological constant [39, 41] or filled with perfect fluid [32,40].

In the present paper we consider the integrable case of a MCM where the non-Ricci-flat space is of constant positive curvature. The main aim consists in the study of solutions possessing Planck scales for the internal spaces during the cosmic evolution. If the Ricci-flat spaces describe the internal spaces, they are assumed to be compact. This can be ensured by appropriate boundary conditions. The simplest example of such Ricci-flat spaces is given by d -dimensional tori.

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As a matter source we consider a homogeneous scalar field minimally coupled to gravity. We shall analyze the case of a free scalar field as well as a field with a special potential. In all these cases the models considered are integrable.

The paper is organized as follows. In Sec.2 we describe the MCM and represent it in appropriate coordinates. Sec.3 is devoted to dynamical compactification, Sec.4 to the spontaneous one. Conclusions and an extended list of references complete the paper.

2. Multidimensional Cosmological Models

Let us start with a D -dimensional space-time manifold

$$M = R \times M_1 \times \dots \times M_n \quad (1)$$

containing n spaces of constant curvature M_i with d_i dimensions. M_1 denotes our three-dimensional space but we do not restrict ourself to this special value and use $d_1 > 1$ as dimension. In general the line element is given by

$$ds^2 = g_{AB} dx^A dx^B, \quad A, B = 0, \dots, D-1. \quad (2)$$

We consider the case of homogeneous spaces where the line element is given by

$$ds^2 = -dt^2 \exp(2\gamma(t)) + \sum_{i=1}^n a_i^2(t) g_{(i)} \quad (3)$$

and the $a_i = e^{\beta^i}$ denote the scale factors of the different factor spaces. The metrics of these spaces are given by

$$g_{(i)} = h_{m_i n_i} dx^{m_i} dx^{n_i}, \quad m_i, n_i = 1, \dots, d_i. \quad (4)$$

With the requirement that M_i are spaces of constant curvature we get

$$g_{(i)} = \frac{\sum (dx^{n_i})^2}{(1 + \frac{1}{4} k_i \sum (x^{n_i})^2)^2}. \quad (5)$$

where $k_i = 0, \pm 1$. The scalar curvature of M_i is given by

$$R[g_{(i)}] \equiv \theta_i = k_i d_i (d_i - 1) \quad (6)$$

Matter sources compatible with this symmetry can be described by a homogeneous matter field or phenomenologically by an energy-momentum tensor of a generalized perfect fluid. We consider the case of a minimally coupled homogeneous scalar field φ with the potential $V(\beta, \varphi)$ depending on φ as well as on the β^i . This gives us the possibility to investigate models with an arbitrary scalar field potential $V(\beta, \varphi) \equiv U(\varphi)$ as well as (for $\varphi = \text{const}$) models with an arbitrary potential $V(\beta, \varphi) \equiv V(\beta)$. Effective potentials of the

form $V(\beta)$ may have their origin in perfect fluid matter source [32,40], for compact inner spaces in the "Casimir effect" [11] or in the "monopole" ansatz [42]. Lastly the general form of the potential $V(\beta, \varphi)$ leads us to new integrable models. An example of this kind of potentials will be presented here.

In the case of a free scalar field and only one of the spaces being non-Ricci-flat the multidimensional models are integrable. In what follows we restrict our consideration to the case $\theta_1 > 0$ and $\theta_i = 0$ for all $i = 2, \dots, n$. We have, therefore, a special class of integrable models. The compactness of the flat internal spaces has to be ensured by appropriate boundary conditions.

The action S for the model with the metric (3) and a minimally coupled homogeneous scalar field can be written in the form [31]

$$S = \int \mathcal{L} dt \quad (7)$$

where the Lagrangian reads:

$$\begin{aligned} \mathcal{L} = & \frac{\mu}{2} \exp\left\{-\gamma + \sum_{i=1}^n d_i \beta^i\right\} \\ & \times \left\{ \sum_{j=1}^n d_j (\dot{\beta}^j)^2 - \left(\sum_{j=1}^n d_j \dot{\beta}^j\right)^2 + \kappa^2 \dot{\varphi}^2 \right\} \\ & + \frac{\mu}{2} \exp\left\{\gamma + \sum_{i=1}^n d_i \beta^i\right\} \theta_1 e^{-2\beta^1} \\ & - \mu \kappa^2 V(\beta, \varphi) \exp\left\{\gamma + \sum_{i=1}^n d_i \beta^i\right\}. \end{aligned} \quad (8)$$

Here κ^2 denotes the gravitational constant and $\mu = \prod_{i=1}^n V_i / \kappa^2$ where V_i is the volume of M_i : $V_i = \int_{M_i} d^{d_i} y (\det(g_{m_i n_i}))^{1/2}$. The metric (3) can be normalized in such a way that $\mu = 1$. For example, this can be done by a transformation $\gamma \rightarrow \gamma + \frac{1}{2} (1 - \frac{D}{2})^{-1} \ln \mu$, $\beta^i \rightarrow \beta^i + \frac{1}{2} (1 - \frac{D}{2})^{-1} \ln \mu$. In what follows we use this property. Further we use natural units with $\kappa^2 = 1$.

The model will be analyzed in two different time gauges. In the harmonic time gauge [31] with the time coordinate τ we have $\gamma = \sum_{i=1}^n d_i \beta^i$, and in the synchronous time gauge with coordinate t we have $\gamma = 0$.

A free choice of the time coordinate leads to a constraint equation which reads

$$\begin{aligned} e^{-\gamma} \left[\sum_{i=1}^n d_i (\dot{\beta}^i)^2 - \left(\sum_{i=1}^n d_i \dot{\beta}^i\right)^2 + \dot{\varphi}^2 \right] \\ - e^{\gamma} \theta_1 e^{-2\beta^1} + 2V(\beta, \varphi) e^{\gamma} = 0. \end{aligned} \quad (9)$$

The described cosmological model was already considered in Refs.[34 - 36, 38], where the main aim was the analysis of the quantized system. In the present

paper we intend to consider mainly problems connected with compactification in Kaluza-Klein theory with respect to this model. The model under consideration belongs to the class of integrable models which makes a detailed consideration of these problems possible.

Let us first consider the models where no scalar potential is present: $V(\beta, \varphi) = 0$. It was shown in [34 - 36, 38] that the field equations for this model can be integrated most easily using the following coordinates:

$$\begin{aligned} qv^0 &= (d_1 - 1)\beta^1 + \sum_{i=2}^n d_i \beta^i, \\ qv^1 &= \left[(D-2) / \left(d_1 \sum_2 \right) \right]^{\frac{1}{2}} \sum_{i=2}^n d_i \beta^i, \\ qv^i &= \left[(d_1 - 1) d_i / \left(d_1 \sum_i \sum_{i+1} \right) \right]^{\frac{1}{2}} \\ &\quad \times \sum_{j=i+1}^n d_j (\beta^j - \beta^i), \quad i = 2, \dots, n-1. \end{aligned} \quad (10)$$

Here we used the notations $D = 1 + \sum_{i=1}^n d_i$, $q^2 = (d_1 - 1)/d_1$, and $\sum_i = \sum_{j=i}^n d_j$. The inverse transformation is given by

$$\begin{aligned} (d_1 - 1)\beta^1 &= qv^0 - \left[\frac{\sum_2 (d_1 - 1)}{D - 2} \right]^{\frac{1}{2}} v^1, \\ \beta^{i+1} &= \left[\frac{d_1 - 1}{(D-2)\sum_2} \right]^{\frac{1}{2}} v^1 + \sum_{k=2}^i \left[\frac{d_k}{\sum_k \sum_{k+1}} \right]^{\frac{1}{2}} v^k \\ &\quad - \left[\frac{\sum_{i+2}}{d_{i+1} \sum_{i+1}} \right]^{\frac{1}{2}} v^{i+1}, \quad i = 1, \dots, n-2, \\ \beta^n &= \left[\frac{d_1 - 1}{(D-2)\sum_2} \right]^{\frac{1}{2}} v^1 + \sum_{k=2}^{n-1} \left[\frac{d_k}{\sum_k \sum_{k+1}} \right]^{\frac{1}{2}} v^k. \end{aligned} \quad (11)$$

In the harmonic time gauge the solutions read [35, 36, 38]

$$e^{qv^0} = \frac{\sqrt{\epsilon/\theta_1}}{\cosh[q\sqrt{\epsilon}\tau]}, \quad -\infty < \tau < +\infty \quad (12)$$

and
$$v^i = \nu^i \tau + c^i, \quad i = 1, \dots, n-1 \quad (13)$$

for the v^i and for the scalar field

$$\varphi = \nu^n \tau + c^n. \quad (14)$$

Here we used

$$\epsilon = \sum_{i=1}^n (\nu^i)^2 > 0 \quad (15)$$

and ν^i , c^i , $i = 1, \dots, n$ are constants of integration. In minisuperspace of vectors $\vec{v} = (v^0, v^1, \dots, v^{n-1}, v^n \equiv \varphi)$ the indices are raised and lowered by the diagonal metric $\eta = (-1, +1, \dots, +1)$ [34]. Thus, we have $v^0 = -v_0$, $v^i = v_i$, $\nu^i = \nu_i$ and $c^i = c_i$, $i = 1, \dots, n$.

We can generalize our model to the case of Einstein spaces M_i with $R[g_i] = \lambda_i d_i$ instead of (6) where λ_i are arbitrary constants. In order to get the forthcoming solutions for this general case we have to perform the substitution $\theta_1 \rightarrow \lambda_1 d_1$ in (12). In chapters 3 and 4 we sometimes use the relation $\theta_1 = d_1(d_1 - 1)$ valid for a space M_1 of positive constant curvature ($k = +1$), but all formulas containing this relation can be trivially rewritten for an Einstein space M_1 .

From (11), (12) and (13) it is easy to get explicit expressions for the scale factors $a_i = e^{\beta^i}$ in the harmonic time gauge for arbitrary $n \geq 2$. These formulas show that the dynamical behaviour of the universe is very complicated in the case of arbitrary n . Some of the factor spaces may expand and others contract at the same time. It depends on the signs of the constants ν_i and relationships between coefficients in Eq.(11). A general analysis with arbitrary ν_i and d_i is hardly possible. Each choice should be considered as a separate case. We shall show how to do this on the example of two particular cases. First of all, we shall consider the two-component model, $n = 2$, i.e. the model where we have only one internal space in addition to the external one and, therefore, only two scale factors. In this case it is possible to perform a general analysis for arbitrary ν_1 , ν_2 and d_1 , d_2 . Here, we shall give explicit expressions for the scale factors in the synchronous time widely used in cosmology. We shall show for this case the occurrence of dynamical compactification. This means a cosmological development where one scale factor monotonically increases while the other one remains on a much smaller scale.

Another important special case which will be considered here is the case with static inner factor spaces. This case will be analyzed for arbitrary $n \geq 2$ and an explicit expression for the scale factor of the external (our) space in the synchronous system will be obtained as well.

3. Dynamical Internal Spaces

For the case of a two-component cosmological model, i.e., with $n = 2$ in (11)-(15) we can easily get an expression for the scale factors in harmonic time gauge. We find $a_{1,2}$ in terms of the harmonic time τ :

$$a_1^{d_1-1} = \frac{a_{(0)1}^{d_1-1} \exp\left[-\sqrt{\frac{d_2(d_1-1)}{D-2}} \nu_1 \tau\right]}{\cosh\left[\sqrt{\frac{(d_1-1)(\nu_1^2 + \nu_2^2)}{d_1}} \tau\right]}, \quad (16)$$

$$a_2^{d_2} = a_{(0)2}^{d_2} \exp\left[\sqrt{\frac{d_2(d_1-1)}{D-2}} \nu_1 \tau\right]. \quad (17)$$

Here $a_{(0)1}$ and $a_{(0)2}$ are connected with the constant of integration c_1 by the expressions

$$a_{(0)1}^{d_1-1} = \sqrt{\frac{\nu_1^2 + \nu_2^2}{d_1(d_1-1)}} \exp\left[-\sqrt{\frac{d_2(d_1-1)}{D-2}} c_1\right], \quad (18)$$

$$a_{(0)2}^{d_2} = \exp\left[\sqrt{\frac{d_2(d_1-1)}{D-2}}c_1\right]. \quad (19)$$

From this we get a connection between $a_{(0)1}$ and $a_{(0)2}$

$$a_{(0)1}^{d_1-1}a_{(0)2}^{d_2} = \sqrt{\frac{\nu_1^2 + \nu_2^2}{d_1(d_1-1)}}. \quad (20)$$

We have three different types of development of the scale factors a_1 and a_2 depending on the relation between the constants $\nu_{1,2}$ and the dimensions of the spaces. Let us consider these cases in more detail.

$$1. \quad \sqrt{d_1d_2\nu_1^2} - \sqrt{(d_1+d_2-1)(\nu_1^2 + \nu_2^2)} > 0$$

In this case the scale factors a_1 and a_2 are permanently in opposite phase: either a_2 is contracting from ∞ to 0 and a_1 is expanding from 0 to ∞ (for $\nu_1 < 0$), or a_2 increases from 0 to ∞ and a_1 decreases from ∞ to 0 (for $\nu_1 > 0$). This type of behaviour can take place also in the case where no scalar field is present ($\nu_2 = 0$). It is clear that condition 1 is not valid for $d_2 = 1$, if only ν_2 is not imaginary.

$$2. \quad \sqrt{d_1d_2\nu_1^2} - \sqrt{(d_1+d_2-1)(\nu_1^2 + \nu_2^2)} < 0$$

In this case we have two types of possible behaviours depending on the sign of ν_1 . While the scale factor a_2 is contracting from ∞ to 0, the scale factor a_1 expands up to a maximal value and then starts to shrink up to 0 (for $\nu_1 < 0$). On the other hand, a_2 increases from 0 to ∞ while a_1 increases up to a maximal value and contracts to 0 (for $\nu_1 > 0$). Solutions with analogous behaviour were described earlier in [24 - 26]. Obviously, this case cannot take place for vanishing scalar field ($\nu_2 = 0$). In the classical Kaluza-Klein case with $d_2 = 1$ the condition 2 is satisfied for any real ν_2 .

$$3. \quad \sqrt{d_1d_2\nu_1^2} - \sqrt{(d_1+d_2-1)(\nu_1^2 + \nu_2^2)} = 0$$

In this case we find a connection between the constants of integration ν_1 and ν_2 :

$$\nu_2^2 = \left(\frac{d_1d_2}{d_1+d_2-1} - 1\right)\nu_1^2 \quad (21)$$

From this expression we can see that $d_2 = 1$ corresponds to the case with vanishing scalar field. With (21), the expression (20) takes the form

$$a_{(0)1}^{d_1-1}a_{(0)2}^{d_2} = \sqrt{\frac{d_2}{(d_1-1)(d_1+d_2-1)}}|\nu_1|. \quad (22)$$

The dependence of the scale factors on harmonic time reads

$$a_2^{d_2} = a_{(0)2}^{d_2} \exp\left[\sqrt{\frac{d_1-1}{d_2(D-2)}}\nu_1\tau\right], \quad (23)$$



Figure 1: The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time gauge for case 3, if $\nu_1 > 0$ ($a = 2^{1/(d_1-1)}a_{(0)1}$).

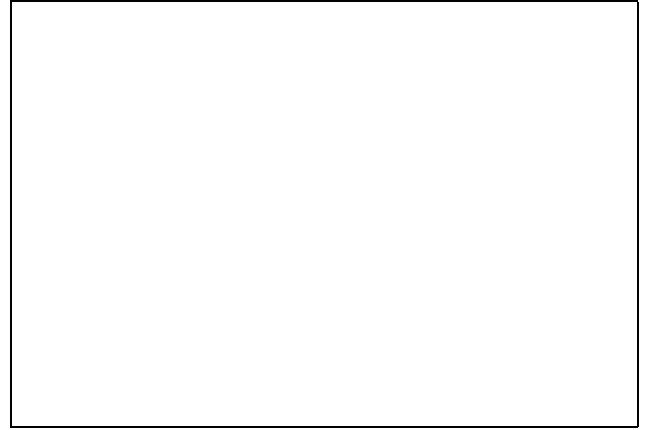


Figure 2: The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time gauge for case 3, if $\nu_1 < 0$ ($a = 2^{1/(d_1-1)}a_{(0)1}$).

$$a_1^{d_1-1} = a_{(0)1}^{d_1-1} \left(1 \pm \tanh \sqrt{\frac{d_2(d_1-1)}{D-2}}|\nu_1|\tau\right) \quad (24)$$

where the upper sign corresponds to the case $\nu_1 < 0$ and the lower sign to $\nu_1 > 0$. The behaviour of the scale factors is shown in Figs.1 and 2. We can see that the phenomenon of dynamical compactification may occur for $\tau > 0$ if $\nu_1 > 0$ and for $\tau < 0$ if $\nu_1 < 0$.

The simple form of Eqs. (23), (24) makes it possible to find an explicit connection between harmonic and synchronous time coordinates. We have the general differential connection

$$dt = \pm e^\gamma d\tau = \pm a_1^{d_1} a_2^{d_2} d\tau \quad (25)$$

We get finally

$$t = \pm \tilde{c} \int \frac{dy}{(y^2 + 1)^{\frac{d_1}{d_1-1}}} + \tilde{c} \quad (26)$$

where

$$y = \exp\left[\pm \sqrt{\frac{d_2(d_1-1)}{D-2}}|\nu_1|\tau\right] \quad (27)$$

(in Eq. (27) the positive sign corresponds to $\nu_1 > 0$, the negative one to $\nu_1 < 0$) and

$$\tilde{c} = \frac{2^{d_1/(d_1-1)}}{d_1-1} a_{(0)1} \quad (28)$$

Let us consider some special cases. If we have $d_1 = 2$, $d_2 \geq 1$, then (26) integrates to give

$$t = \pm \frac{\tilde{c}}{2} \left(\frac{y}{y^2+1} + \arctan y \right), \quad |t| \leq \frac{\tilde{c}\pi}{4}. \quad (29)$$

This case is interesting in the sense that for $d_2 = 1$ this model describes a three-dimensional anisotropic Kantowski-Sachs universe [43] without a scalar field.

A simple expression for the scale factors we find in the most realistic Kaluza-Klein case $d_1 = 3$, $d_2 \geq 1$:

$$t = \pm \tilde{c} \frac{y}{(y^2+1)^{\frac{1}{2}}}, \quad |t| \leq \tilde{c} \quad (30)$$

and with this time coordinate the solution reads

$$\begin{aligned} a_1 &= \sqrt{2} a_{(0)1} \left[1 - \left(\frac{t}{\tilde{c}} \right)^2 \right]^{1/2}, \\ a_2 &= a_{(0)2} \left[\frac{t^2}{\tilde{c}^2 - t^2} \right]^{1/(2d_2)}, \\ \varphi &= \pm \frac{1}{2} \sqrt{\frac{d_2-1}{d_2}} \ln \left(\frac{t^2}{\tilde{c}^2 - t^2} \right) + c_2. \end{aligned} \quad (31)$$

From this solution we see that the point $t = 0$ is a point of maximum for the scale factor a_1 (turning point) and a point of minimum for a_2 (point of repulsion). The derivative da_2/dt is not smooth at $t = 0$. The scale factors a_1 and a_2 are permanently in opposite phase and always time intervals exist where dynamical compactification is realized. On the one hand, a_1 starts to increase from zero at $t = -\tilde{c}$ to some maximum at $t = 0$. At the same time a_2 shrinks from ∞ ($t = -\tilde{c}$) to zero ($t = 0$). This behaviour corresponds to $\nu_1 < 0$. On the other hand, for $\nu_1 > 0$ a_1 monotonically decreases from its maximal value at $t = 0$ to zero at $t = +\tilde{c}$ and a_2 increases from zero ($t = 0$) to ∞ ($t = +\tilde{c}$) (Fig.3). The total space volume is proportional to $V_{3+d_2} = a_1^3 a_2^{d_2} \sim |t|(\tilde{c}^2 - t^2) \rightarrow 0$ if $t \rightarrow 0, \pm\tilde{c}$.

From Eq.(31) it can be seen that the scalar field changes monotonically from $-\infty$ to $+\infty$ as a result of the vanishing potential ($V(\beta, \varphi) = 0$). However, with a nonvanishing potential of the scalar field in the Lagrangian (8), it is in general not possible to separate variables and to integrate the equations.

There is one comparatively easy case when this problem can be overcome. Let us consider the potential

$$V(\varphi, \beta) = U(\varphi) \exp\left(-2 \sum_{i=1}^n d_i \beta^i\right). \quad (32)$$

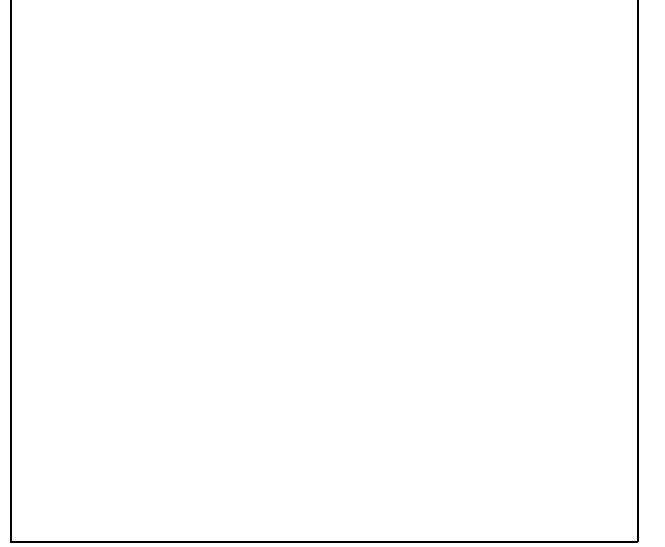


Figure 3: The dynamical behaviour of the scale factors a_1 and a_2 in synchronous time gauge for case 3. with dimensions $d_1 = 3, d_2 \geq 1$. Here we have $t_1 = -\tilde{c}$, $t_2 = \tilde{c}$ and $a = 2^{1/2} a_{(0)1}$.

Then we find for the Lagrangian (8) in v -coordinates and in the harmonic time gauge the expression

$$\mathcal{L} = \frac{1}{2} (\eta_{ik} \dot{v}^i \dot{v}^k + \dot{\varphi}^2) + \frac{1}{2} \theta_1 e^{2qv_0} - U(\varphi) \quad (33)$$

where $\eta_{ik} = \text{diag}(-, +, +, \dots, +)$. The overdot denotes the derivative with respect to the harmonic time and we put $\theta_i = 0$, $i = 2, \dots, n$. In this case we have the possibility to separate the variables and the equations of motion take the form

$$\begin{aligned} \ddot{v}^0 + \theta_1 q e^{2qv_0} &= 0, \\ \ddot{v}^i &= 0, \quad i = 1, \dots, n-1, \\ \ddot{\varphi} + \frac{\partial U}{\partial \varphi} &= 0. \end{aligned} \quad (34)$$

The constraint equation (9) can be written as

$$(\dot{v}^0)^2 + \theta_1 e^{2qv_0} = \epsilon \quad (35)$$

where

$$\begin{aligned} \epsilon &= \sum_{i=1}^{n-1} (\dot{v}^i)^2 + \dot{\varphi}^2 + 2U(\varphi) \\ &= \sum_{i=1}^{n-1} (\nu_i)^2 + \dot{\varphi}^2 + 2U(\varphi). \end{aligned} \quad (36)$$

As before, ν_i denotes the constants of integration of the equations $\ddot{v}^i = 0$ ($i = 1, \dots, n-1$).

Let us now restrict our consideration to a two-component universe (one internal space) and take the potential $U(\varphi) = \frac{1}{2} m^2 \varphi^2$. Then the potential $V(\beta, \varphi)$ can be rewritten in the form $V(\beta, \varphi) = \frac{1}{2} m^2 \Phi^2$, where

$\Phi = \varphi/V_{D'}$ and $V_{D'} = a_1^{d_1} a_2^{d_2}$ is proportional to the volume of the $(D' = d_1 + d_2)$ -dimensional space. For $\varphi = \text{const}$ we have $V(\beta, \varphi) \sim 1/(a_1^{d_1} a_2^{d_2})^2$. This expression is similar to the monopole ansatz of Freund and Rubin [42] but with the difference that their monopole is assumed in the internal space only, but our scalar field is given in the whole universe. Therefore, we have in the expression for $V(\beta, \varphi)$ an additional factor $a_1^{2d_1}$.

With the choice $U(\varphi) = \frac{1}{2}m^2\varphi^2$ we find for Eq. (34) the solution for the scalar field

$$\varphi = \varphi_0 \cos[m(\tau - \tau_0)] \quad (37)$$

where φ_0 and τ_0 are constants of integration.

It can be seen from (34-37) that the behaviour of the scale factors will be the same as in the case $U(\varphi) = 0$ if we put $\nu_2^2 = m^2\varphi_0^2$, because we have once more $\epsilon = \nu_1^2 + \nu_2^2 = \text{const}$. Therefore, for this choice of the potential $U(\varphi)$ we have the same picture for compactification as before. The most simple expression for the scalar field in synchronous time can be obtained for the case 3 (i.e., $\sqrt{d_1 d_2 \nu_1^2} = \sqrt{(d_1 + d_2 - 1)(\nu_1^2 + \nu_2^2)}$) if $d_1 = 3$. Then, we get

$$\varphi = \varphi_0 \cos \left[\frac{m}{2\nu_1} \sqrt{\frac{d_2 + 2}{2d_2}} \ln \frac{t^2}{A(\tilde{c}^2 - t^2)} \right] \quad (38)$$

where \tilde{c} is defined by Eq.(28), $A = \ln[t_0^2/(\tilde{c}^2 - t_0^2)]$ and the constant t_0 corresponds to the harmonic time τ_0 (by Eq.(30)). We can see that the scalar field oscillates. At $t = 0, \pm\tilde{c}$ these oscillations reach infinite frequency.

The quantization procedure changes the constraint equation (35) into the Wheeler-DeWitt equation

$$\left(-\frac{\partial^2}{\partial v^{0^2}} + \theta_1 e^{2qv^0} + \frac{\partial^2}{\partial v^{1^2}} + \frac{\partial^2}{\partial \varphi^2} - m^2 \varphi^2 \right) \Psi = 0 \quad (39)$$

which has the solution

$$\Psi_{\nu_1, l} = e^{i\nu_1 v^1} e^{-\frac{m}{2}\varphi^2} H_l(\varphi\sqrt{m}) C_{i\sqrt{\epsilon}/q} \left(\frac{\sqrt{\theta_1}}{q} e^{qv^0} \right) \quad (40)$$

where H_n are Hermite polynomials and $C = I, K$ denotes the modified Bessel functions. Here we have

$$\begin{aligned} \epsilon &= \nu_1^2 + (2l + 1)m, \\ l &= 0, 1, 2, \dots; \quad -\infty < \nu_1 < +\infty. \end{aligned} \quad (41)$$

In conclusion of this section we would like to mention the existence of a special case of n -component cosmological model ($n > 2$), which can be formally reduced to the two-component case described above. This special case will be given by the following special choice of the constants of integration in Eq. (13):

$$\nu_1 \neq 0, \quad \nu_2 = \dots = \nu_{n-1} = 0.$$

Then we find from the coordinate transformation (11)

$$a_i = e^{B_i} a_2, \quad i = 3, \dots, n \quad (42)$$

where B_i are arbitrary constants and the n -component case is reduced to the two-component one (a_1, a_2) considered above by the replacement of d_2 in Eqs. (16)-(31) by $\sum_2 = \sum_{i=2}^n d_i$. In this particular case all factor spaces $M_i = 3, \dots, n$ have the same dynamical behaviour as M_2 the evolution of which was described above.

4. Static Internal Spaces

Now we turn to models where only one scale factor undergoes dynamical development and will be connected with the scale factor of the external (our) space while all internal space scale factors are fixed. It is usually assumed that these fixed scale factors are of the order of the Planck length $L_{Pl} \sim 10^{-33}$ cm to make internal dimensions in accessible to direct observations.

For the cosmological model under consideration there are solutions of this type corresponding to the special case $\nu_i = 0, (i = 1, \dots, n-1)$. Then we have from (11), (13) $\beta^i = \text{const}$ ($i = 2, \dots, n$), i.e. $a_i = e^{\beta^i} = a_{(0)i} = \text{const}$ ($i = 2, \dots, n$). In this case we have $\epsilon = \nu_n^2 > 0$ and Lorentzian solutions occur only in the presence of a real scalar field $\varphi = \nu_n \tau + c_n$. In the harmonic time gauge the dynamical scale factor behaves like

$$a_1(\tau) = a_{10} \left\{ \cosh \left[(d_1 - 1) \sqrt{\epsilon/\theta_1} \tau \right] \right\}^{-1/(d_1-1)} \quad (43)$$

for $-\infty < \tau < +\infty$, $c = \prod_{i=1}^n a_{(0)i}^{d_i}$ and $a_{10} = \left[\sqrt{\epsilon/\theta_1}/c \right]^{1/(d_1-1)}$.

It is convenient to write the metric (3) in the gauge of conformal time η , connected with the harmonic time τ by the relation

$$\begin{aligned} \cosh \left[(d_1 - 1) \sqrt{\epsilon/\theta_1} \tau \right] &= [\sin(d_1 - 1)\eta]^{-1}, \\ 0 &\leq (d_1 - 1)\eta \leq \pi \end{aligned} \quad (44)$$

Then we have

$$ds^2 = a_1^2(\eta)(-d\eta^2 + g_{(1)}) + a_{(0)2}^2 g_{(2)} + \dots + a_{(0)n}^2 g_{(n)} \quad (45)$$

and the scale factor a_1 depends on the conformal time η in the following way:

$$a_1(\eta) = a_{10} (\sin[(d_1-1)\eta])^{1/(d_1-1)} \quad (46)$$

From (46) it can be seen that for $d_1 = 2$ the scale factor a_1 behaves just as in a closed radiation dominated Friedmann universe, and for $d_1 = 3$ as in a closed Friedmann universe filled with ultra-stiff matter. This example shows once more that extra dimensions may

play the role of a matter source for the external space-time.

We also give an explicit expression for the metric in the synchronous frame:

$$ds^2 = -dt^2 + a_1^2(t)g_{(1)} + a_{(0)2}^2 g_{(2)} + \dots + a_{(0)n}^2 g_{(n)}. \quad (47)$$

The dependence of the scale factor a_1 on the synchronous time is given by

$$t = \int \frac{a_1^{d_1-1} da_1}{\sqrt{\epsilon/(\theta_1 c^2) - a_1^{2(d_1-1)}}} + \text{const.} \quad (48)$$

All these formulas (43), (46), and (48) show that a_1 expands from zero to its maximal value a_{10} and shrinks to zero again. For $a_1 \ll a_{10}$ the scale factor asymptotically behaves like $a_1 \sim t^{1/d_1}$ which corresponds to a closed Friedmann universe filled with radiation for $d_1 = 2$ and ultra stiff matter for $d_1 = 3$.

In the case $d_1 = 2$ the integral (48) can be expressed in elementary functions:

$$a_1 = \sqrt{t(2t_0 - t)}, \quad 0 \leq t \leq 2t_0 \equiv 2\sqrt{\epsilon/\theta_1}/c \quad (49)$$

for $d_1 \geq 3$ it can be expressed in elliptic integrals. For instance, in the case $d_1 = 3$ we have

$$t = \sqrt{t_0} \left\{ \sqrt{2} \left[E\left(\Psi, \frac{\sqrt{2}}{2}\right) - E\left(\frac{\sqrt{2}}{2}\right) \right] - \frac{1}{\sqrt{2}} \left[F\left(\Psi, \frac{\sqrt{2}}{2}\right) - K\left(\frac{\sqrt{2}}{2}\right) \right] \right\} \quad (50)$$

where

$$\Psi = \arccos [a_1/\sqrt{t_0}] \quad (51)$$

and the constant of integration is chosen in such a way that $a_1(t=0) = 0$. It is clear that a potential of the form (32) (with $U(\varphi) = \frac{1}{2}m^2\varphi^2$) would not change the formulas (43) - (51) where we would have to put $\nu_n^2 = m^2\varphi_0^2 = \text{const.}$

The solution (43) has an interesting analytic continuation into the Euclidean region. This solution has the topology of a wormhole [35, 36, 38]. The analytic continuation of the solution (46) symmetric with respect to $\eta = 0$ (this corresponds to the symmetry with respect to the throat of the wormhole in the Euclidean region [44]) leads to

$$a_1(\eta) = a_{10} [\cosh(d_1 - 1)\eta]^{1/(d_1-1)}, \quad -\infty < \eta < +\infty. \quad (52)$$

Now we show that this solution describes indeed a wormhole, that means two asymptotically flat regions connected by a throat [45, 46]. To this end we change to a new "time" coordinate T by the following transformation:

$$a_1(\eta)d\eta = \frac{1}{(d_1 - 1)a_1^{d_1-2}} dT \quad (53)$$

(we see from (53) that the "time" T is the synchronous one in the case $d_1 = 2$). We put (52) into (53) and find

$$T = t_0 \sinh(d_1 - 1)\eta, \quad -\infty < T < +\infty \quad (54)$$

and

$$a_1(T) = [t_0^2 + T^2]^{1/[2(d_1-1)]}. \quad (55)$$

From (55) we can see that the throat has the size $a_{10} = t_0^{1/(d_1-1)}$. In the T-time gauge the metric takes the form

$$ds^2 = \frac{dT^2}{(d_1 - 1)^2 [t_0^2 + T^2]^{(d_1-2)/(d_1-1)}} + [t_0^2 + T^2]^{1/(d_1-1)} g_{(1)} + \sum_{i=2}^n a_{(0)i}^2 g_{(i)}. \quad (56)$$

Asymptotically we find for $T^2 \gg t_0^2$

$$ds^2 \approx \frac{dT^2}{(d_1 - 1)^2 T^{2(d_1-2)/(d_1-1)}} + T^{2/(d_1-1)} g_{(1)} + \sum_{i=2}^n a_{(0)i}^2 g_{(i)}. \quad (57)$$

If we go over to a new coordinate $R = |T|^{1/(d_1-1)}$ the asymptote (57) takes the form

$$ds^2 \approx dR^2 + R^2 g_{(1)} + \sum_{i=2}^n a_{(0)i}^2 g_{(i)}. \quad (58)$$

In this way we get the following result. If $g_{(1)}$ represents the d_1 dimensional sphere the Euclidean region has two asymptotic regions ($T \rightarrow \pm\infty$) with the topology $R^{d_1+1} \times M_2 \times \dots \times M_n$ which are connected by a throat of the size a_{10} . This is a wormhole by the definition of this object.

It is useful to express the metric (56) in the synchronous reference system

$$ds^2 = dt^2 + a_1^2(t)g_{(1)} + \sum_{i=2}^n a_{(0)i}^2 g_{(i)} \quad (59)$$

where the scale factor a_1 depends on the synchronous "time" coordinate due to the expression

$$t = \int \frac{a_1^{d_1-1} da_1}{\sqrt{a_1^{2(d_1-1)} - t_0^2}} + \text{const.} \quad (60)$$

In the special case $d_1 = 2$ this integral can be expressed in elementary functions

$$a_1^2(t) = t^2 + t_0^2 \equiv t^2 + \frac{\epsilon}{\theta_1 c^2}, \quad -\infty < t < +\infty. \quad (61)$$

This follows directly from (56). We can get (61) as well by analytic continuation of the expression (49)

symmetric with respect to $t = 0$. Such type of wormhole solutions was presented by Hawking in [47]. For $d_1 > 2$ the integral (60) can be expressed in elliptic functions. In the case $d_1 = 3$ we find

$$t = \sqrt{t_0} \left\{ -\sqrt{2} E \left(\Psi, \frac{\sqrt{2}}{2} \right) + \frac{1}{\sqrt{2}} F \left(\Psi, \frac{\sqrt{2}}{2} \right) \right\} + \frac{1}{a_1} \left[a_1^4 - t_0^2 \right]^{1/2} \quad (62)$$

where

$$\Psi = \arccos \left[\sqrt{t_0/a_1} \right]. \quad (63)$$

Such wormholes were studied by Giddings and Strominger in [48].

5. Conclusions

We considered integrable multidimensional cosmological models (MCM) with n ($n > 1$) spaces of constant curvature. Only one space was of positive curvature, all the others were assumed to be Ricci-flat. As a matter source we introduced a minimally coupled homogeneous scalar field as a free field or with a special type of potential. It was shown that this models possess solutions describing the process of dynamical compactification of internal dimensions which takes place in the presence of a scalar field and for pure gravity.

The class of solutions with static internal spaces was found as well. In this case only one scale factor undergoes dynamical development and it will be connected with the scale factor of the external space, while all internal space scale factors are fixed. The values of fixed scale factors are free parameters of the theory. It is usually assumed that they are of the order of the Planck length to make them unobservable. For this class of solutions the presence of a real scalar field in the Lorentzian region is a necessary condition. For the case with an imaginary scalar field or without any scalar field our solution belongs completely to the Euclidean region. In the case of a real scalar field the solutions with static internal spaces permit an interesting continuation to the Euclidean region describing Euclidean wormholes.

For the quantized model the Wheeler-DeWitt (WDW) equation was analyzed for the case with a scalar field potential like that described above. The corresponding exact solutions were found. The WDW equation with a free scalar field was considered earlier in [34-36, 38] where exact solutions were presented and analyzed. In this case a special class of solutions satisfies the boundary conditions of quantum wormholes [49].

Of interest is the case where the non-Ricci-flat space is of negative curvature. It is easy to see from the field equations that in this case we have a more rich situation than in the case described in this paper.

There are solutions which belong completely to the Lorentzian region and do not possess an analytic continuation to the Euclidean one, as well as solutions which have such a continuation. Usually, the latter class is connected with models of tunneling universes. A consideration of this case and an analysis of the compactification problem in it will be contained in a separate paper.

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