

ON SPHERICALLY SYMMETRIC SOLUTIONS IN D-DIMENSIONAL DILATON GRAVITY

K.A.Bronnikov

Center for Surface and Vacuum Research, 8 Kravchenko str., Moscow, 117331, Russia¹

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Exact static, spherically symmetric solutions to the Einstein-Abelian gauge-dilaton equations, in D -dimensional gravity with a chain of n Ricci-flat internal spaces are considered, with the gauge field potential having three nonzero components: the temporal, Coulomb-like one, the one pointing to one of the extra dimensions, and the one responsible for a radial magnetic field. For dilaton coupling implied by string theory an $(n+5)$ -parametric family of exact solutions is obtained, while for other dilaton couplings only $(n+3)$ -parametric ones. The geometric properties and special cases of the solutions are discussed, in particular, those when there are horizons in the space-time. Two types of horizons are distinguished: the conventional black-hole (BH) ones and those at which the physical section of the space-time changes its signature (T -horizons). Two theorems are proved, one fixing the BH and T-horizon existence conditions, the other discarding the possibility of a regular center. Different conformal gauges are used to characterize the system from the D -dimensional and 4-dimensional viewpoints.

1. Introduction

The search for and discussion of exact solutions to dilaton gravity equations has recently become the subject of many studies ([18, 22, 23, 24, 29, 30, 31, 35] and many others) mostly because dilaton gravity (more precisely, Einstein-gauge-dilaton-axion gravity) forms the bosonic part of effective low energy string theory [25]. Of the solutions in 4 dimensions existing to-date those found in Refs.[14] and [21] are probably the most general: the former contains 6 independent integration constants (the mass, dilaton, axion, electric and magnetic charges and the Taub-NUT parameter), the latter 5 ones (the mass, electric and magnetic charges, the Taub-NUT and rotation parameters; there are also asymptotic values of the dilaton and axion fields, which may be absorbed by proper re-definitions).

This 4-dimensional approach assumes that the extra dimensions of the original (10-dimensional) theory are compactified and their parameters are constant. To take into account the possible variation of their size from point to point in the physical 4-dimensional space-time it is reasonable to adhere to multidimensional field equations, as is done, e.g., in [5, 6, 8, 37], or to include the extra-dimension scale factors as a separate dynamic variable [15]. Here the former approach is adopted. Moreover, we assume a sufficiently general space-time structure (see (6) and an arbitrary value of the gauge-dilaton coupling constant λ to cover a

wider spectrum of possible multidimensional field theories, such as considered, e.g., in Ref.[39].

Among the multidimensional solutions probably the most general is the solution of Ref.[8] containing $(n+4)$ integration constants, where n is the number of internal spaces (the mass, dilaton and electric charges and $(n+1)$ parameters connected with the extra dimensions). Here the results of [8] are further generalized to include the magnetic charge; some properties of this more general system are discussed.

Unlike [14, 21], where symmetries of the field action were used to obtain the solutions from known, simpler ones, we try to integrate the field equations directly. One of the advantages of this approach is the possibility to consider more general field systems, for instance, with an arbitrary value of the metric-dilaton coupling constant λ .

We will consider static, spherically symmetric Einstein-Abelian gauge-dilaton configurations in D -dimensional gravity with a chain of n Ricci-flat internal spaces. We start from the action

$$S = \int d^D x \sqrt{D}g \left[{}^D R + g^{MN} \varphi_{,M} \varphi_{,N} - e^{2\lambda\varphi} F^2 \right] \quad (1)$$

where g_{MN} is the D -dimensional metric, ${}^D g = |\det g_{MN}|$, φ is the dilaton scalar field and $F^2 = F^a F^a = F^a{}^{MN} F^a{}_{MN}$, $F^a = dW^a$, W^a ($a = 1, 2, \dots$) being Abelian gauge fields of which one is to be interpreted as the electromagnetic field.

Three types of W^a compatible with spherical symmetry will be treated: W^1 , the Coulomb-like one, so

¹e-mail: bron@cvsr.ac.ru

that the vector potential is t -directed, W^2 , pointing to one of the extra dimensions, and W^3 , responsible for a radial magnetic field.

The field-theoretic limit of string theory corresponds to the specific value of the coupling constant $\lambda = \lambda_{\text{string}} = \pm(D-2)^{-1/2}$ [25, 36].

The field equations are written down in Sect.2 and solved in Sect.3. We come through a striking coincidence: if there is more than one nontrivial component of F^a , the field equations are explicitly integrable if and only if $\lambda^2 = 1/(D-2)$, i.e., exactly for the dilaton coupling which follows from string theory.

In Sect.4 some special cases are indicated. In Section 5 we find special cases when the solutions exhibit black-hole or T-hole horizons and prove two theorems, valid for any values of λ . One determining the necessary conditions for horizon existence and the other on the nonexistence of solutions with a regular center in the model under study.

Sects. 6 and 7 discuss different conformal gauges in D and 4 dimensions, respectively. The point is that if the underlying theory is string theory, then a more fundamental role is played by the ‘‘string metric’’, or ‘‘ σ model metric’’ $\hat{g}_{AB} = e^{-2\lambda\varphi}g_{AB}$ rather than g_{AB} from (1) (see, e.g., [1, 36] and references therein). Although mathematically a transition from \hat{g}_{AB} to g_{AB} may be treated as just a substitution simplifying the field equations, such issues as the nature of singularities (if any) and topology are better to discuss in terms of \hat{g}_{AB} . (Strictly speaking, this argument does not apply to $\lambda \neq \lambda_{\text{string}}$ when the underlying more fundamental theory is not definitely fixed).

On the other hand, the observable effects in 4 dimensions depend on how nongravitational matter interacts with the metric and dilaton fields and are described in different ways in different ‘‘conformal gauges’’, or systems of measurement, which are discussed in Sect.7. It should be stressed that such things as horizons and signatures are the same in all the relevant conformal gauges since the conformal factors connecting them are regular at the horizons.

Section 8 contains some concluding remarks.

Throughout the paper capital Latin indices range from 0 to $D-1$, Greek ones from 0 to 3, the index i enumerates subspaces and a gauge field components.

2. Field equations

The set of field equations corresponding to (1) is

$$\nabla^M \nabla_M \varphi + \lambda e^{2\lambda\varphi} F^2 = 0, \quad (2)$$

$$\nabla_N (e^{2\lambda\varphi} F^{aNM}) = 0, \quad (3)$$

$$R_{MN} - g_{MN} R_A^A / 2 = -T_{MN} \quad (4)$$

where T_{MN} is the energy-momentum tensor

$$T_{MN} = \varphi_M \varphi_N - \frac{1}{2} g_{MN} \varphi^A \varphi_A + e^{2\lambda\varphi} \left[-2F_M^a F_{NA}^a + \frac{1}{2} g_{MN} F^2 \right]. \quad (5)$$

Consider a D -dimensional Riemannian or pseudo-Riemannian manifold V^D with the structure

$$V^D = M^4 \times V_1 \times \dots \times V_n; \quad \dim V_i = N_i; \quad D = 4 + \sum_{i=1}^n N_i, \quad (6)$$

where $M^4 = M^2 \times S^2$ is the conventional space-time and V_i are Ricci-flat manifolds of arbitrary dimensions and signatures with the line elements ds_i^2 , $i = 1, \dots, n$. We seek static, spherically symmetric solutions to the field equations, so that the D -dimensional metric is

$$ds_D^2 = g_{MN} dx^M dx^N = e^{2\gamma(u)} dt^2 - e^{2\alpha(u)} du^2 - e^{2\beta(u)} d\Omega^2 + \sum_{i=1}^n e^{2\beta_i(u)} ds_i^2. \quad (7)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the line element on a unit sphere S^2 , while all the scale factors e^{β_i} of the internal spaces V_i depend on the radial coordinates u .

It should be noted that in (7) one could include arbitrary $(d+1)$ -dimensional spheres; however, a nonzero magnetic field W^3 is compatible (at least in the conventional approach) only with $d=1$. Solutions with $W^3=0$ and arbitrary d have been considered in [8].

If we denote $\gamma = \beta_{-1}$, $N_{-1} = 1$, $\beta = \beta_0$, $N_0 = 2$ and choose the harmonic radial coordinate u [3] such that

$$\alpha = \sum_{i=-1}^n N_i \beta_i \equiv \gamma + 2\beta + \sigma, \quad \sigma \equiv \sum_{i=1}^n N_i \beta_i, \quad (8)$$

the Ricci tensor components R_M^N can be written in the highly symmetric form ($x^1 \equiv u$)

$$\begin{aligned} R_u^u &= -e^{-2\alpha} \sum_{i=-1}^n N_i [\beta_i'' + \beta_i'^2 - \beta_i' \alpha']; \\ R_\mu^\mu &= 0 \quad (M > d+2; \mu = 0, \dots, d+2); \\ R_{a_j}^{b_i} &= -\delta_j^i \delta_{a_i}^{b_i} e^{-2\alpha} \beta_i'', \quad i \neq 0 \\ R_\theta^\theta &= R_\phi^\phi = 2e^{-2\beta} - e^{-2\alpha} \beta''; \end{aligned} \quad (9)$$

where primes denote d/du and the indices a_j (b_i) refer to the subspace V_j (V_i).

The fields φ and F compatible with the assumed symmetry are $\varphi = \varphi(u)$, a component of W^a in the t direction (the Coulomb electric field, $W^1 = w^1(u)dt$), a similar component in some internal one-dimensional subspace (if any; this subspace will be denoted V_1 and parametrized by a coordinate v : $N_1 = 1$, $\beta_1 = \nu(u)$, $W^2 = w^2(u)dv$) and a monopole magnetic field of the form $W^3 = \bar{q} \cos \theta d\phi$ where \bar{q} is the magnetic charge.

The gauge field strengths are

$$F^a = F_{AB}^a dx^A \wedge dx^B : \quad F^1 = w^{1'}(u) du \wedge dt;$$

$$F^2 = w^{2'}(u) du \wedge dv; \quad F^3 = -\bar{q} \sin \theta d\theta \wedge d\phi.$$

The vector field equations lead to

$$e^{2\lambda\varphi+2\alpha} F^{1\,ut} = q = \text{const},$$

$$e^{2\lambda\varphi+2\alpha} F^{2\,uv} = q' = \text{const}. \quad (10)$$

Here q is the electric charge and q' is the gauge charge connected with F_{uv}^2 . We will sometimes also use the notations $q = q_1$, $q' = q_2$, $\bar{q} = q_3$.

Now the scalar field equation and some linear combinations of the metric field equations may be written in the form

$$\frac{1}{2}(N+1)\gamma'' = Nq^2 e^{2\omega} - \eta_\nu q'^2 e^{2\psi} + \bar{q}^2 e^{2\chi}, \quad (11)$$

$$\frac{1}{2}(N+1)\nu'' = -q^2 e^{2\omega} + N\eta_\nu q'^2 e^{2\psi} + \bar{q}^2 e^{2\chi}, \quad (12)$$

$$\frac{1}{2}(N+1)\beta_i'' = -q^2 e^{2\omega} - \eta_\nu q'^2 e^{2\psi} + \bar{q}^2 e^{2\chi},$$

$$i = 2, \dots, n; \quad (13)$$

$$\varphi'' = (N+1)\lambda\beta_i'', \quad (14)$$

$$(\alpha - \beta)'' = e^{2\alpha-2\beta}, \quad (15)$$

$$\alpha'^2 - \sum_{i=1}^n N_i \beta_i'^2 - 2e^{2\alpha-2\beta}$$

$$= \varphi'^2 - 2q^2 e^{2\omega} - 2\eta_\nu q'^2 e^{2\psi} - 2\bar{q}^2 e^{2\chi} \quad (16)$$

where we have denoted

$$\omega = \gamma - \lambda\varphi; \quad \psi = \nu - \lambda\varphi; \quad \chi = \gamma + \sigma + \lambda\varphi;$$

$$N = D - 3; \quad \eta_\nu = \text{sign } g_{\nu\nu}. \quad (17)$$

Eq.(16) is the (u) component of the Einstein equations and represents a first integral of (11)-(15).

3. Exact static solutions

Eq.(13) is easily integrated to give

$$e^{\beta-\alpha} = s(k, u) \equiv \begin{cases} k^{-1} \sinh ku, & k > 0; \\ u, & k = 0; \\ k^{-1} \sin ku, & k < 0 \end{cases} \quad (18)$$

where $k = \text{const}$ and an inessential integration constant has been eliminated by shifting the origin of u . Consequently, with no loss of generality one can assert that the harmonic u coordinate is defined for $u > 0$ and $u = 0$ corresponds to spatial infinity. By (18), asymptotically ($u \rightarrow 0$) the conventional flat-space spherical radial coordinate $r = e^\beta$ is connected with u by $u = 1/r$.

The remaining equations may be combined into a set written in terms of the functions ω , ψ , χ in the following matrix form:

$$\frac{N+1}{2} \begin{pmatrix} \omega'' \\ \psi'' \\ \chi'' \end{pmatrix} = \begin{pmatrix} N+\Lambda & \Lambda-1 & 1-\Lambda \\ \Lambda-1 & N+\Lambda & 1-\Lambda \\ 1-\Lambda & 1-\Lambda & N+\Lambda \end{pmatrix} \begin{pmatrix} q^2 e^{2\omega} \\ \eta_\nu q'^2 e^{2\psi} \\ \bar{q}^2 e^{2\chi} \end{pmatrix} \quad (19)$$

where $\Lambda = \lambda^2(N+1) = (\lambda/\lambda_{\text{string}})^2$. The matrix in the right-hand side is nondegenerate for all Λ if $N \geq 2$; at $N = 1$ there is no variable ν (and $\psi \equiv \nu - \lambda\varphi$), nor charge q' , so that the second line in Eq.(19) and the second column in the matrix must be removed.

Eqs.(19) can be explicitly integrated either if there is only one nonzero charge among q , q' , \bar{q} , or in the case $\Lambda = 1$, exactly the one corresponding to the dilaton coupling constant in string theory. The functions γ , ν , φ , β_i are then easily expressed in terms of ω , ψ , χ . We will here present all these solutions for information purposes, although Solutions E (“electric”), I (“internal”) and G (“general”, for $\Lambda = 1$) have been given earlier (see [8] and references therein), the latter only for $\bar{q} = 0$, in slightly different notations. The fields F^a are in all cases expressed by (10). In addition, inessential integration constants are eliminated by rescaling the coordinates in the subspaces V_i : their true scales are thus hidden in ds_i^2 , while the factors e^{β_i} are normalized by $\beta_i(0) = 0$. We impose one more condition: $\varphi(0) = 0$; this is also no generality loss since a nonzero $\varphi(0)$ can be compensated by rescaling the charges q_a .

The number of essential integration constants is in all cases equal to n , the number of internal spaces, plus the number of nontrivial physical fields: scalar, gauge (F^a) and gravitational. The constants are denoted by, respectively, b_i , C , q_a , k , h_a .

Solution E [2, 12, 8, 6]

$$ds_D^2 = e^{2\gamma} dt^2 - e^{-2\gamma/N} \left\{ \left[\frac{e^{-Bu}}{d \cdot s(k, u)} \right]^{2/d} \right.$$

$$\left. \times \left[\frac{du^2}{d^2 s^2(k, u)} + d\Omega_{d+1}^2 \right] - \sum_{i=1}^n e^{2b_i u} ds_i^2 \right\}, \quad (20)$$

$$\varphi = Cu/A - 2\lambda N_+ \omega, \quad (21)$$

$$\gamma = (\omega + \lambda Cu)/A, \quad (22)$$

$$e^{-\omega} = Q_1 s(h_1, u + u_1); \quad u_1 = \text{const}; \quad \omega(0) = 0. \quad (23)$$

The integration constants are connected by the relation due to (16)

$$2k^2 \text{sign } k = 2N_+ h_1^2 \text{sign } h_1 + \frac{C^2}{A} + B^2 + \sum_{i=1}^n N_i b_i^2. \quad (24)$$

The notations are

$$Q_1 = \sqrt{q^2/N_+}, \quad A = 1 + \Lambda/N,$$

$$N_+ = (N + 1)/(N + \Lambda), \quad B = \sum_{i=1}^n N_i b_i \quad (25)$$

The last condition from (23) is the requirement that $\gamma = 0$ at infinity, i.e., dt is a time interval measured by a distant observer at rest with respect to our static configuration. One may notice that no function ν is distinguished among the scale factors β_i .

Solution I [8]

The solution can be found by proper substitutions in Solution E. Namely, we obtain:

$$ds_D^2 = \eta_\nu e^{2\nu} dv^2 + e^{2\nu/N} \left\{ e^{2b_0 u} dt^2 - \frac{e^{-2Bu}}{s^2(k, u)} \left[\frac{du^2}{s^2(k, u)} + d\Omega^2 \right] + \sum_{i=2}^n e^{2b_i u} ds_i^2 \right\} \quad (26)$$

$$\varphi = Cu/A - 2\lambda N_+ \psi, \quad (27)$$

$$\nu = (\psi + \lambda Cu)/A, \quad (28)$$

$$e^{-\psi} = \begin{cases} Q_2 s(h_2, u + u_2) & \text{if } \eta_\nu = +1, \\ (Q_2/h_2) \cosh h_2(u + u_2) & \text{if } \eta_\nu = -1, \end{cases}$$

$$\psi(0) = 0 \quad (29)$$

where $Q_2 = \sqrt{q^2/N_+}$. The integration constants are constrained by

$$2k^2 \text{sign } k = 2N_+ h_2^2 \text{sign } h_2 + \frac{C^2}{A} + B^2 + b_0^2 + \sum_{i=2}^n N_i b_i^2. \quad (30)$$

In (26) and (30) B is expressed as $B = b_0 + \sum_{i=1}^n N_i b_i$.

Solution M

$$ds_D^2 = e^{2\gamma} dt^2 - \frac{e^{-2N\gamma - 2bu}}{s^2(k, u)} \left[\frac{du^2}{s^2(k, u)} + d\Omega^2 \right] + e^{2\chi/(AN)} \sum_{i=1}^n e^{2b_i u} ds_i^2; \quad (31)$$

$$\varphi = Cu + 2\lambda N_+ \chi; \quad (32)$$

$$\gamma = b_0 u + \chi/(AN); \quad (33)$$

$$e^{-\chi} = \bar{q}^2 s(h_3, u + u_3), \quad \chi(0) = 0, \quad (34)$$

$$2k^2 \text{sign } k = 2N_+ h_3^2 \text{sign } h_3 + C^2(1 + \lambda^2) + b_0^2 + \sum_{i=1}^n N_i b_i^2. \quad (35)$$

where $b_0 = -\lambda C - b$, $b = \sum_{i=1}^n N_i b_i$, $Q_3 = \sqrt{\bar{q}^2/N_+}$.

Solution G ($\lambda^2 = \lambda_{\text{string}}^2 = N + 1$)

$$ds_D^2 = e^{2\lambda\varphi} \left\{ e^{2\omega} dt^2 - \frac{e^{-2\chi}}{s^2(k, u)} \left[\frac{du^2}{s^2(k, u)} + d\Omega^2 \right] \right.$$

$$\left. + \eta_\nu e^{2\psi} dv^2 + \sum_{i=2}^n e^{2b_i u} ds_i^2 \right\} \quad (36)$$

$$e^{-\omega} = Q_1 s(h_1, u + u_1), \quad \omega(0) = 0; \quad (37)$$

$$e^{-\psi} = \begin{cases} Q_2 s(h_2, u + u_2), & \eta_\nu = +1, \\ (Q_2/h_2) \cosh h_2(u + u_2), & \eta_\nu = -1; \end{cases} \quad (38)$$

$$e^{-\chi} = Q_3 s(h_3, u + u_3), \quad \chi(0) = 0; \quad (39)$$

$$\lambda\varphi = (\chi - \omega - \psi - bu)/(N + 1); \quad (40)$$

$$2k^2 \text{sign } k \sum_{a=1}^3 h_a^2 \text{sign } h_a + \sum_{i=2}^n N_i b_i^2. \quad (41)$$

where $Q_a = \sqrt{2q_a^2}$, $a = 1, 2, 3$, $\psi(0) = 0$ and $b = \sum_{i=2}^n N_i b_i$.

The ‘‘intermediate’’ solutions for $\Lambda = 1$ and two nonzero charges (to be labelled EM, EI, IM) are easily obtained from Solution G in the corresponding limits. The only subtle point is that the limit, say, $Q_1 \rightarrow 0$ can be realized only if $h_1 > 0$ and, moreover, the constant u_1 must vary along with $q_1 \rightarrow 0$ so as to maintain $\omega(0) = 0$. Thus in the limit $Q_1 = 0$ we obtain $\omega = -h_1 u$, just what could be obtained directly from the equations with $q = 0$. The situation is the same with all limits $q_a \rightarrow 0$, for instance, when obtaining the purely scalar-vacuum solution from either of the solutions E, I, M. In particular, the EI solution obtained from Solution G in this way coincides with Solution C from Ref. [8] up to notations of some constants.

4. Special cases

Let us enumerate some special cases of the solutions.

- (a) Either of the solutions yields the well-known scalar-vacuum solution in D dimensions if the gauge fields F^a are switched off [9, 19]. Their specialization to 4 dimensions leads to the scalar-vacuum [20] and vacuum (Schwarzschild) solutions of general relativity.
- (b) When $\lambda = 0$, Solution E reduces to the generalized Reissner-Nordstrom (RN) one for linear scalar and electromagnetic fields [5]; at $D = 4$, $n = 0$ it coincides with the Penney solution [33] which in turn reduces to the RN one when the scalar field is eliminated.

A new feature implied by nonzero gauge fields as compared with item (a) is that the constants k and (or) h_a can have either sign and the functions $s(h_a, u + u_a)$ can be sinusoidal, leading to $u_{\text{max}} < \infty$. In Solution E that means the appearance of a RN-like repelling singularity at the center of the configuration.

- (c) Solution E, if deprived of extra dimensions, yields the solution for interacting scalar and electromagnetic fields in general relativity, first obtained in Ref.[13].

- (d) The case $D = 5$ ($n = N_1 = 1$) with no dilaton and gauge fields was considered in Refs.[32, 16] and many others; it also coincides with the ‘‘Kaluza-Klein soliton’’ considered by Gross and Perry [26]. Similar solutions for $D = 6$ and $D = 7$ are presented in [39]; see also references therein.
- (e) There are special cases of the above solutions when the space-time exhibits horizons. They are discussed in the next section.

5. Horizons: black holes and time holes

5.1. Horizons in Solution E

The behavior of the metric for different combinations of integration constants is rather various. However, calculations show that, in particular, Solution E has a naked singularity at $u = u_{\max}$ or $u = \infty$ in all cases except

$$b_i = -\frac{k}{N}; \quad h_1 = k > 0; \quad C = -\lambda k \frac{N+1}{N}, \quad (42)$$

when the sphere $u = \infty$ is a Schwarzschild-like event horizon: at finite radius $r = e^\beta$ of a coordinate sphere the metric coefficient $g_{tt} = 0$ and the light travel time $\int e^{\alpha-\gamma} du$ diverges [6].

In this black-hole (BH) case only two independent integration constants remain, say, k and q , and the coordinate transformation

$$e^{-2ku} = 1 - 2k/R \quad (43)$$

brings the solution to the form [6]

$$\begin{aligned} ds_D^2 &= \frac{(1-2k/R)dt^2}{(1+p/R)^{2/A}} \\ &- (1+p/R)^{2/AN} \left[\frac{dR^2}{1-2k/R} + R^2 d\Omega^2 - \sum_{i=1}^n ds_i^2 \right]; \\ e^{\lambda\varphi} &= (1+p/R)^{\Lambda/(\Lambda+N)}; \\ F &= F^1 = q(R+p)^{-2} dR \wedge dt; \\ p &= (k^2 + q^2/N_+)^{1/2} - k. \end{aligned} \quad (44)$$

This extends the well-known dilaton black-hole solution (see, e.g., [24, 22, 28]) to spaces of the form (6). For the first time it was obtained in 4 dimensions [13] where for $\lambda = 0$ it reduces to the RN one. More frequently used notations are connected with ours by

$$R + p = r; \quad p = r_-; \quad 2k = r_+ - r_-. \quad (45)$$

In this family of BH solutions a nonzero dilaton field exists solely due to the interaction ($\lambda \neq 0$). When $\lambda = 0$, i.e., the φ field becomes minimally coupled, a horizon is compatible only with $\varphi = \text{const}$. This conforms with the well-known ‘‘no-hair’’ theorems and the properties of the general-relativistic scalar-vacuum and scalar-electrovacuum configurations.

5.2. Horizons in Solution I

There are counterparts of the above BH solutions among those of Class I. Namely, under the same conditions (42) (where just h_1 is replaced by h_2) the sphere $u = \infty$ is a horizon as well but now in the (v, u) subspace instead of (t, u) in the previous, conventional case. The solution is

$$\begin{aligned} ds_D^2 &= \frac{(1-2k/R)\eta_v dv^2}{(1+p'/R)^{2/A}} + (1+p'/R)^{2/AN} \\ &\times \left[dt^2 - \frac{dR^2}{1-2k/R} - R^2 d\Omega_2^2 + \sum_{i=2}^n ds_i^2 \right], \\ e^{\lambda\varphi} &= (1+p'/R)^{\Lambda/(\Lambda+N)}; \\ F &= q'(R+p')^{-2} dR \wedge dv, \\ p' &= (k^2 + \eta_v q'^2/N_+)^{1/2} - k. \end{aligned} \quad (46)$$

The main feature of these configurations is that the physical space-time M^4 changes its signature at $R = 2k$: it is $(+ - - -)$ at $R > 2k$ and $(+ + - -)$ at $R < 2k$. This evidently means that the anomalous domains should contain quite unconventional physics whose possible consequences and observational manifestations are yet to be studied. It has been suggested [8] to call the domains with an unusual space-time signature *time holes* or *T-holes* and the corresponding horizons *T-horizons*.

Each BH configuration of any dimension $D > 4$ has a family of T-hole counterparts (a family since the subspaces V_i may have different dimensions and signatures) and *vice versa*. However, if a BH possesses an external field, such as the Coulomb field of a RN black hole, its T-hole analog has a field modified by the $t \leftrightarrow v$ interchange, as is the case with the above solutions: the Coulomb-like field becomes the one pointing in the v direction which from the 4-dimensional viewpoint looks like a scalar field interacting with the dilaton (see Sect.7).

Unlike a BH-horizon, a T-horizon is not in absolute past or future from a distant observer’s viewpoint, it is visible since it takes a finite time for a light signal to come from it ($\int e^{\alpha-\gamma} du < \infty$, independently of a conformal gauge). T-hole properties are further discussed in Sect.6.

5.3. Horizons in Solutions M and EM

Solution M admits a BH horizon if and only if

$$h_3 = k > 0, \quad b_i = k/(N+\Lambda), \quad \lambda C = \Lambda k/(N+\Lambda) \quad (47)$$

After the same substitution (43) the solution takes the form

$$\begin{aligned} ds_D^2 &= (1+\bar{p}/R)^{-2/AN} \sum_{i=1}^n ds_i^2 + \frac{(1-2k/R)dt^2}{(1+\bar{p}/R)^{2/AN}} \\ &- (1+\bar{p}/R)^{2/A} \left[\frac{dR^2}{1-2k/R} + R^2 d\Omega^2 \right]; \end{aligned}$$

$$\begin{aligned}
e^{\lambda\varphi} &= (1 + \bar{p}/R)^{-\Lambda/(\Lambda+N)}; \\
F &= F^3 = -\bar{q} \sin\theta d\theta \wedge d\phi; \\
\bar{p} &= (k^2 + \bar{q}^2/N_+)^{1/2} - k.
\end{aligned} \tag{48}$$

Like (44), this solution is well-known for $D = 4$ ($N = 1$, $n = 0$). It is easy to notice that just for $N = 1$ the metrics (44) and (48) coincide, while the φ fields are different. At higher dimensions the distinctions are more complicated.

The same solution turns into a T-hole one if one replaces, as before, $t \leftrightarrow v$. This case is simpler than that of Solutions E and I since the interchange does not affect the magnetic field.

Solution EM (with $\lambda = \lambda_{\text{string}}$, obtained from Solution G in the limit $q' \rightarrow 0$) also has a BH horizon under the conditions

$$h_1 = h_3 = k > 0; \quad b_i = 0 \quad (i = 1, \dots, n). \tag{49}$$

The substitution (43) leads to the following form of the solution:

$$\begin{aligned}
ds_D^2 &= e^{2\lambda\varphi} \left\{ \frac{1 - 2k/R}{(1 + p/R)^2} dt^2 \right. \\
&\quad \left. - (1 + \bar{p}/R)^2 \left[\frac{dR^2}{1 - 2k/R} + R^2 d\Omega^2 \right] + \sum_{i=1}^n ds_i^2 \right\}; \\
e^{2\lambda\varphi} &= \left(\frac{1 + p/R}{1 + \bar{p}/R} \right)^{2/(N+1)}; \\
F &= q(R + p)^{-2} dR \wedge dt - \bar{q} \sin\theta d\theta \wedge d\phi; \\
p &= (k^2 + 2q^2)^{1/2} - k; \quad \bar{p} = (k^2 + 2\bar{q}^2)^{1/2} - k.
\end{aligned} \tag{50}$$

This is a dyon BH solution in dilaton gravity of arbitrary dimension, which naturally passes to (44) and (48) with $\lambda = \lambda_{\text{string}}$ in the limits $\bar{q} \rightarrow 0$ and $q \rightarrow 0$, respectively.

To obtain the T-hole counterpart of (50) (a special case of Solution IM) one has to change $t \leftrightarrow v$, $q \rightarrow q'$, $p \rightarrow p'$.

5.4. The general case. Non-existence theorems

It can be directly verified that among the Class G solutions with three nonzero charges q_a there are no special cases with horizons, either BH- or T-hole ones. This observation can be generalized to include the cases $\lambda \neq \lambda_{\text{string}}$ when exact solutions are hard to obtain. Namely, we will prove a theorem generalizing Theorem 1 of Ref. [8] (the latter concerned configurations with nonzero q and q' but $\bar{q} = 0$). The present proof, employing directly the horizon regularity condition, is much simpler than that in [8].

Let us previously adopt a convenient horizon definition for our static, spherically symmetric configurations. Namely, we will call a *BH-horizon* (i.e., a conventional black-hole horizon) a nonsingular sphere in a space with the metric (7) where the metric functions β and β_i are finite while $g_{tt} = e^{2\gamma} \rightarrow 0$. Similarly, a

T-horizon is a nonsingular sphere where β , γ , β_i ($i = 2, \dots, n$) are finite while $g_{vv} = e^{2\nu} \rightarrow 0$ where v parametrizes the one dimensional subspace V_1 , one of the internal subspaces. (We ought to require that, in addition, the travel time $\int e^{\alpha-\gamma} du$ of a light signal approaching it be infinite; however, this condition is irrelevant for our argument.)

Theorem 1. *The static, spherically symmetric field system (1,6,7,10) has no BH horizon if $q' \neq 0$ and has no T-horizon if $q \neq 0$.*

Proof. Assume that $q' \neq 0$ and there is a BH horizon at some $u = u^*$. By (18) and (8), $\beta = -\gamma - \sigma - \ln s(k, u)$, whereas β and σ must be finite at $u = u^*$ and $\gamma \rightarrow \infty$. This implies

$$\ln s(k, u) = -\gamma(u) + O(1), \tag{51}$$

whence $s \rightarrow \infty$ at $u \rightarrow u^*$. By definition of $s(k, u)$, this is possible only if $k \geq 0$, $u^* = \infty$.

The metric regularity condition at $u \rightarrow \infty$ implies, in particular, that the invariant $|R_N^M R_M^N| < \infty$. In our case the Ricci tensor R_N^M is diagonal and connected with the energy-momentum tensor by (4). Consequently, the above invariant is just a sum of squares and its finiteness implies the finiteness of each summand. As the invariant R_M^M is also finite, each component of T_N^M is finite as well. Recalling the explicit expressions for T_N^M , we obtain the requirements:

$$\begin{aligned}
\text{(a)} \quad &e^{-2\alpha} \varphi'^2 < \infty; \quad \text{(b)} \quad q^2 e^{-2\alpha+2\omega} < \infty; \\
\text{(c)} \quad &q'^2 e^{-2\alpha+2\psi} < \infty; \quad \text{(d)} \quad \bar{q}^2 e^{-2\alpha+2\chi} < \infty.
\end{aligned} \tag{52}$$

If $k > 0$, from (51) it follows $\gamma = -ku + O(1)$ at $u \rightarrow \infty$ and $\alpha(u)$ has the same asymptotic. Then by (52.a) φ' decays exponentially and $|\varphi(\infty)| < \infty$, so that $|\psi| = |\nu - 2\lambda\varphi| < \infty$. As $\alpha \rightarrow \infty$, we can satisfy (52.c) only if $q' = 0$, in contrast to what was assumed.

The case $k = 0$ is somewhat more involved. At $u \rightarrow \infty$ by (51) $\gamma = -\ln u + O(1)$ and the same asymptotic has $\alpha(u)$. By (52.a) then $\varphi' = O(1/u)$ and $\varphi = C \ln u + O(1)$ where C is a constant. The finiteness condition (52.c) only implies then that either $q' = 0$, or $\lambda C \geq 1$.

In the latter case (52.d) can be satisfied only if $\bar{q} = 0$. Therefore let us assume that $q' \neq 0$, $\bar{q} = 0$, $\lambda C \geq 1$ and address to Eqs.(11) and (12) at the asymptotic $u \rightarrow \infty$. As $e^\omega \sim u^{-1-\lambda C}$ and $e^\psi \sim u^{-\lambda C}$, in both equations the leading terms are those with q'^2 and, subtracting them, one can write: $N\gamma'' - \nu'' = O(u^{-2-2\lambda C})$. Since $2 + 2\lambda C \geq 4$, we conclude that $N\gamma - \nu$ at the asymptotic $u \rightarrow \infty$ can be either a constant, or a linear function of u , both possibilities being inconsistent with $|\nu(\infty)| < \infty$ and $\gamma = -\ln u + O(1)$.

There is still one more possibility, namely, that the map with the coordinate u is incomplete in the present static frame of reference. This may happen if $u = \infty$ is

a regular surface and another coordinate must be used to penetrate beyond it, where anything may be found, in particular, a horizon. This is, however, not the case for our system. Indeed, assume that β , β_i , ν and γ are finite at $u = \infty$. Then from (18) it follows that $s(k, u)$ must have a finite limit at $u \rightarrow \infty$, contrary to the definition of $s(k, u)$.

Thus a BH-horizon is inconsistent with $q' \neq 0$. By symmetry of our equations with respect to γ and ν , a T-horizon is inconsistent with $q \neq 0$. (The only asymmetry, the possibility of $\eta_v = -1$, is insignificant for the above argument). The theorem is proved. •

Another general statement valid for all λ is

Theorem 2. *The field system (1,6,7,10) cannot form a static, spherically symmetric configuration with a regular center.*

A proof makes use of the same type of argument as that of Theorem 1. •

Remark. A regular center assumes the regularity of g_{MN} and φ . Consequently, the statement of Theorem 2 is valid as well for all other conformal gauges connected with g_{MN} by factors of the form $\exp(\text{const} \cdot \varphi)$. The same is true for Theorem 1 if the (natural) additional requirement is adopted that φ should be finite at a horizon.

Thus, although our system consists of interacting fields, they cannot create a particle model with a regular center. On the other hand, regular spherical configuration with no center at all, like wormholes or ‘‘cornucopions’’ [1] are certainly not excluded, as seen, e.g., from the next section.

6. String metric

6.1. Properties of Solution G

As already pointed out, in the case $\lambda = \lambda_{\text{string}} = \pm\sqrt{N+1}$ it is more adequate to study the field behavior in terms of the so-called string metric

$$\hat{g}_{MN} = e^{-2\lambda\varphi} g_{MN} \quad (53)$$

rather than the metric g_{MN} in the ‘‘Einstein gauge’’ (such that the coefficient by ${}^D R$ in the Lagrangian is constant), the most convenient one for solving the equations. For other λ the same can be done by analogy, leading to the action (1) in terms of \hat{g}_{MN}

$$S = \int d^D x \sqrt{\hat{g}^D} e^{(N+1)\lambda\varphi} \left\{ \hat{R}^D + [1 - \lambda^2(N+1)(N+2)] \hat{g}^{MN} \varphi_{,M} \varphi_{,N} - \hat{F}^2 \right\} \quad (54)$$

where symbols with hats denote quantities obtained with or corresponding to \hat{g}_{MN} .

Nevertheless, we will discuss some features of the solutions only in the case $\lambda = \lambda_{\text{string}}$ for which the

metric \hat{g}_{MN} is manifestly meaningful. Moreover, as we saw in Sect.3, $\lambda = \lambda_{\text{string}}$ is the condition under which solutions with more than component of F^a can be obtained.

For $\lambda = \lambda_{\text{string}}$ we can address to the general Solution G, all the others being its special cases. It is described by Eqs.(37)-(41) with the string metric given by the expression in curly brackets in (36).

For $\eta_v = 1$ Solution G may be written as follows:

$$d\hat{s}_D^2 = \frac{dt^2}{2q^2 s^2(h_1, u + u_1)} - \frac{2\bar{q}^2 s^2(h_3, u + u_3) \left[\frac{du^2}{s^2(k, u)} + d\Omega^2 \right]}{s^2(k, u)} + \frac{\eta_v dv^2}{2q^2 s^2(h_2, u + u_2)} + \sum_{i=2}^n e^{2b_i u} ds_i^2; \quad (55)$$

$$2k^2 \text{sign } k = \sum_{a=1}^3 h_a^2 \text{sign } h_a + \sum_{i=2}^n N_i b_i^2; \quad (56)$$

the fields φ and F^a are determined by (40) and (10), respectively.

The coordinate u is defined in the range

$$0 < u \leq u_{\text{max}} = \min\{z(s_0), z(s_1), z(s_2), z(s_3), \infty\} \quad (57)$$

where $u = 0$ corresponds to spatial infinity, $s_0 = s(k, u)$, $s_a = s(h_a, u + u_a)$ ($a = 1, 2, 3$) and $z(s)$ is the smallest positive zero of the function s .

Thus the solution behaviors for different sets of integration constants are naturally classified by the variants of u_{max} by (57). We will label them by boldface figures from **0** to **4**, respectively. For instance, **4** labels solutions with $u_{\text{max}} = \infty$ (i.e., $k \geq 0$, $h_a \geq 0$, $u_a > 0$); **12** corresponds to the special case when the first zeros of s_1 and s_2 coincide and are smaller than those of s_0 and s_3 (if the latter exist), etc.

Let us briefly outline the properties of the solution.

- 0.** The value $u = u_{\text{max}} = \pi/|k|$ corresponds to the second spatial infinity: $r^2 \equiv \hat{g}_{\theta\theta} \rightarrow \infty$. The metric is regular; moreover, the asymptotic $u \rightarrow u_{\text{max}}$ is flat as well as $u \rightarrow 0$ (the proper asymptotic radius/circumference relation is valid). Thus the physical space section of the space-time forms a wormhole with (in general) different values of $\hat{g}_{tt} \equiv e^{2\omega}$, $\hat{g}_{vv} \equiv \eta_v e^{2\psi}$ and the scale factors $e^{b_i u}$ at the two asymptotics.
- 1.** $u = u_{\text{max}}$ is a singular sphere of a finite radius where $\hat{g}_{tt} = e^{2\omega} \rightarrow \infty$.
- 2.** The same as **1** with \hat{g}_{tt} replaced by $\hat{g}_{vv} = \eta_v e^{2\psi}$. In this case the 4-dimensional part of the metric is regular, so that the D -curvature singularity is connected with the 5th dimension.
- 3.** At $u = u_{\text{max}}$ the radius r is zero. The space-time has a singular center.

- 01.** The same geometry as in the case **0** (the 3-dimensional section forms a wormhole) but $\hat{g}_{tt} \rightarrow \infty$ at $u \rightarrow u_{\max}$.
- 02.** The same as **01** with \hat{g}_{tt} replaced by \hat{g}_{vv} .
- 03.** $u = u_{\max}$ corresponds to a finite radius; moreover, the integral $\int \sqrt{|\hat{g}_{uu}|} du$ diverges, meaning that the 3-dimensional space section forms an infinitely long tube, or “horn” like that described e.g. in [1]. The solution as a whole is nonsingular.
- 12.** The same as **1** and **2** but both \hat{g}_{tt} and \hat{g}_{vv} are infinite at $u = u_{\max}$.
- 13, 23.** Space-times with a singular center where, respectively, \hat{g}_{tt} or \hat{g}_{vv} is infinite.
- 012, 013, 023, 123, 0123.** The triple and quadruple combinations are also easily described, thus, in each case the figure **1** involved corresponds to a singularity of \hat{g}_{tt} , **2** to that of \hat{g}_{vv} and the combination **03** to a “horn”.

Evidently the cases **0**, **1**, **2**, **3** are general, the double combinations require an additional relation among the integration constants, the triple and quadruple ones are still more special.

The possibility **0** can be realized only if all $h_a < 0$, as follows from (56). This in turn means that all the charges q , q' , \bar{q} must be nonzero.

On the other hand, the possibility **2** requires $h_2 < 0$ or/and $u_2 < 0$. This is possible only if $\eta_v = +1$. In the case $\eta_v = -1$ the function $s(h_2, u + u_2)$ is replaced by the positive-definite function $h_2^{-1} \cosh h_2(u + u_2)$, with $h_2 > 0$. Along with the aforesaid, that means that all the variants involving **0** and **2** are eliminated for configurations where the fifth coordinate v is spacelike: from the above diversity of behaviors only the singular variants **1**, **3** and **13** survive.

In all cases when $u_{\max} < \infty$ the extra-dimension scale factors $e^{b_i u}$ are regular in the whole space.

- 4.** $u_{\max} = \infty$ if all $h_a > 0$, $u_a > 0$, whence by (56) $k \geq 0$. At the asymptotic $u \rightarrow \infty$ the metric coefficients $\hat{g}_{tt} \rightarrow 0$ (unless $q = 0$, $h_1 = 0$) and $\hat{g}_{vv} \rightarrow 0$ (unless $q' = 0$, $h_2 = 0$). (Recall that at $q = 0$ the function s_1 is replaced by $e^{h_1 u}$ and similarly for $q' = 0$ and $\bar{q} = 0$.) The behavior of the radius $r = \sqrt{|\hat{g}_{\theta\theta}|}$ depends on the relation between k and h_3 : $r \sim e^{(h_3 - k)u}$. In general, the surface $u = \infty$ is singular, with an exception deserving a separate description: solutions with BH and T-horizons.

6.2. Dyon black holes

As follows from Theorem 1 and is directly verified for (55), the most general solution with a BH horizon is

(50), with three free parameters k , q and \bar{q} . The solution looks more transparent in curvature coordinates, with the notations

$$\begin{aligned} r &= R + \bar{p}; & r_e &= p = \sqrt{k^2 + 2q^2} - k; \\ r_m &= \bar{p} = \sqrt{k^2 + 2\bar{q}^2} - k; & r_+ &= 2k + r_m. \end{aligned} \quad (58)$$

Namely, the string metric and the dilaton field are

$$\begin{aligned} d\hat{s}_D^2 &= \frac{(r-r_+)(r-r_m)}{(r+r_e-r_m)^2} dt^2 \\ &\quad - \frac{r^2 dr^2}{(r-r_+)(r-r_m)} - r^2 d\Omega^2 + \sum_{i=1}^n ds_i^2, \quad (59) \\ e^{2\lambda\varphi} &= \left(1 + \frac{r_e - r_m}{r}\right)^{2/(D-2)}. \quad (60) \end{aligned}$$

In (59) the extra dimensions are “frozen” and exert no influence on the 4-dimensional part of the metric, which is thus universal for all $D \geq 4$. The only trace of multidimensionality is the exponent in (60). Another observation of interest is the striking asymmetry between the electric and magnetic fields represented here by the parameters r_e and r_m . This distinguishes dilaton field theory from the 4-dimensional Einstein-Maxwell theory.

The space-time structure described by (59) depends on the values of the three parameters $r_e \geq 0$, $r_m \geq 0$ and $r_+ \geq r_m$:

- (a) $r_e = r_m = 0$, corresponding to $q = \bar{q} = 0$: the Schwarzschild metric.
- (b) $r_e = 0$, $r_+ = r_m > 0$: $\hat{g}_{tt} \equiv 1$; $r = r_+$ corresponds to an infinitely long regular “horn”, just the case described in [1] and papers cited therein.
- (c) $r_e = 0$, $r_+ > r_m > 0$: the sphere $r = r_+$ is a Schwarzschild-like horizon, $r = r_m$ is a singularity inside it, with $\hat{g}_{tt} \rightarrow \infty$.
- (d) $r_e > 0$, $r_m = r_+ = 0$: $r = 0$ is a naked singularity, with $\hat{g}_{tt} \rightarrow 0$.
- (e) $r_e > 0$, $r_m = 0$, $r_+ > 0$: $r = r_+$ is a Schwarzschild-like horizon, $r = 0$ is a central singularity, again with $\hat{g}_{tt} \rightarrow 0$.
- (f) $r_+ = r_m > r_e > 0$: $r = r_+$ is a horizon of extreme RN type, $r = r_m - r_e$ is a singular sphere inside it ($\hat{g}_{tt} \rightarrow \infty$).
- (g) $r_+ > r_m > r_e > 0$: $r = r_+$ and $r = r_m$ are analogs of the outer and inner RN horizons; the sphere $r = r_m - r_e$ is like that in (f).
- (h) $r_e \geq r_m > 0$; $r_+ = r_m$: $r = r_+$ is an extreme RN-like horizon, $r = 0$ is a central singularity, where \hat{g}_{tt} is infinite if $r_e = r_m$ and finite if $r_e > r_m$.

- (i) $r_e \geq r_m > 0$; $r_+ > r_m$: $r = r_+$ and $r = r_m$ are outer and inner RN-like horizons; $r = 0$ is like that in (h).

The dilaton field φ is regular at all horizons, including inner ones, and singular at the singularities $r = 0$ or $r = r_m - r_e$. In the case $r_m = r_e$, corresponding to $q = \bar{q}$, the φ field is constant and the 4-metric is just the RN one; in its usual notation [$\hat{g}_{tt} = (-\hat{g}_{rr})^{-1} = r^{-2}(r^2 - 2Mr + Q^2)$] its parameters M and Q are connected with ours as follows:

$$Q^2 = 2q^2 = q^2 + \bar{q}^2; \quad M = \sqrt{k^2 + 2q^2} \geq |Q|.$$

We see that, although the metric (59) was obtained in search for solutions with horizons, one of its special cases (d) has a naked singularity, while another one, (b), is nonsingular and, in the opinion of some researchers, may describe the final state of evaporated BHs. To this end it should be emphasized that these ‘‘horned particles’’ form a very special subset in the set of solutions, with a single parameter, the magnetic charge.

6.3. T-holes

In addition to the above family of BH solutions, there is a similar family of T-hole ones, obtained from the former by the substitution $t \leftrightarrow v$, $q \rightarrow q'$, so that the horizons occur in the (u, v) subspace. However, there are certain problems connected with the compactification of extra dimensions, which are most clearly understood on the following simple example.

Putting $q' = 0$ in (46), we come to a direct analog of the Schwarzschild solution (to be called *T-Schwarzschild*) which for $D = 5$ coincides with the zero dipole moment soliton in the terminology of [26]:

$$ds_D^2 = (1 - 2k/R)\eta_v dv^2 + dt^2 - (1 - 2k/R)^{-1}dR^2 - R^2 d\Omega_2^2 + \sum_{i=2}^n ds_i^2 \quad (61)$$

while both fields φ and F are zero, so that $e^{\lambda\varphi} \equiv 1$ and $\hat{g}_{AB} = g_{AB}$.

At $R = 2k$ the signs of g_{uu} and g_{vv} simultaneously change. Moreover, if $\eta_v = 1$, i.e., this direction is timelike at big R , the overall signature of V^D is preserved but in the opposite case, $\eta_v = -1$, it is changed by four: two spacelike directions become timelike. However, as is directly verified, a T-horizon is not a curvature singularity, either for the D -dimensional metric or for its 4-dimensional section.

If $\eta_v = 1$, the surface $R = 2k$ is a Schwarzschild-like horizon in the (R, v) subspace and an analytic continuation to $R < 2k$ with the corresponding Kruskal picture exists. However, if some points on the v axis are identified, as should be done to compactify V_1 , then the corresponding sectors (wedges) are cut out in the Kruskal diagram, so that the T -domain

and R -domain sectors join each other only in a single point, namely, the horizon intersection point.

Another thing happens if $\eta_v = -1$. A further study is again possible after a transition to coordinates in which the metric is manifestly nonsingular at $R = 2k$. Let us perform it for (61) in the vicinity of the T-horizon, $R \rightarrow 2k$ (the more general case is treated similarly):

$$R - 2k = x^2 + y^2/(8k); \quad v = 4k \arctan(y/x);$$

$$ds_2^2(R, v) \approx \frac{R - 2k}{2k} dv^2 + \frac{2k}{R - 2k} dR^2 = dx^2 + dy^2. \quad (62)$$

Thus the (R, v) surface metric is locally flat near the T-horizon $R = 2k$, transformed to the origin $x = y = 0$, while the v coordinate behaves like an angle.

This transformation could be conducted as a conformal mapping of the complex plane with the aid of the analytic function $\ln z$, $z = x + iy$, as was done in Ref.[4] for some cylindrically symmetric Einstein-Maxwell solutions; then v is proportional to $\arg z$.

Consequently, in the general case the (R, v) surface near $R = 2k$ behaves like the Riemann surface having a finite or infinite (if v varies in an infinite range) number of sheets, with a branching point at $x = y = 0$ (a branching-point singularity [4]). If V_1 is compactified, v is naturally described as an angular coordinate ($0 \leq v < 2\pi l$, where $v = 0$ and $v = 2\pi l$ are identified and l is the compactification radius at the asymptotic $R \rightarrow \infty$). $R = 2k$ is just the center of symmetry in the (R, v) surface; the latter has the shape of a tube having a constant thickness at $R \rightarrow \infty$, becoming narrower at smaller R and ending at $R = 2k$ either smoothly (if the regular center condition $l = 2(2k + p)$ is satisfied), or with a conic or branching-point singularity (otherwise). This suggests that there is no way to go beyond $R = 2k$.

In the singular case the geodesic completeness requirement is violated at the horizon, so it is reasonable to require $l = 2(2k + p)$, or, more generally, $l = 2j(2k + p)$ where j is a positive integer, so that $R = 2k$ is a j -fold branching point. In this case a radial geodesic, whose projection to the (R, v) surface hits the point $R = 2k$, passes through it and returns to greater values of R but with another value of v , thus leaving the particular 4-dimensional section of the D -dimensional space-time. However, if the quantum wave function of the corresponding particle is v -independent, the particle does not disappear from an observer’s sight and can look as if reflected from a mirror. It can be concluded that a T-horizon with $\eta_v = -1$ looks observationally like a mirror.

Another thinkable possibility is to consider a continuation beyond $R = 2k$, similarly to the way a cone is continued through its vertex. In this case, however, the space-time as a whole is no longer a manifold and, moreover, the neighborhoods of points belonging to

the horizon must be specially defined to preserve the Hausdorff nature of the space-time. Whether or not it is possible, is yet to be studied. Physically such a continuation would mean that a particle getting to $R = 2k$ "has a choice" either to return to greater R , or to penetrate to smaller R , to the domain with another signature. One can assume that its probabilistic behavior is describable in terms of quantum concepts.

If such an exotic possibility is not considered, the T-horizons are regular, although peculiar, surfaces of the D -dimensional space-time.

The 4-dimensional sections of T-hole space-times are curved but nonsingular in the range $R \geq 2k$; a possible continuation to $R < 2k$ is discussed above and depends on η_ν or, if $\eta_\nu = -1$, on additional assumptions.

7. D -dimensional solutions from the 4-dimensional viewpoint

A 4-dimensional version of (1) is obtained by integrating out the extra-dimension coordinates, so that (up to a constant factor and a divergence)

$$S = \int d^4x \sqrt{4} g e^\sigma \left[R^{(4)} - \sigma^{;\mu} \sigma_{;\mu} + \sum_{i=1}^n N_i \beta_{i,\mu} \beta_i^{;\mu} + \varphi^{;\mu} \varphi_{;\mu} - e^{2\lambda\varphi} F^{\mu\nu} F_{\mu\nu} - 2\eta_\nu e^{-2\nu+2\lambda\varphi} W^{;\mu} W_{;\mu} \right] \quad (63)$$

where Greek indices range from 0 to 3, 4R is derived from the 4-dimensional part $g_{\mu\nu}$ of g_{MN} , and, as before, $\sigma = \sum_{i=1}^n N_i \beta_i$. The Maxwell tensor $F_{\mu\nu}$ includes both F^1 and F^3 , i.e., the electric and magnetic fields, while F^2 is re-formulated in terms of an effective scalar field W : $F^2 = dW^2$, $W^2 = W(x^\alpha) dv$ where, as before, v parametrizes the one-dimensional subspace V_1 and $\nu = \beta_1$. Both $F_{\mu\nu}$ and $W(x^\alpha)$ are coupled to φ and all β_i . The field W is minimally coupled to $g_{\mu\nu}$ and the sign of its kinetic term depends on η_ν : it is normal if v is spacelike and anomalous if v is timelike.

Eq.(63) is written in the original D -Einstein conformal gauge. The 4-dimensional Einstein gauge with the metric $\bar{g}_{\mu\nu}$ is obtained by the conformal mapping similar to that used by Dicke [17] and Wagoner [40]

$$\bar{g}_{\mu\nu} = e^\sigma g_{\mu\nu} \quad (64)$$

after which the action takes the form

$$S = \int d^4x \sqrt{4\bar{g}} \left[{}^4\bar{R} + \frac{1}{2} \sigma^{;\mu} \sigma_{;\mu} + \sum_{i=1}^n N_i \beta_{i,\mu} \beta_i^{;\mu} + \varphi^{;\mu} \varphi_{;\mu} - e^{\sigma+2\lambda\varphi} F^{\mu\nu} F_{\mu\nu} - 2\eta_\nu e^{-2\nu+2\lambda\varphi} W^{;\mu} W_{;\mu} \right] \quad (65)$$

where indices are raised and lowered using $\bar{g}_{\mu\nu}$.

The actions (1) and (65) are equally convenient for solving the field equations due to the constant effective gravitational coupling. Noteworthy, the coordinate u as introduced in (8) is harmonic with respect

to both the D -metric g_{MN} and the 4-metric $\bar{g}_{\mu\nu}$, i.e., $\nabla^M \nabla_M u = \bar{\nabla}^\mu \bar{\nabla}_\mu u = 0$, but not with respect to the 4-dimensional part $g_{\mu\nu}$ of the D -metric: $\nabla^\mu \nabla_\mu u \neq 0$.

The metric $\bar{g}_{\mu\nu}$ thus corresponds to the so-called *gravitational system of measurement* [17, 38]. However, real space-time measurements, such as solar-system experiments, rest on the constancy of atomic quantities (the *atomic system of measurements*). Thus, the modern definition of reference length is connected with a certain spectral line, determined essentially by the Rydberg constant and ultimately by the electron and nucleon masses. Therefore observational properties of various theoretical models are most reasonably described in a conformal gauge where masses of bodies of nongravitational matter, such as atomic particles, do not change from point to point.

In other words, in the gauge to be selected (to be denoted $g_{\mu\nu}^*$) the nongravitational matter Lagrangian L_m should enter into the action with no σ - or φ -dependent factor. However, the choice of $g_{\mu\nu}^*$ depends on how L_m appears in the original action, that is, how matter is coupled to the metric and dilaton fields in the underlying fundamental theory.

In [25], where the effective field-theoretic limit of string theory in 10 dimensions is given in a form similar to (1) (Eq.(13.1.49)), some quadratic fermion terms do not contain the dilaton. If those terms are associated with matter, then in our simplified model it is reasonable to write L_m just as an additional term in the brackets of Eq.(1). Then, passing over to the 4-dimensional formulation, it is easy to check that the metric in the "atomic gauge" should have the form

$$g_{\mu\nu}^* = e^{\sigma/2} g_{\mu\nu} \quad (66)$$

The term L_m would enter (63) and (65) with the factors e^σ and $e^{-\sigma}$, respectively, whereas in terms of $g_{\mu\nu}^*$ the matter part of the action is just $\int d^4x \sqrt{g^*} L_m$. The same metric $g_{\mu\nu}^*$ would be obtained if we wrote the action for a point particle moving in D dimensions in the conventional form $-\int m ds$ and required that it move along geodesics of the 4-dimensional metric $g_{\mu\nu}^*$.

The notion of active gravitating mass of an isolated object in a space with the structure (6) is to be also introduced with the aid of $g_{\mu\nu}^*$, by comparing g_{tt}^* far from the source with the expression $(1 - 2GM/r)$ of the Schwarzschild metric, so that, with an arbitrary radial coordinate u [3]

$$GM = -|g_{\theta\theta}^*|^{3/2} \frac{\partial_u g_{tt}^*}{\partial_u g_{\theta\theta}^*} \Big|_{u \rightarrow u_\infty} = \frac{r^2 \gamma^{*t}}{r'} \Big|_{u \rightarrow u_\infty} \quad (67)$$

where $e^{2\gamma^*} = g_{tt}^*$, $r^2 = -g_{\theta\theta}^*$ and u_∞ is the value of u where $r \rightarrow \infty$ and $\gamma^* \rightarrow 0$.

As in Sect.6, let us consider only solutions for $\lambda = \lambda_{\text{string}}$. In the general case of Solution G the mass is

expressed in terms of the constants k , h_a , q_a , b_i :

$$GM = \frac{1}{4(D-2)} \left[\sum_{a=1}^3 c_a \sqrt{2q_a^2 + h_a^2 \text{sign } h_a} + 2b \right];$$

$$c_1 = 3D - 8, \quad c_2 = -2, \quad c_3 = D. \quad (68)$$

For the dyon solution (50) the mass is connected with the charges and the other constants in the following way:

$$GM = k + \frac{1}{4(D-2)} [(3D-8)p + D\bar{p}]$$

$$= \frac{r_+}{2} + \frac{1}{4(D-2)} [(3D-8)r_e + (D-4)r_m] \quad (69)$$

The expression (69) can be obtained from (68) by substituting $q' = h_2 = b = 0$, $h_1 = h_3 = k$. Noteworthy, the mass (69) is nonnegative, while in the general case (68) the sign is not fixed.

For T-holes with a magnetic charge, obtained from (50) by the substitution $t \leftrightarrow v$, $q \rightarrow q'$, $p \rightarrow p'$, we obtain:

$$GM = \frac{k}{4} + \frac{1}{4(D-2)} (D\bar{p} - 2p'). \quad (70)$$

For other solutions GM can be easily found as well using (66) and (67).

Eq.(69) generalizes the corresponding relations for dilatonic BHs in 4 dimensions and Eq. (67) of Ref.[8] for electrically charged BHs in D dimensions. Thus, extreme BHs, those with the smallest mass for given charges, correspond to $k = 0$. At $k \rightarrow 0$ the horizon is squeezed to a point (the center) and becomes a singularity.

As for T-holes, Eq.(70) (generalizing Eq.(68) from [8]) shows that they can have negative gravitational masses, i.e., be felt by test particles as repellers. That happens if $\eta_v = +1$ for large q' (see (46)).

In conformal gauges other than (66) test particle trajectories $x^j(t)$ ($j = 1, 2, 3$) are certainly the same but they are no longer geodesics: particles fell both the metric and the scalar field. A similar situation was discussed by Dicke [17].

8. Discussion

8.1. The solutions obtained here are less general as compared with those of Refs. [21, 14] in that they do not contain the Taub-NUT parameter, axion charge and Kerr rotation parameter. However, they are more general in that they include extra dimensions (the corresponding n parameters are b_i and the additional gauge field component F^2 (the parameter $q_2 = q'$)).

Another point of interest is connected with the possible non-string couplings: the remarkable symmetries of string theory, which enabled the authors of Refs. [21, 14] and some earlier papers to obtain string-theory

solutions without actually solving the equations, do not work when $\lambda \neq \lambda_{\text{string}}$. Some such solutions are presented here but they are less general than those with $\lambda = \lambda_{\text{string}}$. Apparently the same symmetries explain the total integrability of the field equations which follow from string theory.

The approach connected with a direct study of the field equations has made it possible not only to obtain some solutions within less symmetric versions of field theory, but also to prove certain statements (Theorems 1 and 2) for situations when it is hard to obtain exact solutions.

8.2. The stability of Solution E under small perturbations preserving spherical symmetry was studied in Refs. [7, 12, 2]. The system with two dynamic degrees of freedom (the dilaton field and a single extra-dimension scale factor) was considered and three cases when the perturbation equations decouple were studied in detail. It was concluded that solutions with naked singularities are catastrophically unstable, while the BH ones are stable. These results generalized the earlier ones from [10] and [11] where the instability of static scalar-(electro)vacuum configurations in conventional general relativity, in particular, black holes with scalar charge, was established.

In Ref. [8] the earlier results were confirmed and slightly extended; in particular, the simplest example of a T-hole (the uncharged one, Eq.(61) with $a_0 = 0$, $a_1 = 1$) was investigated and shown to be unstable.

Among the static, spherically symmetric configurations in dilaton gravity studied to-date only BHs turned out to be stable under monopole perturbations. It would be of importance to extend the stability investigation to other configurations, in particular, those described here and in the papers [21, 14].

8.3. We have seen that T-holes, the possible windows to space-time domains with unusual physics, appear as solutions to multidimensional field equations as frequently as do black holes.

As in the present state of the Universe extra dimensions are generally believed to be compactified to a very small size, it is hard to imagine how T-holes might now form from ordinary matter. However, in the early Universe where all dimensions could be equally relevant, T-holes could form on equal grounds with primordial black holes and consequently their relics might play a certain role at the present stage, e.g., be one of the forms of dark matter (as suggested, e.g. by Wesson [41]). However, such a possibility looks questionable due to the instability of these objects. The latter is so far established only for the above simplest, uncharged T-hole solution. However, very probably the same is true for charged T-holes since perturbations near a T-horizon behave just as they do near a singularity [8], while all singular solutions studied so far [7, 12, 2] turned out to be unstable. Nevertheless, T-hole solutions in other field models, which

exist for sure, may turn out to be stable, although, on the other hand, it may happen that there is a kind of “censorship” like Hawking’s chronology protection conjecture [27], confirmed, in particular, by the instability of Cauchy horizons.

8.4. Certain difficulties in the T-hole description arise due to the compactification of extra dimensions. However, the latter may be invisible to 3-dimensional observers for a reason other than their small size, such as, e.g., the behavior of field potentials, as discussed in Ref. [34] (we may live within a 3-dimensional “membrane” at the bottom of a potential “trench” in a multidimensional world). A possible BH and T-hole existence in such models may be a subject of interest for further study.

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