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MULTIDIMENSIONAL INTEGRABLE COSMOLOGICAL MODELS WITH NEGATIVE EXTERNAL SPACE CURVATURE

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Multidimensional cosmological models with n $(n > 1)$ Einstein spaces are discussed classically and with respect to canonical quantization. These models are integrable in the case of Ricci-flat internal spaces. For negative external space curvature we find exact classical solutions modelling both dynamical and static internal spaces. Solutions with static internal spaces turn out to be an attractor. Solutions of the quantum Wheeler-DeWitt equation are also obtained. Some of them describe the tunnelling process to be interpreted as the birth of the universe from "nothing".

1. Introduction

Everyday experience seems to show evidently that we are living in a four dimensional space-time. Why should we speak about a multidimensional universe? There are good reasons to do so. First of all, we know that consistent theories unifying fundamental interactions take place in multidimensional spaces only, and may be this is not purely a question of mathematical methods and extra dimensions are a physical reality. Second, extra dimensions are actually not observable, because they are extremely small at present time. If they are of the scale of Planck's length ($L_{PL} \sim 10^{-33}$cm) their observation is impossible due to the necessary super-high frequency (energy), and beyond the Planck length quantum uncertainties forbid the observation. Nevertheless, all dimensions, including ours and internal ones, might have been of the same scale at early stages of the universe. Moreover, extra dimensions could be much larger then ours at that time. Thus, there is a reason to investigate multidimensional cosmological models and the observable consequences of the possible existence of extra dimensions.

In all multidimensional cosmological models (MCM) a mechanism of dimensional reduction or, in other words, compactification of the extra dimensions should be present. There are two approaches to realize compactification. In the first case the internal dimensions become much smaller than our external ones during the evolution of the universe. These are the MCM with dynamical compactification. Observable consequences of extra dimensions are in this case possible variations of effective constants of nature (like the gravitation constant) [1–4], imprints in cosmic rays of ultrahigh energy [5] or in the spectrum of gravitational waves [6]. Another possibility consists in the proposal that all extra dimensions are static and small from the very beginning. The presence of extra dimensions leads in this case to the generation of particle masses [7–9]. In both of these approaches the presence of extra dimensions very strongly affects the evolution of our external space-time. Compactification takes place for pure gravity as well as for gravity coupled to different matter fields. There is a large amount of papers devoted to these questions (see e.g. the references in [10]).

In our paper we consider MCM which consist of M_i $(i = 1, \ldots, n)$ spaces of constant curvature. This model was investigated from the classical as well as from quantum points of view in papers [11–21]. The model can be generalized to the case of all spaces being Einstein spaces. The multidimensional Einstein equations as well as the quantum Wheeler-DeWitt equation (WDW) can be integrated for this model if one of the spaces M_i is not Ricci-flat [11, 14, 17]. This property is unchanged if the model contains in addition a massless minimally coupled scalar field. From the point of view of internal space compactification this integrable model was considered in [10] in case the non-Ricci-flat space is of positive constant curvature. This space, let it be M_1, was considered to be our external space and all other factor spaces were internal ones. Both types of compactification were found. In the case of a real scalar field as a matter source in the Lorentzian domain the solution with spontaneous compactification permits an interesting continuation to the Euclidean

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domain describing Euclidean wormholes.

In the present paper we investigate these integrable models on both the classical and quantum levels for the case when the non-Ricci-flat space M_1 is of constant negative curvature. The main problem consists in the investigation of compactification of internal spaces. In the case of positive curvature of M_1 the parameter playing the role of energy may take positive values only in the Lorentzian domain [14]. In contrast to this case, the model with negative curvature permits both positive and nonpositive values of this parameter. This feature of the models with negative curvature leads to a richer and more interesting picture than in the former case.

The paper is organized as follows. In Section 2. we describe our MCM and represent the classical Einstein equations for this model in appropriate coordinates. Section 3. is devoted to the investigation of both dynamical and static internal spaces on the classical level. In Section 4. we consider the quantum properties of the model. The results of the paper are summarized in the Conclusions.

2. Minisuperspace cosmological models

Let us consider the metric

$$ds^2 = -d\tau^2 \mathrm{e}^{2\gamma(\tau)} + \sum_{i=1}^{n} e^{2\beta^i(\tau)} ds_i^2 \tag{1}$$

on a D-dimensional space-time manifold

$$M = R \times M_1 \times \ldots \times M_n \tag{2}$$

where the M_i are d_i-dimensional compact spaces of constant curvature with line elements ds_i^2. The connection to the scale factors a_i is given by $a_i = e^{\beta^i}$. The scalar curvature of M_i is normalized in such a way that we can write

$$R[g_{(i)}] = \theta_i = k_i d_i(d_i - 1), \qquad i = 1, \ldots, n, \tag{3}$$

where $k_i = 0, \pm 1$. In the case of nonpositive curvature the compactness condition for the internal spaces can be achieved by appropriate periodicity conditions for the coordinates [22].

As mentioned in the Introduction, this model can be generalized to the case of Einstein spaces M_i for which $R[g_{(i)}] = \lambda_i d_i$ instead of (3), and λ_i are arbitrary numbers. In formulas obtained later on this generalization is achieved by the trivial substitution $\theta_i \to \lambda_i d_i$.

We restrict our consideration to the important case when only one of the spaces M_i is not Ricci-flat with negative curvature: $\theta_1 < 0,\ \theta_i = 0,\ i = 2, \ldots, n$. In this case the cosmological model is a completely integrable system [14, 17]. This can be generalized by taking into account a minimally coupled free scalar field φ.

The action S for the model with the metric (1) and a minimally coupled scalar field can be written in the form [11]

$$S = \int \mathcal{L} dt \tag{4}$$

where the Lagrangian has the form

$$\begin{aligned} \mathcal{L} &= \frac{\mu}{2} \mathrm{e}^{-\gamma+\sigma} \left\{ \sum_{j=1}^{n} d_j (\dot\beta^j)^2 - \dot\sigma^2 + \kappa^2 \dot\varphi^2 \right\} \\ &- \frac{\mu}{2} e^{\gamma+\sigma} |\theta_1| e^{-2\beta^1}, \qquad \sigma \equiv \sum_{i=1}^{n} d_i \beta^i \end{aligned} \tag{5}$$

Here κ^2 denotes the gravitational constant and $\mu = \prod_{i=1}^{n} V_i/\kappa^2$ where V_i is the volume of M_i: $V_i = \int_{M_i} d^{d_i} y (\det(g_{m_i n_i}))^{1/2}$. The metric (1) can be normalized in such a way that $\mu = 1$. After normalizing μ we may also consider noncompact spaces. We shall also use natural units with $\kappa^2 = 1$.

The analysis of this system will be done mostly in two time gauges, in the gauge of harmonic time τ [11] where $\gamma = \sigma = \sum_{i=1}^{n} d_i \beta^i$, and in the gauge of synchronous time t with $\gamma = 0$.

The possibility of a free choice of gauge implies the following constraint equation:

$$e^{-\gamma} \left[\sum_{i=1}^{n} d_i (\dot\beta^i)^2 - \dot\sigma^2 + \dot\varphi^2 \right] + e^{-\gamma} |\theta_1| e^{-2\beta^1} = 0 \tag{6}$$

It was shown in [14] that the field equations for this model can be integrated most easily using the following coordinates:

$$\begin{aligned} q v^0 &= (d_1 - 1)\beta^1 + \sum_{i=2}^{n} d_i \beta^i, \\ q v^1 &= [(D-2)/(d_1 \Sigma_2)]^{1/2} \sum_{i=2}^{n} d_i \beta^i, \\ q v^i &= [(d_1 - 1) d_i / (d_1 \Sigma_i \Sigma_{i+1})]^{1/2} \sum_{j=i+1}^{n} d_j (\beta^j - \beta^i), \\ &\qquad i = 2, \ldots, n-1. \end{aligned} \tag{7}$$

Here we used the notation $D = 1 + \sum_{i=1}^{n} d_i$, $q^2 = (d_1 - 1)/d_1$, and $\Sigma_i = \sum_{j=i}^{n} d_j$. In the Lorentzian domain and in harmonic time gauge the Lagrangian and the constraint equation take the form

$$\mathcal{L} = \frac{1}{2} \left(-(\dot v^0)^2 + \sum_{i=1}^{n-1} (\dot v^i)^2 + \dot\varphi^2 \right) - \frac{1}{2} |\theta_1| e^{2 q v^0} \tag{8}$$

and

$$-(\dot v^0)^2 + \sum_{i=1}^{n-1} (\dot v^i)^2 + \dot\varphi^2 + |\theta_1| e^{2 q v^0} = 0 \tag{9}$$

The dot denotes a derivative with respect to the harmonic time τ. The consideration of the system can be generalized to the Euclidean domain by analytic continuation.

The equations of motion corresponding to the Langrangian (8) read

$$\begin{aligned} \ddot{v}^0 - q|\theta_1|e^{2qv^0} &= 0 \\ \ddot{v}^i &= 0, \qquad i = 1, \ldots, n-1 \\ \ddot{\varphi} &= 0. \end{aligned} \tag{10}$$

The last two equations are easily integrated. We find

$$\begin{aligned} v^i &= \nu^i \tau + c^i, \qquad i = 1, \ldots, n-1 \\ \varphi &= \nu^n \tau + c^n \end{aligned} \tag{11}$$

where the ν^i and c^i are constants of integration. In the minisuperspace of vectors $\vec{v} = (v^0, v^1, \ldots, v^{n-1}, v^n \equiv \varphi)$ the indices are raised and lowered by the diagonal metric $\eta = (-1, +1, \ldots, +1)$ [14]. Thus, we have $v^0 = -v_0,\ v^i = v_i,\ \nu^i = \nu_i$ and $c^i = c_i,\ i = 1, \ldots, n$. Now the constraint equation may be rewritten as

$$(\dot{v}^0)^2 - |\theta_1|e^{2qv^0} = \epsilon \tag{12}$$

with

$$\epsilon = \sum_{i=1}^{n} (\nu^i)^2 \tag{13}$$

It can be seen from (12) that ϵ can be treated as an energy. This was shown in more detail in [14].

3. Classical solutions

Equations (10 - 13) are written in the Lorentzian domain. The parameters ν_i $(i = 1, \ldots, n)$ are momenta in minisuperspace. Thus, $E \equiv \epsilon/2$ plays the role of energy [14]. Equation (12) shows that we can consider both cases, $\epsilon \leq 0$ and $\epsilon > 0$. We have to demand the metric to be real in the Lorentzian domain. In what follows, the momenta ν_i $(i = 1, \ldots, n-1)$ should be real there (see Eq. (11). The case $\epsilon = 0$ is treated as the ground state where all momenta are put equal to zero: $\nu_i = 0$ $(i = 1, \ldots, n)$. For $\epsilon > 0$ all ν_i $(i = 1, \ldots, n)$ are considered to be arbitrary real numbers. In the case $\epsilon < 0$ the demand of a real metric in the Lorentzian domain leads to the condition that all ν_i $(i = 1, \ldots, n-1)$ are real and the condition $\epsilon < 0$ and Eq. (13) are compatible only for a purely imaginary ν_n. This means that the scalar field in this case must be imaginary in the Lorentzian domain and we have the following condition:

$$\sum_{i=1}^{n-1} (\nu_i)^2 - |\, \nu_n \,|^2 < 0 \tag{14}$$

Let us consider the three special cases $\epsilon = 0,\ \epsilon < 0$, and $\epsilon > 0$ separately.

3.1. The case $\epsilon = 0$

As mentioned above, in this case we put $\nu_i = 0$ $(i = 1, \ldots, n)$. Then it can be seen from Eqs. (11) and (7) that all scale factors are static $(a_i = e^{\beta^i} = a_{0(i)},\ i = 2, \ldots, n)$. In this case we have only one scale factor with dynamical behaviour (in our case a_1) and the corresponding factor space M_1 will be associated with the external (our real) space. The fixed scale factors are free parameters of the model assumed to be of the order of the Planck length $a_{(0)i} \sim L_{Pl} \sim 10^{-33}\,\mathrm{cm}$ $(i = 2, \ldots, n)$ to make internal dimensions unobservable.

The dynamical behaviour of the remaining developing scale factor can be determined from the solution of Eq. (12)

$$e^{qv^0} = ((d_1 - 1) \mid \tau \mid)^{-1}, \qquad -\infty < \tau < \infty \tag{15}$$

With the help of transformation (7) we find the expression for the scale factor a_1 in the harmonic time gauge

$$a_1(\tau) = [1/C(d_1 - 1) \mid \tau \mid]^{1/(d_1 - 1)} \tag{16}$$

where

$$C = \prod_{i=2}^{n} a_{(0)i}^{d_i}. \tag{17}$$

Then the metric in harmonic time takes the form

$$g = -e^{2\gamma(\tau)} d\tau \otimes d\tau + a_1^2(\tau) g_{(1)} + \sum_{i=2}^{n} a_{(0)i}^2 g_{(i)} \tag{18}$$

where

$$e^{\gamma(\tau)} = C a_1^{d_1}(\tau). \tag{19}$$

It makes sense to rewrite the metric in the synchronous time coordinate t. In this case we have $a_1(t) = \mid t \mid$ and the metric is given by

$$g = -dt \otimes dt + t^2 g_{(1)} + \sum_{i=2}^{n} a_{(0)i}^2 g_{(i)}. \tag{20}$$

From this expression we see that the dynamical part of the universe is described by the Milne model [23]. In this way we find for the physically interesting case of Kaluza-Klein theory with $d_1 = 3$ and M_1 being an open hyperbolic space that the solution with $\epsilon = 0$ describes the following topology of a spontaneously compactified universe

$$M^4 \times T^{d_2} \times \ldots \times T^{d_n}. \tag{21}$$

Here M^4 is the four dimensional Minkowski space-time and the T^{d_i} are d_i- dimensional tori (or other compact spaces of constant zero curvature). The tori are frozen and assumed to have scales of Planck size.

3.2. The case $\epsilon > 0$

As explained above, in this case all parameters $\nu_i, (i = 1, \ldots, n)$ are considered to be real. The solution to Eq. (12) takes the form

$$e^{qv^0} = \frac{\epsilon_1}{\left|\sinh\left[(d_1-1)\epsilon_1(\tau-\tau_0)\right]\right|}, \quad -\infty < \tau < +\infty \tag{22}$$

where τ_0 is a constant of integration and we have denoted $\epsilon_1 = \sqrt{\epsilon/|\theta_1|}$. Choosing the initial value of the harmonic time coordinate in an appropriate way we can put $\tau_0 = 0$.

Eqs.(22) and (11) give a general solution to the system (10) with constraint (12). But in cosmology the synchronous time coordinate t is conventionally used and it is quite difficult to find the dependence of the scale factors $a_i = e^{\beta^i}$ on this time coordinate. But the explicit dependence on t can be found in some interesting special cases which we now consider.

3.2.1. The 2-component universe. Dynamical compactification

Let us consider the special case where only two factor spaces are included in the model. We shall show that in this case solutions with dynamical compactification occur, which means solutions with both scale factors depending on time but with one, let it be a_1, increasing and the other one (a_2) shrinking to Planck scales. In this case M_1 is treated as our external space and M_2 describes an unobservable internal space. For two component cosmological models (i.e. for $n = 2$ in Eqs. (11) and (22)) it is easy to get explicit expressions for the scale factors as functions of harmonic time:

$$a_1^{d_1-1} = \frac{{a_{(0)}}_1^{d_1-1} \exp\left[-\sqrt{\frac{d_2(d_1-1)}{D-2}}\ \nu_1\tau\right]}{\sinh\left[\sqrt{\frac{(d_1-1)(\nu_1^2+\nu_2^2)}{d_1}}\ |\tau|\right]}; \tag{23}$$

$$a_2^{d_2} = {a_{(0)}}_2^{d_2} \exp\left[\sqrt{\frac{d_2(d_1-1)}{D-2}}\ \nu_1\tau\right]. \tag{24}$$

Here ${a_{(0)}}_1$ and ${a_{(0)}}_2$ are connected with the constant of integration c_1 in (11) by the expressions

$${a_{(0)}}_1^{d_1-1} = \sqrt{\frac{\nu_1^2+\nu_2^2}{d_1(d_1-1)}} \exp\left[-\sqrt{\frac{d_2(d_1-1)}{D-2}}\ c_1\right], \tag{25}$$

$${a_{(0)}}_2^{d_2} = \exp\left[\sqrt{\frac{d_2(d_1-1)}{D-2}}\ c_1\right]. \tag{26}$$

Hence we get the relation between ${a_{(0)}}_1$ and ${a_{(0)}}_2$

$${a_{(0)}}_1^{d_1-1}{a_{(0)}}_2^{d_2} = \sqrt{\frac{\nu_1^2+\nu_2^2}{d_1(d_1-1)}}. \tag{27}$$

It can be seen from (23) that a_1 has a discontinuity at $\tau = 0$. Therefore, we have different solutions in the ranges $-\infty < \tau \leq 0$ and $0 \leq \tau < \infty$.

Figure 1: The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in Case 1 if $\nu_1 > 0$.

It follows from Eq.(23) that we have three different types of development of the scale factors a_1 and a_2, depending on the sign of the expression

$$\sqrt{d_1 d_2 \nu_1^2} - \sqrt{(d_1+d_2-1)(\nu_1^2+\nu_2^2)}.$$

Let us consider these cases in more detail.

1. $\sqrt{d_1 d_2 \nu_1^2} - \sqrt{(d_1+d_2-1)(\nu_1^2+\nu_2^2)} > 0$.

Here, depending on the sign of ν_1, two types of solutions exist for the scale factors (see the qualitative pictures in Figs. 1, 2). From Fig. 1 ($\nu_1 > 0$) we see that for $\tau > 0$ dynamical compactification may occur

Figure 2: The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in Case 1 if $\nu_1 < 0$.

when $\tau \to +\infty$. For $\tau < 0$ the dynamical behaviour of the scale factors is more complicated. Nevertheless, there are also time intervals where one of the scale factors is much bigger than the other one, for example in

Figure 3: The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in Case 2 if $\nu_1 > 0$.

the limit $\tau \to -0$. For the case of Fig. 2 ($\nu_1 < 0$) dynamical compactification may take place for $\tau < 0$ when $\tau \to -0$ or for $\tau > 0$ if $\tau \to +\infty$. Moreover, depending on the sign of ν_1, the space M_1 or the space M_2 can play the role of the external space. We have also to mention that Case 1 can be realized if no scalar field is present ($\nu_2 = 0$). It is clear that the condition of item 1 is not valid for $d_2 = 1$.

2. $\sqrt{d_1 d_2 \nu_1^2} - \sqrt{(d_1 + d_2 - 1)(\nu_1^2 + \nu_2^2)} < 0$.

The qualitative behaviour of the scale factors is shown in Figs. 3, 4. It can be seen that in this case there are also regions with dynamical compactification. Because of $d_1, d_2 \geq 1$ Case 2 cannot be realized if $\nu_2 = 0$, i.e. without a scalar field.

Figure 4: The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in Case 2 if $\nu_1 < 0$.

Figure 5: The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in Case 3 if $\nu_1 > 0$.

3. $\sqrt{d_1 d_2 \nu_1^2} - \sqrt{(d_1 + d_2 - 1)(\nu_1^2 + \nu_2^2)} = 0$.

In this case we have a connection between the two constants of integration ν_1 and ν_2:

$$\nu_2^2 = \left(\frac{d_1 d_2}{d_1 + d_2 - 1} - 1 \right) \nu_1^2. \tag{28}$$

From this expression we see that for $d_2 = 1$ there is no scalar field: $\nu_2 = 0$. With (28), Eq.(27) reads:

$$a_{(0)1}^{d_1 - 1} a_{(0)2}^{d_2} = \sqrt{\frac{d_2}{(d_1 - 1)(d_1 + d_2 - 1)}} |\nu_1|. \tag{29}$$

In this case we find for the scale factors the following expressions:

$$a_1^{d_1 - 1} = \frac{2 a_{(0)1}^{d_1 - 1}}{\left| \exp\left[\pm 2 \sqrt{\frac{d_2 (d_1 - 1)}{D - 2}} \; |\nu_1| \tau \right] - 1 \right|}, \tag{30}$$

$$a_2^{d_2} = a_{(0)2}^{d_2} \exp\left[\sqrt{\frac{d_2 (d_1 - 1)}{D - 2}} \nu_1 \tau \right] \tag{31}$$

where the upper sign in the expression for a_1 corresponds to $\nu_1 > 0$ and the lower one to $\nu_1 < 0$. The qualitative behaviour of the scale factors in the case $2^{1/(d_1 - 1)} a_{(0)1} > a_{(0)2}$ is given in Figs. 5, 6 showing the existence of regions with dynamical compactification, too.

Due to the simple form of Eq.(30) we can find an explicit connection between the conformal time τ and the synchronous time t, given by [11]

$$dt = \pm e^{\gamma} d\tau = \pm a_1^{d_1} a_2^{d_2} d\tau. \tag{32}$$

Putting (30) and (31) into (32), we find:

$$t = \pm \tilde{c} \int \frac{dy}{|y^2 - 1|^{d_1/(d_1 - 1)}} + \tilde{\tilde{c}} \tag{33}$$

Figure 6: The dynamical behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon > 0$ in Case 3 if $\nu_1 < 0$.

where

$$y = \exp\Big[\pm\sqrt{\frac{d_2(d_1-1)}{D-2}}\Big]. \tag{34}$$

The upper sign corresponds to $\nu_1 > 0$ the lower one to $\nu_1 < 0$. The constant $\tilde{c}$ is defined by

$$\tilde{c} = \frac{2^{d_1/(d_1-1)}}{d_1-1} a_{(0)1} \tag{35}$$

and the initial value of the synchronous time will be taken so that $\tilde{\tilde{c}} = 0$.

In the case $d_1 = 2, d_2 \geq 1$ the integration of (33) gives

$$t = \pm\frac{\tilde{c}}{2}\left\{\begin{array}{ll} y/(1-y^2) + \coth^{-1} y, & |y| > 1, \\ y/(1-y^2) + \tanh^{-1} y, & |y| < 1. \end{array}\right. \tag{36}$$

This case is of special interest because for $d_2 = 1$ it describes a 3-dimensional anisotropic Bianchi III universe without scalar field [26]. As we could see above, there is no isotropization in this case.

The integral (33) can be easily calculated in the case $d_1 = 3$ (M_1 can be looked at as our external space). Then we have two types of solutions:

$$a_1 = \frac{\sqrt{2}a_{(0)1}}{\tilde{c}}(t^2-\tilde{c}^2)^{1/2}, \tag{37}$$

$$a_2 = a_{(0)2}\left[\frac{t^2}{t^2-\tilde{c}^2}\right]^{1/(2d_2)}, \tag{38}$$

$$\varphi = \pm\frac{1}{2}\sqrt{\frac{d_2-1}{d_2}}\ln\frac{t^2}{t^2-\tilde{c}^2} \tag{39}$$

where $|t| \geq \tilde{c}$ and

$$a_1 = \frac{\sqrt{2}a_{(0)1}}{\tilde{c}}(t^2+\tilde{c}^2)^{1/2}, \tag{40}$$

$$a_2 = a_{(0)2}\left[\frac{t^2}{t^2+\tilde{c}^2}\right]^{1/(2d_2)}, \tag{41}$$

$$\varphi = \pm\frac{1}{2}\sqrt{\frac{d_2-1}{d_2}}\ln\frac{t^2}{t^2+\tilde{c}^2} \tag{42}$$

Figure 7: The dynamical behaviour of the scale factors a_1 and a_2 in synchronous time for $\epsilon > 0$ in Case 3 if $d_1 = 3$ (formulas (37), (38). The line $a_1 = |t|$ is an attractor for the scale factor a_1.

Figure 8: The dynamical behaviour of the scale factors a_1 and a_2 in synchronous time for $\epsilon > 0$ in Case 3 if $d_1 = 3$ (formulas (40), (41). The line $a_1 = |t|$ is an attractor for the scale factor a_1.

where $-\infty < t < +\infty$ (in (39) and (42) we put $c_2 = 0$). In Figs. 7,8 the behaviour of the scale factors corresponding to (37), (38) and (40), (41) is presented. In both these cases there are ranges where dynamical compactification is possible. For the solutions (37), (38) the scale factors are always in opposite phases and the role of the external space can be played by M_1 as well as by M_2. For instance, in the region $t \geq \tilde{c}$ the scale factor a_2 shrinks from $+\infty$ to $a_{(0)2}$ (asymptotically). For $a_{(0)2} \sim L_{Pl}$ M_2 becomes unobservable. At the same time $a_1 \sim t (t \gg \tilde{c})$ and M_1 describes the exterior space of Milne type. In this way the topology of the universe tends asymptotically to $M^4 \times T^{d_2}$.

Figure 9: Case 3. $(d_1 = 3)$. The behaviour of the scalar field for Eq.(39) (with positive sign).

Fig. 8 is drawn for the case $\sqrt{2}a_{(0)1} > a_{(0)2}$. For $|t| \gg \tilde{c}$ we have $a_2 \to a_{(0)2}$ and $a_1 \sim |t|$, i.e., the factor space M_2 becomes static and the factor space M_1 behaves asymptotically as a Milne universe. Thus, for $|t| \gg \tilde{c}$ the topology of the universe tends asymptotically again to $M^4 \times T^{d_2}$.

Figs. 9 and 10 show the behaviour of the scalar field for the upper sign in Eqs.(39) and (42), respectively. It is interesting to note that $|\varphi| \to 0$ if $|t| \gg \tilde{c}$.

Figure 10: Case 3. $(d_1 = 3)$. The behaviour of the scalar field for Eq.(42) (with positive sign).

Here, we have investigated the two-component model $(n = 2)$ in the case $\epsilon > 0$. There exists one particular case for $n > 2$ which may be reduced to the two-component model considered above. This exceptional case corresponds to a special choice of the integration constants in Eq.(11)

$$\nu_1 \neq 0,$$

$$\nu_2 = \ldots = \nu_{n-1} = 0. \tag{43}$$

Then it follows from the coordinate transformation (7) that

$$a_i = e^{B_i} a_2, \qquad i = 3, \ldots, n, \tag{44}$$

where B_i is an arbitrary constant. Therefore, all factor spaces M_i $(i = 3, \ldots, n)$ have identical dynamical behaviour, the same as M_2 in the two component universe. Further, Eqs.(23) - (42)) remain the same and define the dynamics of the factor spaces M_1 and M_2 with the only difference that we have to make the change $d_2 \to \sum_2 = \sum_{i=2}^n d_i$.

3.2.2. The n-component universe. Static internal spaces

Just as in the case $\epsilon = 0$, we have a solution with static internal spaces for $\epsilon > 0$. Obviously, this special case corresponds to the following choice of contants of integration: $\nu_i = 0$ $(i = 1, \ldots, n-1)$, $\nu_n \neq 0$. Therefore, this case is realized only in the presence of a real scalar field. All scale factors are frozen $(e^{\beta^i} = a_i = a_{(o)i}, i = 2, \ldots, n)$ except one $(e^{\beta^1} = a_1)$. Then the metric in harmonic time τ takes the form (18) where the scale factor a_1 is

$$a_1(\tau) = \left(\frac{\epsilon_1}{C}\right)^{1/(d_1-1)} \left\{\sinh\left[(d_1-1)\epsilon_1|\tau|\right]\right\}^{-1/(d_1-1)} \tag{45}$$

where C is defined by (17) and, as before, $\epsilon_1 = \sqrt{\epsilon/|\theta_1|}$. The interval $(-\infty, -0]$ describes the expanding universe and $[+0, +\infty)$ the contracting universe. It is easy to obtain for $|\varphi|$ (taking $c_n = 0$ in (11)) the formula

$$|\varphi| = \left\{|\nu_n|/[\epsilon_1(d_1 - 1)]\right\} \sinh^{-1}\left(\epsilon_1/C a_1^{d_1-1}\right). \tag{46}$$

It is convenient to rewrite the metric in the conformal time η connected with the harmonic time τ by

$$\sinh\left[(d_1 - 1)\epsilon_1(-\tau)\right] = \{\sinh[(d_1 - 1)\eta]\}^{-1}. \tag{47}$$

Then

$$g = a_1^2(\eta)\left[-d\eta \otimes d\eta + g_{(1)}\right] + a_{(0)2}^2 g_{(2)} + \ldots + a_{(0)n}^2 g_{(n)} \tag{48}$$

and the scale factor a_1 as a function of the conformal time is given by

$$a_1(\eta) = (\epsilon_1/C)^{1/(d_1-1)} \left\{\sinh\left[(d_1-1)|\eta|\right]\right\}^{1/(d_1-1)}. \tag{49}$$

In the synchronous time t the metric takes the form

$$g = -dt \otimes dt + a_1^2(t) g_{(1)} + \sum_{i=2}^n a_{(0)i}^2 g_{(i)} \tag{50}$$

where the scale factor a_1 and the time coordinate t are connected by

$$t = \int \frac{a_1^{d_1-1} da_1}{\sqrt{\epsilon_1^2/C^2 + a_1^{2(d_1-1)}}} + \text{const.} \tag{51}$$

If $a_1 \ll [\epsilon_1/C]^{1/(d_1-1)}$, the scale factor a_1 has the asymptotic behaviour $a_1 \sim |t|^{1/d_1}$. It corresponds to the open Friedmann universe filled with radiation for $d_1 = 2$ and filled with ultrastiff matter for $d_1 = 3$. If $a_1 \gg [\epsilon_1/C]^{1/(d_1-1)}$, the scale factor a_1 has the asymptotic behaviour corresponding to a Milne universe $a_1 \sim |t|$ for all d_1. Therefore, in the case of static internal spaces the topology of the universe asymptotically tends to $M^{d_1+1} \times T^{d_2} \times \ldots \times T^{d_n}$, where M^{d_1+1} denotes the d_1+1 dimensional observable Milne universe (for $d_1 = 3$ this is the Minkowski space-time) and T^{d_i} represent the frozen (unobservable) internal spaces which are d_i dimensional tori or other compact spaces of constant zero curvature (later on we shall call this topology $M \times T$ topology).

For $d_1 = 2$ the integral (51) can be expressed in elementary functions

$$a_1 = (\epsilon_1/C)\left\{[(C/\epsilon_1)|t| + 1]^2 - 1\right\}^{1/2} \tag{52}$$

but for $d_1 > 2$ we get elliptic integrals.

3.3. The case $\epsilon < 0$

In this case, as we can see from Eq.(12), there are classically allowed and forbidden domains. Classically forbidden domains are usually treated as those with Euclidean signature and classically allowed ones as those with Lorentzian signature. As we shall see in the next section, on the quantum level there are tunnelling solutions which describe processes with metric signature alteration [18], for example, universe nucleation from "nothing"[27]. If $\epsilon < 0$, at least some ν_i should be imaginary. This leads to complex metric and scalar fields. As stressed above, it is possible to have a real metric and an imaginary scalar field in a Lorentzian domain if we demand that ν_i $(i = 1, \ldots, n-1)$ be real, ν_n be imaginary and ν_i should satisfy the condition (14).

Let us consider the equations (10), (12) in the classically allowed region $\exp(2qv^0) \geq \epsilon/\theta_1$. The solutions of the equations of motion in the harmonic time gauge are

$$\begin{aligned} e^{qv^0} &= \frac{\sqrt{\epsilon/\theta_1}}{\cos[(d_1-1)\sqrt{\epsilon/\theta_1}\tau]}, \\ |\tau| &\leq \frac{\pi}{2(d_1-1)}\sqrt{\frac{\theta_1}{\epsilon}} \equiv \tau_1 \end{aligned} \tag{53}$$

where the constant of integration τ_1 is fixed by a proper choice of the origin of the time coordinate τ_0, and

$$\begin{aligned} v^i &= \nu_i\tau + c_i, \qquad i = 1, \ldots, n-1 \\ \varphi &= \nu_n\tau + c_n. \end{aligned} \tag{54}$$

The solution in the classically forbidden region can be found with the help of an analytic continuation $\tau \to -i\tau$ in the expressions (53), (54).
As in the case $\epsilon > 0$ we consider here two special cases.

Figure 11: The qualitative behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon < 0$ (Eq. (55) for $\nu_1 > 0$).

Figure 12: The qualitative behaviour of the scale factors a_1 and a_2 in harmonic time for $\epsilon < 0$ (Eq.(55) for $\nu_1 < 0$). It can be seen that in the case $a_{2min} \sim L_{Pl}$ for $\tau \to \tau_1$ dynamical compactification takes place.

3.3.1. The 2-component universe. Dynamical compactification

In this case the scale factors as functions of the harmonic time coordinate read

$$\begin{aligned} a_1^{d_1-1} &= \frac{a_{(0)1}^{d_1-1}\exp\left[-\sqrt{\frac{d_2(d_1-1)}{D-2}}\nu_1\tau\right]}{\cos\left[\sqrt{\frac{d_1-1}{d_1}\left(|\nu_2|^2 - \nu_1^2\right)}\,\tau\right]}, \\ a_2^{d_2} &= a_{(0)2}^{d_2}\exp\left[\sqrt{\frac{d_2(d_1-1)}{D-2}}\,\nu_1\tau\right] \end{aligned} \tag{55}$$

where $|\tau| \leq \tau_1$ and the constants $a_{(0)1}$ and $a_{(0)2}$ are defined by Eqs. (25) and (26) with the substitution $\nu_1^2 + \nu_2^2 \to |\nu_2|^2 - \nu_1^2$. According to (11), the scalar field is given by $\varphi = i|\nu_2|\tau + c_2$. It can be seen from

(55) that there are different types of development of the scale factors dependence on the sign of ν_1. The qualitative picture is shown in Figs. 11 and 12. Fig. 11 corresponds to $\nu_1 > 0$ where the scale factor a_2 increases monotonically from the minimal value at $\tau = -\tau_1$ to the maximal value at $\tau = \tau_1$, while a_1 decreases from $+\infty$ at $\tau = -\tau_1$ down to $a_{1\min}$ and after that tends to $+\infty$ at $\tau = \tau_1$. If the value $a_{2\max}$ does not exceed $L_{\rm Pl}$ too much, then for $\tau \to \tau_1$ dynamical compactification takes place. Fig. 12 corresponds to $\nu_1 < 0$. Here, a_2 monotonically decreases from $a_{2\max}$ at $\tau = -\tau_1$ to $a_{2\min}$ at $\tau = \tau_1$ and a_1 once more decreases from $+\infty$ at $\tau = -\tau_1$ down to $a_{1\min}$ and after that tends to $+\infty$ at $\tau = \tau_1$. In this case the region $0 \le \tau \le \tau_1$ gives an example of dynamical compactification for $\tau \to \tau_1$.

In contrast to the case $\epsilon > 0$, for $\epsilon < 0$ we are unable to give an explicit expression for the scale factors as functions of the synchronous time coordinate if more than one scale factors have a dynamical behaviour.

As was shown in [18], quantum solutions with a fixed value of $\epsilon < 0$ describe transitions from a classically forbidden Euclidean domain to a classically allowed Lorentzian one (see next section). The universe is created by quantum tunnelling with the scale $e^{qv_0} = \sqrt{\epsilon/\theta_1} = \epsilon_1$, which cooresponds to the time $\tau = 0$ in Eqs. (53) - (55). After that the evolution of the universe is described by the classical equations (53) - (55). Figs. 11, 12 show that we have two possible pictures for the evolution of the universe. Due to the first possibility (Fig. 11) the universe is created by quantum tunnelling with the scale factors $a_1 = a_{(0)1}$ and $a_2 = a_{(0)2}$. After that the space M_1 shrinks to $a_1 = a_{1\min}$ and then increases to $+\infty$. The space M_2 expands monotonically to $a_2 = a_{2\max}$. The second possibility (Fig.12) describes a universe created with $a_1 = a_{(0)1}$ and $a_2 = a_{(0)2}$. Here M_1 expands monotonically to $+\infty$ while M_2 contracts to $a_2 = a_{2\min}$.

In conclusion of this section we mention the special case (43) and (44) of an n ($n > 2$) component model which may be reduced to the two-component model (55) considered above.

3.3.2. The n-component universe. Static internal spaces

From the considerations above we conclude that in the case of static internal spaces we should take $\nu_i = 0$ ($i = 1, \dots, n-1$), $\nu_n \neq 0$. It follows from the condition $\epsilon < 0$ that the scalar field should be purely imaginary (ν_n imaginary). All the scale factors $a_i = a_{(0)i}$ ($i = 2, \dots, n$) are frozen and only $a_1 = e^{\beta^1}$ has a dynamical behaviour. From (53), (54) and the transformation (7) we find in the Lorentzian domain the scale factor a_1 as a function of the harmonic time coordinate

$$a_1(\tau) = (\epsilon_1/C)^{1/(d_1-1)} \left\{\cos[(d_1-1)\epsilon_1\tau]\right\}^{-1/(d_1-1)} \quad (56)$$

where C is defined by (17) and $\epsilon_1 = \sqrt{\epsilon/\theta_1}$. In the interval $\left[-\pi\epsilon_1/[2(d_1-1)],\ 0\right]$ the universe contracts from infinity to the classical turning point where $a_1(\tau = 0) = (\epsilon_1/C)^{1/(d_1-1)}$ and after that in the interval $\left[0, \pi/[2(d_1-1)\epsilon_1]\right]$ expands to infinity again.

It is not difficult to obtain the absolute value of the scalar field depending on the scale factor a_1 (taking $c_n = 0$ in (54))

$$|\varphi| = |\nu_n| \frac{\epsilon_1}{d_1 - 1} \arccos\left[\frac{\epsilon_1}{C a_1^{d_1-1}}\right]. \quad (57)$$

This formula shows that $|\varphi|$ has its minimum at the turning point and tends asymptotically to $\pi|\nu_n|\epsilon_1/[2(d_1-1)]$ when $a_1 \to \infty$. The space-time metric takes in the harmonic time gauge the form (18), where $a_1(\tau)$ is defined by (56).

The harmonic time τ and the conformal time η are connected by

$$\cos[(d_1-1)\epsilon_1\tau] = \frac{1}{\cosh[(d_1-1)\eta]}, \quad -\infty < \eta < +\infty \quad (58)$$

and the metric in conformal time takes the form (48) where a_1 depends on η as

$$a_1(\eta) = [\epsilon_1/C]^{1/(d_1-1)} \left\{\cosh\left[(d_1-1)\eta\right]\right\}^{1/(d_1-1)}, \quad -\infty < \eta < +\infty. \quad (59)$$

To investigate the asymptotic behaviour of the scale factor at large times, let us consider synchronous coordinates with the metric of the form (50). We find for the time dependence of the scale factor a_1:

$$t = \int \frac{a_1^{d_1-1} da_1}{\sqrt{a_1^{2(d_1-1)} - \epsilon/(\theta_1 C^2)}} + \text{const}. \quad (60)$$

Then asymptotically, when $a_1 \gg [\epsilon_1/C]^{1/(d_1-1)}$, we have $a_1 \sim |t|$. Thus, the universe behaves asymptotically like a Milne Universe with respect to the scale factor a_1. Therefore, as in the case of spontaneous compactification for $\epsilon > 0$ (paragraph 3.2.2), the topology of the universe asymptotically tends to $M \times T$.

In the particular case $d_1 = 2$ we have from (60)

$$a_1^2(t) = t^2 + \epsilon/(\theta_1 C^2), \qquad -\infty < t < +\infty. \quad (61)$$

For $d_1 > 2$ this integral can be expressed in terms of elliptic integrals. For example, in the case $d_1 = 3$ we have

$$t = \sqrt{\epsilon_1/C}\left\{\frac{1}{\sqrt{2}} F\left(\Psi, \frac{\sqrt{2}}{2}\right) - \sqrt{2} E\left(\Psi, \frac{\sqrt{2}}{2}\right)\right\} + \frac{1}{a_1}\left[a_1^4 - \epsilon_1^2/C^2\right]^{1/2} \quad (62)$$

where

$$\Psi = \arccos\left(\sqrt{\epsilon_1/C}/a_1\right) \quad (63)$$

and F and E are the elliptic integrals of the first and second kind, repectively.

The classical expressions for the Euclidean domain $e^{qv^0} < \epsilon_1$ can be found by analytic continuation of the formulas obtained here. Then the point

$$a_1 = (\epsilon_1/C)^{1/(d_1-1)} \quad (64)$$

is the classical turning point. The nucleation of the universe can be considered as the quantum tunnelling process through the potential barrier [27]. The universe is nucleated with a finite size $a_1 = (\epsilon_1/C)^{1/(d_1-1)}$ and zero speed $(da_1/dt = 0)$ and its further evolution is described by the classical formulas (56), (59) and (60).

4. Solutions to the quantized model

At the quantum level the constraint equation (9) turns over to the Wheeler-DeWitt equation (WDW). The WDW equation is covariant with respect to minisuperspace coordinate transformations and can be written in the harmonic time gauge in the following form [14]:

$$\left(-\frac{\partial^2}{\partial {v^0}^2}+\frac{\partial^2}{\partial {v^1}^2}+\ldots+\frac{\partial^2}{\partial {v^{n-1}}^2}+\frac{\partial^2}{\partial \varphi^2}-|\theta_1|e^{2qv^0}\right)\Psi = 0. \tag{65}$$

It is easy to obtain solutions of the WDW equation (65) by separation of variables

$$\Psi = \Psi_0(v^0)\ldots\Psi_{n-1}(v^{n-1})\Psi_n(\varphi) \tag{66}$$

where

$$\Psi_i(v^i) = e^{i\nu_i v^i},\ i=1,\ldots,n-1;\ \Psi_n(\varphi) = e^{i\nu_n\varphi} \tag{67}$$

and Ψ_0 satisfies the equation

$$\left(-\frac{d^2}{d{v_0}^2} - |\theta_1|e^{2qv^0}\right)\Psi_0 = \epsilon\Psi_0. \tag{68}$$

Here ϵ and the arbitrary numbers ν_i are related to each other by

$$\epsilon = \sum_{i=1}^{n}\nu_i^2. \tag{69}$$

The solutions to equation (68) are

$$C^{(m)}_{i\sqrt{\epsilon}/q}\left[\left(\sqrt{|\theta_1|}/q\right)e^{qv^0}\right] \tag{70}$$

where $C^{(m)}$ denotes the Bessel function of the first $(m=1)$, second $(m=2)$ or third $(m=3)$ kind. It was shown in [14] that ϵ can be interpreted as an energy. From this point of view the states with $\epsilon = 0$ (with all $\nu_i = 0, i=1,\ldots,n$) are treated as ground states. The general solution can be written in the form

$$\Psi = \sum_{m=1}^{3}\int d^n\nu B^{(m)}(\nu)C^{(m)}_{i\sqrt{\epsilon}/q}\prod_{j=1}^{n}e^{i\nu_j v^j} \tag{71}$$

where $B^{(m)}(\nu)$ are arbitrary coefficients depending on the quantum numbers ν_i.

The solutions (66) are eigenstates of the quantum-mechanical operators

$$\Pi_{v^i} = -\frac{i}{N}\frac{\partial}{\partial v^i},\ i=1,\ldots,n-1,\quad \Pi_\varphi = -\frac{i}{N}\frac{\partial}{\partial\varphi} \tag{72}$$

with the eigenvalues ν_n/N, where we have $N=1$ for a Lorentzian space-time and $N=i$ for a Euclidean one. The classical equations corresponding to the states (66) are [14]

$$\dot{v}^i = N^{-1}\nu_i,\quad i=1,\ldots,n-1,\quad \dot\varphi = N^{-1}\nu_n \tag{73}$$

where the dot denotes differentiation with respect to the harmonic time τ. Evidently, equations (73) coincide with (10), (11). Thus, for the classical equations corresponding to (66) the constants of integration ν_i in (11) should coincide with the quantum numbers ν_i in (67).

Let us consider the wave function (66) in more detail. In the same way as in the classical case we distinguish three special cases: $\epsilon = 0, > 0, < 0$.

4.1. The case $\epsilon = 0$

This case was considered earlier in [14], where the state with $\epsilon = 0$ was treated as a ground state. For all ν_i to be real the condition $\epsilon = 0$ leads to the demand $\nu_i = 0$ $(i=1,\ldots,n)$. Thus, there are no excitations in the directions $v^1,\ldots,v^n$. That is why the case $\epsilon = 0$ is treated as a ground state. As shown above, this corresponds in the classical limit to a universe with static internal spaces and the $M\times T$ topology. Once more, this describes the product of a (d_1+1)-dimensional Milne Universe and static d_i-dimensional tori or other compact spaces of constant zero curvature. The wave function

$$\begin{aligned}\Psi_0^{(0)} &= J_0\left[\left(\sqrt{|\theta_1|}/q\right)e^{qv^0}\right]\\ &= J_0\left[(\sqrt{|\theta_1|}/q)a_1^{d_1-1}a_{(0)2}^{d_2}\ldots a_{(0)n}^{d_n}\right]\end{aligned} \tag{74}$$

is related to the Hartle-Hawking boundary conditions [28] and describes a superposition of expanding and contracting universes [14]. The wave function

$$\Psi_0^{(2)} = H_0^{(2)}\left[\left(\sqrt{|\theta_1|}/q\right)e^{qv^0}\right] \tag{75}$$

describes an expanding universe and satisfies the Vilenkin boundary conditions [14, 29]. The wave function

$$\Psi_0^{(1)} = H_0^{(1)}\left[\left(\sqrt{|\theta_1|}/q\right)e^{qv^0}\right] \tag{76}$$

describes a contracting universe. Here J_0 and $H_0^{(1,2)}$ are the Bessel functions of first and third kind respectively.

The Hartle-Hawking ground state wave function (74) is nonsingular. The vacuum wave functions (75) and (76) go to infinity as $\ln a_1$ when $a_1 \to 0$.

4.2. The case $\epsilon > 0$

Let us assume all ν_i to be real. This corresponds to real metric and scalar field in the Lorentzian domain.

The excited states (66) can be written in the form

$$\Psi^{(0)}_{\nu_1,\ldots,\nu_n} = J_{ik}\left[\left(\sqrt{|\theta_1|}/q\right)e^{qv^0}\right]\prod_{j=1}^{n}e^{i\nu_j(v^j-c^j)}, \tag{77}$$

Figure 13: For solutions with static internal spaces it is shown how the solution $a_1 = |t|$ for $\epsilon = 0$ acts as an attractor. The curves are drawn for the simple case $d_1 = 2$. Here we have $a_1 = \left[t^2 + \epsilon/(\theta_1 C^2)\right]^{1/2}$ for $\epsilon < 0$ and $a_1 = \sqrt{\epsilon/(|\theta_1| C^2)}\left[(C\sqrt{|\theta_1|/\epsilon}|t| + 1)^2 - 1\right]^{1/2}$ for $\epsilon > 0$. For all dynamical solutions $\epsilon \gtrless 0$ we have asymptotically $a_1 \sim |t| \to +\infty$, $a_2, \dots, a_n \to$ const, $\varphi \to$ const and therefore the solution $a_1 = |t|$, $\epsilon = 0$ is an attractor too (see, e.g., Figs. 7–10).

$$\Psi^{(1,2)}_{\nu_1,\dots,\nu_n} = H^{(1,2)}_{ik}\left[\left(\sqrt{|\theta_1|}/q\right)e^{qv^0}\right]\prod_{j=1}^{n} e^{i\nu_j(v^j - c^j)} \quad (78)$$

where the energy $\epsilon = \sum_1^n \nu_i^2 \equiv q^2k^2, -\infty < k < +\infty$, $\varphi \equiv v^n$, and the translation invariance of the WDW equation (65) was used for the v^i $(i = 1, \dots, n)$.

All these wave functions oscillate infinitely many times when the spatial geometry degenerates $(a_i \to 0)$. This singular behaviour reflects the initial and final singularities of the classical solutions.

As shown above, the solutions to the classical Lorentzian equations have the following asymptotic behaviour. When $\tau \to 0$, we have $|t| \to \infty$ and $v^i \to c^i$, so that $a_i \to a_{(0)i}$ $(i = 2, \dots, n)$ and $a_1 \sim |t|$. This corresponds asymptotically to freezing the internal dimensions. Therefore, solutions with static internal spaces act as an attractor solution for those with dynamical compactification (Fig.13). The corresponding limit for the wave functions (77), (78) may be achieved for $v^0 \to \infty, v^i \to c^i (i = 1, \dots, n)$. In this case the asymptotic bahaviour of the solutions (77), (78) is given by

$$\Psi^{(0)} = \cos\left[\left(\sqrt{|\theta_1|}/q\right) a_1^{d_1 - 1} a_{(0)2}^{d_2} \dots a_{(0)n}^{d_n}\right], \quad (79)$$

$$\Psi^{(j)} = \exp\left[(-1)^{j+1} i \left(\sqrt{|\theta_1|}/q\right) a_1^{d_1 - 1} a_{(0)2}^{d_2} \dots a_{(0)n}^{d_n}\right] \quad (80)$$

where $j = 1, 2$ and (79), (80) describe in the classical limit asymptotically a universe with the $M \times T$ topology.

Evidently, (77 and 78) are not the unique solutions to Eq.(65), other wave functions can also be found. Some of these wave functions may be free of the above mentioned singularities. For the one-component model with a scalar field this was analyzed in [30]. By analogy to the papers [15, 16], where the case $\theta > 0$ was considered, we can give here an example of such a solution. We write the wave function (78) in the form

$$\Psi^{(j)}_{k,\gamma_1,\dots,\gamma_{n-1}} = \exp\left[ik\sum_{l=1}^{n} q(v^l - c^l)\Gamma_l\right] \times H^{(j)}_{ik}\left[\left(\frac{\sqrt{|\theta_1|}}{q}\right)e^{qv^0}\right] \quad (81)$$

where the quantum numbers ν_i are written in the form

$$\nu_i = kq\Gamma_i \quad (82)$$

and the n-dimensional unit vector Γ_i is defined by

$$\Gamma_i = \begin{pmatrix} \cos\gamma_1 & & & \\ \sin\gamma_1 & \cos\gamma_2 & & \\ \sin\gamma_1 & \sin\gamma_2 & \cos\gamma_3 & \\ \vdots & & & \\ \sin\gamma_1 & \dots & \sin\gamma_{n-2} & \cos\gamma_{n-1} \\ \sin\gamma_1 & \dots & \sin\gamma_{n-2} & \sin\gamma_{n-1} \end{pmatrix}. \quad (83)$$

Using the integral transformation [31]

$$\Psi^{(j)}_{\lambda,\gamma_1,\dots,\gamma_{n-1}} = \int_{-\infty}^{+\infty} dk C^{(j)}_k(\lambda)\Psi^{(j)}_{k,\gamma_1,\dots,\gamma_{n-1}} \quad (84)$$

with

$$C^{(j)}_k(\lambda) = \frac{i}{2}(-1)^{j+1}\exp\left[(-1)^j\frac{\pi k}{2}\right]\exp(-ik\lambda) \quad (85)$$

where $-\infty < \lambda < \infty$, we get

$$\Psi^{(j)}_{\lambda,\gamma_1,\dots,\gamma_{n-1}} = \exp\left\{(-1)^{j+1} i\frac{\sqrt{|\theta_1|}}{q}e^{qv^0}\cosh\left[\sum_{i=1}^{n} q(v^i - c^i)\Gamma_i - \lambda\right]\right\}. \quad (86)$$

This wave function has the same asymptotic behaviour at $v^i \to c^i$ $(i = 1, \dots, n)$, $v^0 \to +\infty$, as (81). It corresponds asymptotically to the contracting $(j = 1)$ and expanding $(j = 2)$ $(d_i + 1)$ dimensional Milne universe. The wave function (86) cannot describe a state of fixed energy ϵ.

4.3. The case $\epsilon < 0$

In this case the wave functions (66) can be written in the form

$$\Psi^{(0)}_{\nu_1,\dots,\nu_n} = J_k\left[\frac{\sqrt{|\theta_1|}}{q}e^{qv^0}\right]\prod_{j=1}^{n} e^{i\nu_j(v^j - \tilde{c}^j)}, \quad (87)$$

$$\Psi^{(1,2)}_{\nu_1,\dots,\nu_n} = H^{(1,2)}_k\left[\left(\frac{\sqrt{|\theta_1|}}{q}\right)e^{qv^0}\right]\prod_{j=1}^{n} e^{i\nu_j(v^j - \tilde{c}^j)} \quad (88)$$

where $\varphi \equiv v^n, j = 1, 2, \epsilon \equiv -q^2k^2, -\infty < k < \infty$ and for v^j $(j = 1, \ldots, n)$ the translation invariance of the WDW equation (65) was used.

As a consequence of the condition $\epsilon < 0$, the quantum numbers ν_i (or part of them) become imaginary. Then, in the classically allowed region we would get a complex metric. In order to avoid this we demand by analogy to the classical case all ν_i $(i = 1, \ldots, n-1)$ to be real and ν_n to be imaginary and the condition (14) to be valid. Consequently, with these conditions the metric of the classically accessible region is a real Lorentz metric, while the scalar field in this region is imaginary.

It can be seen from Eq.(68) (the quantum analogue of (12)) that in the case $\epsilon < 0$ there exist classically accessible as well as forbidden regions. The solutions (87), (88) are solutions with fixed energy ϵ and describes transitions between the classically allowed and forbidden regions due to tunnelling processes. As a consequence, it becomes possible to analyze processes with changes of the metric signature [18]. For instance, the solution $\Psi^{(2)}$ in (88) describes quantum tunnelling through a potential barrier and is usually interpreted as creation of the universe from "nothing" [14, 18, 27]. In this way for $\nu_i = 0$ $(i = 1, \ldots, n-1)$ an n component universe with static internal spaces will be created, while for $\nu_1 \neq 0, \nu_i = 0$ $(i = 2, \ldots, n-1)$ the creation of an n-component universe with dynamical compactification is described. The solution $\Psi^{(1)}$ describes the opposite process, the transition into "nothing" as the final stage of the evolution of the universe.

By analogy to the case $\epsilon > 0$, we have also solutions to Eq.(65), which no more describe wave functions with a fixed energy. For instance, with an integral transformation of $\Psi^{(0)}$ we find [31]

$$\begin{aligned}\Psi^{(0)}_{\lambda,\gamma_1,\ldots,\gamma_{n-1}} &= \int_{-\infty}^{+\infty} dk C_k(\lambda) \Psi^{(0)}_{k,\gamma_1,\ldots,\gamma_{n-1}} \\ &= \exp\left\{ i\frac{\sqrt{|\theta_1|}}{q} e^{qv^0} \sin\left[\sum_{i=1}^{n-1} q(v^i - \tilde{c}^i)\Gamma_i \sinh\gamma_{n-1} \right.\right. \\ &\left.\left. \qquad + qi(v^n - \tilde{c}^n)\cosh\gamma_{n-1} + \lambda \right] \right\} \end{aligned} \tag{89}$$

where $C_k(\lambda) = \exp(ik\lambda)$, λ has a continuous spectrum of width 2π and the wave function (87) $\Psi^{(0)}_{\nu_1\ldots\nu_n}$ was refwritten in the form

$$\begin{aligned}&\Psi^{(0)}_{k,\gamma_1,\ldots,\gamma_{n-1}} \\ &= \exp\left\{ ik \left[\sum_{i=1}^{n-1} q(v^i - \tilde{c}^i)\Gamma_i \sinh\gamma_{n-1} \right.\right. \\ &\left.\left. \qquad + qi(v^n - \tilde{c}^n)\cosh\gamma_{n-1} \right] \right\} \times J_k\left[\frac{\sqrt{|\theta_1|}}{q} e^{qv^0} \right] \end{aligned} \tag{90}$$

where we used for the quantum numbers ν_i the representation

$$\nu_i = kq\Gamma_i \sinh\gamma_{n-1}, \qquad i = 1, \ldots, n-1, \tag{91}$$
$$\nu_n = ikq\cosh\gamma_{n-1}. \tag{92}$$

Here Γ_i represents an $(n-1)$-dimensional unit vector specified by a formula like (83) and the arbitrary constants $\tilde{c}^i$ can be taken best as the classical limits for the v^i $(i = 1, \ldots, n)$ in Eq. (54) for $|\tau| \to \tau_1$, i.e., $\tilde{c}^i = \nu^i\tau_1 + c^i$ or $\tilde{c}^i = -\nu^i\tau_1 + c^i$. In this limit v^i $(i = 1, \ldots, n)$ tend to their maximal (or minimal) fixed values, the corresponding scale factors a_i $(i = 2, \ldots, n)$ are frozen out and $a_1 \sim |t| \to \infty$. So it can be seen that in the limit $v^i \to \tilde{c}^i$ $(i = 1, \ldots, n)$, $v^0 \to \infty$, the wave functions $\Psi^{(0)}_{k,\ldots}$ and $\Psi^{(0)}_{\lambda,\ldots}$ have the same asymptotic behaviour and describe asymptotically in the classical limit a universe with the same $M \times T$ topology as was in the case $\epsilon > 0$.

5. Conclusions

We have investigated multidimensional cosmological models (MCM) with n $(n > 1)$ Einstein spaces for the case when these spaces are of constant curvature. The integrable case was considered where only one of these spaces has a negative constant curvature, while all others were assumed to be Ricci-flat. As a matter source we introduced a massless minimally coupled homogeneous scalar field. The main attention was paid to the problem of compactification of extra dimensions. The non-Ricci-flat space M_1 was considered to be our external space, while all the others, with zero curvature, described internal spaces. But our general solutions do not exclude the possiblility that one of the Ricci-flat spaces plays the role of our external space.

The problem of compactification for the model with positive curvature of the non-Ricci-flat space was considered in detail in our previous paper [10] where solutions with dynamical and static internal spaces were found. In that case the parameter ϵ defined by formula (13) plays the role of energy [14] and should be positive in the Lorentzian domain. For the model with negative curvature of the non-Ricci-flat space considered here, in the Lorentzian domain we can have for this parameter both $\epsilon \geq 0$ and $\epsilon < 0$. This feature leads to a more complex picture than in the former case. For all values of the parameter ϵ solutions with both dynamical and static internal spaces are found. By a proper choice of the free parameters of the model we can get reasonably compactified dynamical or static internal spaces. In the latter case the values of the fixed scale factors of the internal spaces are free parameters and it is usually assumed that they are of the order of the Planck length to make them unobservable.

In the case $\epsilon = 0$ the solution with static internal spaces is an attractor for all kinds of solutions with both $\epsilon > 0$ and $\epsilon < 0$. In the limit of large geometry all solutions tend to $a_1 \sim |t| \to \infty$ for the scale factor of the space M_1, while all other scale factors and the scalar field become frozen (see Figs. 7–10, 13). Thus, asymptotically all solutions describe the universe with the topology $M^{d_1+1} \times T^{d_2} \times \ldots \times T^{d_n}$ where M^{d_1+1} is the $(d_1 + 1)$-dimensional Milne universe and T^{d_1} are d_i-dimensional frozen tori or other compact spaces of constant zero curvature. Solutions to the quantum Wheeler-DeWitt equation were also obtained. In the case $\epsilon < 0$ some of them describe the process of tunnelling from the Euclidean domain to the Lorentzian one, often called birth of the universe from "nothing" [29]. For all values of ϵ these wave functions asymptotically describe in the classical limit a universe with the topology $M^{d_1+1} \times T^{d_2} \times \ldots \times T^{d_n}$.

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