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INTEGRABLE WEYL GEOMETRY IN COSMOLOGY

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Vacuum cosmological models are considered in the context of a gravitation theory based on integrable Weyl geometry. A family of exact solutions with a chain of internal spaces is obtained. Models with one internal space are considered in more detail; nonsingular models are selected and their dynamics is illustrated graphically. Their Eucledean counterparts are also considered; a correspondence betweeen nonsingular Lorentzian models and Euclidean wormholes is demonstrated.

1. Multidimensional theories of gravity are very attractive in the context of the unification of fundamental interactions. Moreover, most of modern unification theories require space-time to have more than four dimensions [1–8]. The unobservability of extra dimensions in such theories is often explained by dynamical contraction of internal manifolds in the course of the Universe expansion [9–13]. To realize such a mechanism one or more additional scalar fields are usually introduced and the problem of its origin appears. This problem may be more naturally solved in space-time models with generalized geometries where such fields have a geometric nature.

One of the simplest generalizations of Riemannian geometry is integrable Weyl geometry with the connection components

$$\Gamma^A_{BC} = \widetilde{\Gamma}^A_{BC} - \frac{1}{2}\left(\omega_B \delta_{CA} + \omega_C \delta^A_B - g_{BC}\omega^A\right) \qquad (1)$$

where $\widetilde{\Gamma}^A_{BC}$ are the Christoffel symbols, $\omega_A = \omega_{,A}$, ω is a scalar field, δ^A_B are the Kronecker symbols, g_{AB} is the metric tensor of the D-dimensional ($D \geq 4$) space-time; capital Latin indices range from 0 to $D-1$. The curvature scalar corresponding to the connection (1) is

$$R = \widetilde{R} + (D-1)\widetilde{\Box}\omega - \frac{(D-1)(D-2)}{4}\omega^C\omega_C \qquad (2)$$

where the tildes denote quantities calculated with the connection $\widetilde{\Gamma}^A_{BC}$ and $\widetilde{\Box}$ is the d'Alembert operator with this connection.

Thus, the gravitational field in a Weyl integrable space-time (WIST) is determined by the tensor g_{AB} and the scalar ω, just as in scalar-tensor theories (STT) of gravity. The difference between these two cases is determined by Eq. (1). Namely, both in STT and in WIST there is a conformal gauge in which test particles move along geodesics; however, in WIST, unlike STT, even in this frame the motion in general depends on both the metric and the scalar field. Thus a gravitation theory on the basis of WIST is in general not a special case of STT due to a nonminimal coupling between the matter and the scalar field.

However, field equations in STT and WIST-based theories in many cases coincide, in particular, for all vacuum space-times.

The description of cosmological models in STT is often reduced to that of Einsteinian cosmologies with scalar fields. The latter were considered by many authors [9,12,14–22] in both 4-dimensional and (4+d)-dimensional space-times. Cosmological models in 4-dimensional WIST were recently considered by Novello et al. [23], where the existence of nonsingular open cosmologies was demonstrated.

In this paper we consider the evolution of multidimensional cosmological models based on integrable Weyl geometry. We begin with finding exact solutions for some simplest cases of empty spaces. The main characteristic features of the solutions are illustrated graphically. Keeping in mind the possible applications of the results to the description of quantum stages of the universe evolution we also consider WIST with the Euclisean signature. After that we investigate numerically some models with a nonminimal scalar field as toy models of a nonempty WIST. Some 4-dimensional models are considered for comparision.

2. As is the case with STT, the gravitational field Lagrangian may in general contain various invariant combinations of g_{AB} and ω. Let us restrict ourselves to Lagrangians which are (a) linear in the scalar curvature and (b) quadratic in ω_A. Then the general form of the Lagrangian satisfying (a) and (b) is

$$L = A(\omega)R + B(\omega)\omega^A\omega_A - 2\Lambda(\omega) + L_m \qquad (3)$$

where R is the Weyl scalar curvature corresponding to

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the connection (1), A, B and Λ are arbitrary functions and L_m is the nongravitational matter Lagrangian.

Using the expression (2) for R in terms of the Riemannian curvature $\widetilde{R}$ corresponding to the metric g_{AB}, the conformal mapping well-known in STT [25], modified for D dimensions [22, 26]:

$$g_{MN} = A^{-2/(D-2)}\overline{g}_{MN}. \tag{4}$$

and omitting a total divergence, we obtain the following form of the Lagrangian:

$$\begin{aligned}\overline{L} = A(\omega)\overline{R} + F(\omega)\overline{g}^{AB}\omega_A\omega_B \\ +A^{-D/(D-2)}[-2\Lambda(\omega) + L_m]\end{aligned} \tag{5}$$

where

$$\begin{aligned}F(\omega) = \frac{1}{A(\omega)^2}\Big[A(\omega)B(\omega) \\ -(D-1)A(\omega)\Big(A_\omega + \frac{D-2}{4}\Big) + \frac{D-1}{D-2}A_\omega^2\Big].\end{aligned} \tag{6}$$

3. Let us consider vacuum cosmological models with the following structure of the space-time W_D:

$$W_D = \mathrm{R} \times M_1 \times \ldots \times M_n; \quad \dim M_i = N_i; \tag{7}$$

where the subspaces M_i are assumed to be maximally symmetric. The component R corresponds to the time τ; besides, we assume $\omega = \omega(\tau)$. Thus, the effective Riemannian metric is written in the form

$$d\overline{s}^2 = \overline{g}_{AB}dx^A dx^B = \mathrm{e}^{2\gamma(\tau)}d\tau^2 - \sum_{i=1}^{n}\mathrm{e}^{2\beta_i(\tau)}ds_i^2 \tag{8}$$

where ds_i^2 are τ-independent metrics of the N_i-dimensional spaces of constant curvatures K_i; with no loss of generality one can put $K_i = 0,\ \pm 1$.

Making use of the freedom to choose the time coordinate τ, let us introduce the harmonic time by putting

$$\gamma = \sum_{i=1}^{n} N_i\beta_i. \tag{9}$$

Then the Ricci tensor for $\overline{g}_{AB}$ has the following nonzero components:

$$\begin{aligned}\overline{R}^\tau_\tau &= \mathrm{e}^{-2\gamma}\Big(\ddot{\gamma} - \dot{\gamma}^2 + \sum_{i=1}^{n} N_i\dot{\beta}_i^2\Big), \\ \overline{R}^{m_i}_{n_i} &= \delta^{m_i}_{n_i}\Big[\mathrm{e}^{-2\gamma}\ddot{\beta}_i + (N_i-1)K_i\mathrm{e}^{-2\beta_i}\Big]\end{aligned} \tag{10}$$

where the indices $m_i,\ n_i$ belong to the subspace M_i.

4. The field equations take an especially simple form under the additional condition $\Lambda \equiv 0$:

$$\overline{R}_{MN} + F(\omega)\omega_M\omega_N = 0, \tag{11}$$

$$2\overline{\nabla}_M[F(\omega)\omega^M] - F_\omega\omega^M\omega_M = 0. \tag{12}$$

They can be integrated completely under one of the above assumptions: (i) if all the subspaces M_i are Ricci-flat and (ii) if one of M_i (for instance, M_1) is a space of nonzero constant curvature (K_1). Indeed, putting $K_i = 0$ ($i > 1$), we obtain:

$$(F\dot{\omega}^2)^{\cdot} = 0 \quad \Rightarrow \quad F\dot{\omega}^2 = S = \text{const}; \tag{13}$$

$$\ddot{\beta}_i = 0 \quad \Rightarrow \quad \beta_i = \beta_{i0} + h_i\tau,\ i > 1; \tag{14}$$

$$\ddot{\gamma} - \ddot{\beta}_1 = -K_1 d^2\mathrm{e}^{2\gamma - 2\beta_1} \tag{15}$$

where $d+1 = N_1 = \dim M_1$. Eq.(15) leads to different results for different K_1: for $K_1 = 0$ (case (i)) Eq.(14) may be regarded to include $i = 1$; for $K_1 \neq 0$ (case (ii)) we get:

$$\mathrm{e}^{\beta_1 - \gamma} = \frac{d}{k}\cosh k\tau, \quad k > 0 \quad (K_1 = +1), \tag{16}$$

$$\begin{aligned}\mathrm{e}^{\beta_1-\gamma} &= d\cdot s(k,\tau) \\ &\equiv \begin{cases} (d/k)\sinh k\tau, & k > 0, \\ d\cdot\tau, & k = 0, \quad (K_1 = -1) \\ (d/k)\sin k\tau, & k < 0, \end{cases}\end{aligned} \tag{17}$$

where $k = $ const and another integration constant is eliminated by a particular choice of the origin of τ. Lastly, a combination of components of (11) representing the temporal component of the Einstein equations (the initial data equation) leads to the following relation among the integration constants:

$$\Big(\sum_{i=1}^{n} N_i h_i\Big)^2 - \sum_{i=1}^{n} N_i h_i^2 = S, \quad K_1 = 0; \tag{18}$$

$$\begin{aligned}\frac{d+1}{d}k^2\text{sign }k = \frac{1}{d}\Big(\sum_{i=2}^{n} N_i h_i\Big)^2 + \sum_{i=2}^{n} N_i h_i^2 + S, \\ K_1 \neq 0.\end{aligned} \tag{19}$$

Thus, the set of equations (11),(12) has been integrated in quadratures.

As the original functions $A(\omega)$ and $B(\omega)$ and hence $F(\omega)$ are arbitrary, it is difficult to describe the physical properties of the models in a general form. Therefore, here we would like to restrict ourselves to some simple special cases.

Thus, we will assume $A \equiv 1$ while $B(\omega)$ remains arbitrary, so that the metrics $\overline{g}_{AB}$ and g_{AB} coincide.

5. As the first step consider 4-dimensional homogeneous isotropic cosmologies. For this purpose we must put $n = 1$, $d = 2$, $\beta_1 \equiv \beta(\tau)$. The condition that τ is a harmonic coordinate takes the form $\gamma = 3\beta$ and for the scale factor we get:

$$\mathrm{e}^{-2\beta} = a^{-2}(\tau) = \begin{cases} 2s(k,\tau), & K_1 = 1, \\ \mathrm{e}^{k\tau}, & K_1 = 0, \\ 2\cosh k\tau, & K_1 = -1 \end{cases} \tag{20}$$

where $s(k,\tau)$ is defined by (17) and the physical time is determined by the integral $t = \pm\int \mathrm{e}^{\gamma(\tau)}d\tau$. The constant k is connected with the “scalar charge” S according to (18), (19) where one should substitute $h_i = 0$ $(i > 1)$ and $h_1 = k/2$:

$$2S = \begin{cases} 3k^2 \operatorname{sign} k, & K_1 = \pm 1, \\ 3k^2, & K_1 = 0. \end{cases} \tag{21}$$

It is easy to obtain that in the case of a spherical world ($K_1 = 1$) the values $\tau = \pm\infty$ correspond to finite times t_1 and t_2 at which $a = 0$ (the initial and final singularities). For a flat world ($K_1 = 0$) at $k \neq 0$ and a hyperbolic one ($K_1 = -1$) at $k > 0$ an initial or final singularity is observed at infinite τ. In the special case $K_1 = -1,\ k = 0$ we obtain the Milne vacuum model which is known to describe a domain in flat space-time (in this case $S = 0$, so that the scalar field is trivial).

Lastly, in the case $K_1 = -1,\ k < 0$ we see that the limits $\tau \to 0,\ \pi/|k|$ correspond to $t \to \pm\infty$; the scale factor $a(t)$ decreases in an asymptotically linear manner in the remote past ($t \to -\infty$), reaches a minimum at $\tau = \pi/2|k|$ and grows in an asymptotically linear manner at $t \to \infty$, while the scalar field ω changes monotonically from one limiting value ω_- at $t \to -\infty$ to another limiting value ω_+ at $t \to +\infty$. The model is time-symmetric with respect to the maximum contraction instant.

By (21) a necessary condition for the existence of nonsingular solutions is the restriction $F < 0$ on the function (6), or, in terms of the initial function $B(\omega)$: $B > 3/2$.

These results confirm those of Ref. [23].

6. Consider now the metric $\overline{g}_{AB}$ for $n = 2$: let $a(t) \equiv e^{\beta_1(\tau)}$ be the scale factor of the ordinary physical space ($N_1 = 3$), while $b(t) \equiv e^{\beta_2(\tau)}$ that of the internal space ($N_2 = N$).

6.1. In the case $K_1 = 0$ (spatially flat models) we obtain:

$$d\overline{s}^2 = e^{2(3h_1+Nh_2)\tau} d\tau^2 - e^{2h_1\tau} ds_1^2 - e^{2h_2\tau} ds_2^2 \tag{22}$$

where with no loss of generality the scales in M_1 and M_2 are chosen so that $\beta_{10} = \beta_{20} = 0$. Herewith

$$6(h_1 + Nh_2/2)^2 = N(N+1/2) + S \tag{23}$$

In the special case $3h_1 + Nh_2 = 0$ the time coordinate τ is synchronous, i.e., physical. The metric (22) is nonsingular at finite τ and describes an exponential expansion (inflation) of one of the spaces (e.g., the physical one, M_1) and a simultaneous exponential contraction of the other, M_2, since h_1 and h_2 have different signs. However, by (23) and (13)

$$S = F\dot{\omega}^2 = -h_1^2(2N+1)/N < 0. \tag{24}$$

So a necessary condition for the existence of the special solution (22) is the restriction

$$B(\omega) < (D-1)(D-2)/4, \tag{25}$$

more general than $B < 3/2$ from Sect. 4.

In the more general case $3h_1 + Nh_2 = H \neq 0$ a transition to the physical time $dt = e^{H\tau} d\tau$ leads to the metric

$$d\overline{s}^2 = dt^2 - t^{2h_1/H} ds_1^2 - t^{2h_2/H} ds_2^2 \tag{26}$$

which is singular at $t = 0$ if at least one of the constants h_1 or h_2 is nonzero. At $h_1 = h_2 = 0$ the metric is static and (24) implies that either $\dot{\omega} = 0$ (the solution is trivial), or $F \equiv 0$, a special choice of B such that $\omega(\tau)$ has no dynamics.

6.2. For a spherical world ($K_1 = 1$) the metric is

$$d\overline{s}^2 = \frac{e^{-Nh\tau}}{2\cosh k\tau}\left[\frac{d\tau^2}{4\cosh^2 k\tau} - ds_1^2\right] - e^{2h\tau} ds_2^2 \tag{27}$$

where ds_1^2 is the line element on a unit sphere. A consideration like that in Sect. 5.1 leads to the following conclusions:

(a) The model behavior is classified by the values of the constant $h = h_2$ as compared with $k > 0$. The physical time $t = \pm\int e^{\gamma(\tau)} d\tau$ varies either within a finite segment $[t_1,\ t_2]$ (if $|Nh| < 3k$), or within a semi-infinite range (if $|Nh| \geq 3k$).

(b) At any finite boundary of the range of t at least one of the scale factors $a(t)$ or $b(t)$ vanishes, i.e., a singularity takes place.

(c) At $t \to \pm\infty$ either $a \to 0,\ b \to \infty$, or conversely, $a \to \infty,\ b \to 0$.

The value $S = -F\dot{\omega}^2$ is determined at $K_1 = \pm 1$ from

$$3k^2 \operatorname{sign} k = N(N+2)h^2 + 2S. \tag{28}$$

6.3. For hyperbolic models ($K_1 = -1$) the metric has the form

$$d\overline{s}^2 = \frac{e^{-Nh\tau}}{2s(k,\tau)}\left[\frac{d\tau^2}{4s^2(k,\tau)} - ds_1^2\right] - e^{2h\tau} ds_2^2 \tag{29}$$

(the same as (27) but the function $\cosh k\tau$ is replaced by $s(k,\tau)$ defined in (17)). Preserving generality, let us assume $\tau > 0$.

The model behavior may be briefly described as follows:

(a) At $k > 0,\ Nh \leq -3k$ or $k = 0,\ h < 0$ the physical time $t = \pm\int e^{\gamma(\tau)} d\tau$ ranges from $-\infty$ to $+\infty$. The factor $b(t) = e^{h\tau}$ varies from a finite value at $\tau = 0$ ($t = -\infty$) to zero at $\tau \to \infty$ ($t \to \infty$). The factor $a(t)$ describes a power-law contraction from infinity (at $t \to -\infty$) to a regular minimum and an infinite (in general, power-law) expansion at $t \to \infty$. There is no singularity at finite t.

Figure 1: The scale factors $a(t)$ and $b(t)$ in a 5-dimensional open model

(b) At $k \geq 0,\ Nh > 3k$ the model is singular at finite t corresponding to $\tau \to \infty$. In the special case $h = k = 0$ we come again to the Milne model (see Sect.4) supplemented with the space M_2 with a constant scale factor.

(c) At $k < 0$ the time t ranges again from $-\infty$ to $+\infty$. The factor $a(t)$ behaves as it did in item (a), however, its variation at $t \to \pm\infty$ is linear (but in general with unequal slopes at the two asymptotics). The factor $b(t)$ changes monotonically between two finite boundary values. The typical time dependence of the scale factors $a(t)$ and $b(t)$ in this case is shown in Fig.1.

It is necessary to note that, unlike the 4-dimensional models (Sect. 4), the nonsingular multidimensional ones with $h \neq 0$ exhibit a time-asymmetric behavior of $a(t)$.

It is seen in a straightforward manner that in all the nonsingular models the requirement (25) is imposed on $B(\omega)$, which, as it could be formulated in general relativity, means the negative scalar field energy density.

Some properties of the above models have been discovered in numerical calculations for a number of special cases with $D = 5$ and $D = 6$ [24].

7. Keeping in mind possible applications of our models to quantum stages of the universe evolution, let us continue them to the Eucledean sector. For this purpose let us replace the metric (8) by a slightly more general one

$$d\overline{s}^2 = \overline{g}_{AB}dx^A dx^B = \mathrm{e}^{2\gamma(\tau)}d\tau^2 + \sum_{i=1}^{n} \varepsilon_i \mathrm{e}^{2\beta_i(\tau)} ds_i^2 \quad (30)$$

where $\varepsilon_i = \pm 1$. Then in Eqs.(10) and consequently in the field equations the only change is that K_i are replaced by $\varepsilon_i K_i$. If we put, as before, $K_i = 0$ for $i \geq 2$, the equations depend only on $\varepsilon_1 K_1$. That means that the evolution of the Lorentzian open model ($K_1 = -1,\ \varepsilon_1 = -1$) coincides with that of the Eucludean closed model ($K_1 = 1,\ \varepsilon_1 = 1$) and vice versa, and the evolution of models with a flat 3-space ($K_1 = 0$) does not depend on the metric signature. In particular, the nonsingular Lorentzian model with an open 3-space, whose external and internal scale factors are shown in Fig.1, corresponds to the Eucledean four-dimensional wormhole $\mathrm{S}^3 \times \mathrm{R}^1$ whose radius is $a(t)$ and the interior scale factor is $b(t)$.

8. As was pointed out in the introduction, the fundamental difference between STT and WIST-based models consists in the nonminimal coupling of matter with a scalar field. As a toy model consider a theory with the Lagrangian

$$L = R(1+\zeta\varphi^2) + \xi\omega^A\omega_A + \eta\varphi^A\varphi_A, \qquad \zeta = \frac{1}{2(D-1)} \quad (31)$$

where R is determined by (2), φ is a real scalar field, $\eta = \pm 1$ and ξ =const, as before. In the limiting case φ =const the Lagrangian (31) coincides with (3) with $A = 1,\ B = \xi$. This particular value of the coefficient ζ is chosen just for computational convenience.

The substitution of (2) into (31) gives after simplification

$$L = (1+\zeta\varphi^2)\widetilde{R} + \Big(F + \frac{D-2}{8}\varphi^2\Big)\omega^A\omega_A - \varphi\varphi^A\omega_A + \eta\varphi^A\varphi_A \quad (32)$$

where total derivatives are omitted and

$$F = \xi + \frac{1}{4}(D-1)(D-2)$$

is just the quantity defined in (6) in the case $A = 1,\ B = \xi$.

Variation of (32) with respect to g_{MN}, ω and φ yields the equations

$$\begin{aligned}
&\big(\widetilde{R}_{AB} - \tfrac{1}{2}g_{AB}\widetilde{R} + \widetilde{\nabla}_A\widetilde{\nabla}_B - g_{AB}\widetilde{\Box}\big)(1+\zeta\varphi^2) \\
&\quad + \Big(F + \frac{D-2}{8}\varphi^2\Big)\big(\omega_A\omega_B - \tfrac{1}{2}g_{AB}\omega^C\omega_C\big) \\
&\quad - \tfrac{1}{2}\varphi\big(\varphi_A\omega_B + \varphi_B\omega_A - g_{AB}\varphi^C\omega_C\big) \\
&\quad + \eta\big(\varphi_A\varphi_B - \tfrac{1}{2}g_{AB}\varphi^C\varphi_C\big) = 0
\end{aligned} \quad (33)$$

$$\Big(F + \frac{D-2}{4}\varphi^2\Big)\widetilde{\Box}\omega + \frac{1}{2}(D-2)\varphi\varphi^A\omega_A - \varphi\widetilde{\Box}\varphi - \varphi_A\varphi^A = 0, \quad (34)$$

and

$$\eta\widetilde{\Box}\varphi - \Big(\frac{1}{2}\widetilde{\Box}\omega + 2\zeta\widetilde{R} + \frac{1}{8}(D-2)\omega^A\omega_A\Big)\varphi = 0. \quad (35)$$

Eq.(35) shows that the non-Riemannian nature of the space-time geometry leads in our model to an effective mass generation for the scalar field φ.

Eqs.(33)-(35) are too complicated for solving analytically. Therefore we briefly reproduce the main results of their numerical investigation. More details may be found in [24].

8.3. Let us first consider 4-dimensional models. Evidently, the coefficients before $\ddot{a}/a$, $\ddot{\omega}$ and $\ddot{\varphi}$ in Eqs. (33) - (35) depend on both the parameters ξ and η and on the scalar field φ. The determinant of the matrix of coefficients before $\ddot{a}/a$, $\ddot{\omega}$ and $\ddot{\varphi}$ is

$$\Delta = \left(\frac{\eta}{2} - \frac{2}{3}\right)\varphi^4 + \left(2\eta + \frac{4\xi}{3} - \frac{2\eta\xi}{3} - 4\right)\varphi^2 + 6\eta - 4\eta\xi.$$

The points where $\Delta = 0$ are singular points of the set (33)-(35). These points are not described by this set of equations because for fixed η and ξ the equation Δ =const defines no more than four fixed values of φ and the set (33)-(35) reduces to a set of first-order equation. Therefore the initial value of the field φ must belong to the open set $\Delta \neq 0$.

For $\eta = 1$ the equation $\Delta = 0$ defines two curves which divide the half-plane (ξ, φ^2) into three regions, to be denoted by A, B and C, while for $\eta = -1$ there are only two regions A and B (Fig. 2a,b). The behavior of the model depends on the region where the point (ξ, φ_0^2) is situated.

A numerical investigation of Eqs. (33) - (35) shows that for closed ($k = 1$) and flat ($k = 0$) cosmological models only singular solutions exist under any initial conditions. For hyperbolic models ($k = -1$), if the pair $(\xi,\ \varphi_0^2)$ defines a point in the domain B (for both $\eta = 1$ and $\eta = -1$) or C (for $\eta = 1$), then only singular solutions exist as well. If the pair $(\xi,\ \varphi_0^2)$ defines a point in the domain A, then the solutions may be both regular and singular. A numerical study is unable to specify the exact regularity conditions but shows that both regular and singular solutions are stable against finite perturbations of initial conditions. Typical qualitative behaviors of the universe scale factor $a(t)$, the Weyl field ω and the matter scalar field φ are shown in Fig.3.

The universe scale factor $a(t)$ in a typical nonsingular solution evolves from infinity at $t = -\infty$ to its minimal value $a_0 = a(0)$ and then grows to infinity at $t \to \infty$ Both scalar fields, ω and φ, evolve between two finite limiting values. A difference in their evolutions is that the field ω evolves monotonically, while φ near $t = 0$ (i.e., near the minimum of $a(t)$) may have several intermediate extrema with one absolute maximum if $\eta = 1$ or absolute minimum if $\eta = -1$. As $\varphi(t)$ for big $|t|$ tends asymptotically to constants, the model evolves asymptotically as an empty Weyl cosmological model considered in Sect.5. It is also noteworthy that the evolution of $a(t)$ has a small time asymmetry, unlike the vacuum case. This asymmetry results from a non-symmetric evolution of the matter field φ, since the field equations (33)-(35) contain both φ and $\dot{\varphi}$.

Figure 2: The set $\Delta = 0$ for $\eta = 1$ (a) and $\eta = -1$ (b) in 4 dimensions

Figure 3: Qualitative behaviors of $a(t)$, $\omega(t)$ and $\varphi(t)$ in an open 4-dimensional WIST model

8.4. Consider now the 5-dimensional case. In this case

Figure 4: The sets $\Delta = 0$ for $\eta = 1$ (a) and $\eta = -1$ (b) in 5 dimensions

Figure 5: Qualitative behavior of the scale factors $a(t)$ and $b(t)$, the matter scalar field $\varphi(t)$ and the Weyl field $\omega(t)$ in a nonsingular 5-dimensional WIST model

the determinant of the matrix of coefficients before $\ddot{a}/a$, $\ddot{s}/s$, $\ddot{\omega}$ and $\ddot{\varphi}$ in the equations (33)-(35) is

$$\Delta = \Big(\frac{9}{256}\eta - \frac{3}{32}\Big)\varphi^6 + \Big(\frac{27}{32}\eta + \frac{1}{8}\xi - \frac{3}{32}\eta\xi - \frac{3}{2}\Big)\varphi^4 + \Big(\frac{27}{4}\eta + \xi - \frac{3}{2}\eta\xi - 6\Big)\varphi^2 + 18\eta - 6\eta\xi.$$

The qualitative features of the function $\Delta(\xi, \eta, \varphi)$ are the same as have been in 4-dimensional case: for $\eta = 1$ the equation $\Delta = 0$ defines two curves which divide the half-plane $(\xi, \varphi^2 > 0)$ into three domains denoted by A, B and C, while for $\eta = -1$ there are only two regions A and B (Fig. 4a,b). The behavior of the model depends on the domain where the point (ξ, φ_0^2) is situated.

A numerical investigation shows that, just as in 4 dimensions, only singular solutions exist under any initial conditions for closed ($k = 1$) and flat ($k = 0$) cosmological models. For open models ($k = -1$), if the point (ξ, φ_0^2) is in the domain B (both for $\eta = 1$ and $\eta = -1$) or C (for $\eta = 1$), then only singular solutions exist, while if it belongs to the domain A, the solution may be both regular and singular, depending on integration constants which may be considered as the initial conditions at $t = 0$.

A typical nonsingular solution behavior for $\eta = 1$ is shown in Fig.5 for the case of a contracting internal space.

In the general nonsingular solution the radius of the universe changes monotonically from infinity at $t = -\infty$ to a minimum value a_0 and then grows to infinity, while the internal space radius starts from some b_-, passes through one or two intermediate extrema (occurring near the minimum of $a(t)$ and absent in some cases) evolves to some b_+. Note that b_+ and b_- may be of the same or different orders of magnitude. The field φ evolves similarly to the 4-dimensional case. The extremal points of the functions $a(t)$, $b(t)$ and $\varphi(t)$ do not coincide in the general case and the function $a(t)$ is time asymmetric, especially near its minimum. Lastly, the Weyl field ω changes monotonically between its two limiting values ω_- and ω_+. In the case $\eta = -1$ the model evolves in the same way but the extremal points of the field φ change their types: a minimum become a maximum and vice versa.

9. In conclusion, we have seen that many of the multidimensional Weyl cosmologies with flat (toroidal) additional spaces are nonsingular: there are special flat-space models with eternally increasing or decreasing scale factors (such models are absent in 4 dimensions) and there are more general hyperbolic models with a cosmological bounce (generalizing the 4-dimensional ones [23]) which realize the dimensional reduction scenario. It has been shown that in the multidimensional case the evolution of the scale factor of the universe $a(t)$ becomes time-asymmetric, unlike the 4-dimensional case. In particular the evolution of $a(t)$ for big $|t|$ is asymptotically linear.

Furthermore, in nonsingular models the Weyl scalar field $\omega(t)$ is asymptotically constant. The models thus tend to purely Einsteinian ones and the change of the collapse into expansion may be interpreted as a cosmological phase transition induced by the transition of scalar field $\omega(t)$ from one stationary state $\omega = \omega_-$ to another stationary state $\omega = \omega_+$. At the late stages of the evolution the field $\omega(t)$ is unobservable.

A numerical investigation shows that the models are stable against the introduction of sources in the form of a perfect fluid and unstable against a mass perturbation of the Weyl scalar field. In particular, nonsingular models may become singular under such perturbations.

The models considered show that the real space-time structure may be of a non-Riemannian nature which is, however, essential only near $t = 0$ and becomes unobservable at later stages of the evolution. Therefore, studies of generalized geometric structures in multidimensional cosmology are of considerable interest [27], especially for the description of the most violent epochs.

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