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CONFORMAL MINISUPERSPACE QUANTIZATION AND EQUIVALENCE OF MULTIDIMENSIONAL CLASSICAL AND QUANTUM MODELS[1]

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We consider a reduction of the superspace of geometries to a finite-dimensional minisuperspace $\mathcal{M}$, which, in the case of multidimensional models $M = R \times M_1 \times \cdots \times M_n$, is given by the scale exponents of Einstein spaces M_i. Additional scalar fields yield an enlarged minisuperspace $\mathcal{MS}$. We analyze the effect of conformal equivalence of geometries for Lagrangian models on M with different couplings of geometry and a scalar field, and compare it to conformal transformations on the minisuperspaces $\mathcal{MS}$ of corresponding quantum models. Canonical quantization of minisuperspace is performed in a way which is generally covariant under coordinate transformations and yields equivariance with respect to conformal representations. The conformal coupling $\xi_c = (D-2)/[4(D-1)]$ is analyzed number-theoretically, yielding distinguished dimensions $D = 3, 4, 6, 10$ and 26. From the known solution of a minimally coupling model (MCM) we obtain a solution of the corresponding conformal coupling model (CCM), able to describe the birth of the Universe and the internal spaces at different times without singularities.

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1. Introduction

Multidimensional geometric models [1, 2, 3] are an interesting class for studies of cosmology. On the one hand, they are rich enough to model features of phenomenological interest, on the other they provide a well defined minisuperspace. The latter is a convenient starting point for covariant conformal quantization of the energy constraint yielding the Wheeler-DeWitt (WDW) equation. Nevertheless several mathematical questions concerning the (mini-)superspace construction are still open, although [4, 5] indicate some progress. In this paper we put emphasis on the comparison between conformal transformations of minisuperspace geometry, conformal transformations of ordinary geometry and coordinate transformations.

In Sec.2 we sketch the reduction of the superspace of geometries to a midi-superspace and further to a minisuperspace.

In Sec.3 we compare conformal transformations on three structural levels: conformal coordinate transformations (1), conformal transformations of ordinary geometry (2), and conformal transformations of the (mini-)superspace geometry (3).

In Sec.4 we describe multidimensional Lagrangian models for Einstein-Hilbert-Gibbons-Hawking gravity coupled to a scalar field, obtaining a well defined formulation on the minisuperspace.

Sec. 5 deals with conformal quantization on a minisuperspace. The first quantization of the energy constraint to the WDW equation is performed in a manner both generally covariant and conformally equivariant.

In Sec. 6 we derive the unique conformal Laplace (-Beltrami) operator on a (pseudo) Riemannian manifold $\mathcal{M}$ of dimension n. Though this had been given already by construction of a conformal WDW equation in [6], and in the mathematical literature there is agreement on a linear coupling $\Delta + aR$ of Laplacian Δ, generalized from the flat case, and Ricci curvature scalar R on the underlying manifold, there is sometimes some confusion [7] about the proper choice of the coupling a on an arbitrarily curved manifold. Therefore here we prove that $\Delta + aR$ is conformal if and only if $n > 1$ and $a = -\xi_c$.

Sec. 7 is devoted to the distinguished conformal coupling number $\xi_c = (D-2)/[4(D-1)]$ in different dimensions D, which can be found already in the classical paper [8]. Number theoretical analysis of ξ_c, gives some hint, that it might play a crucial role in dimensional reduction, since ξ_c is especially simple for $D = 3, 4, 6, 10$ and 26.

In Sec. 8 we examine conformal equivalence transformations between Lagrangian models for D-dimen-

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sional geometry on M coupled to a spatially homogeneous scalar field. We consider as example of special interest the conformal transformation between a model with minimally coupled scalar field (MCM) and an equivalent conformal model with a conformally coupled scalar field (CCM), thus generalizing the previous results from [9, 10], obtained for $n=1$ and $D=4$.

In Sec. 9 we apply the results of Sec. 8 to obtain classical multidimensional cosmological solutions of a CCM, performing characteristic dynamics of external, internal and hidden factor spaces.

2. Reduction of the Superspace of Geometries

Let $\mathrm{Met}^s(M)$ be the space of (pseudo-)Riemannian metrics of signature s (s be the number of negative eigenvalues, $s=0$ Euclidean, $s=1$ Lorentzian) on a smooth manifold M, and $\mathrm{Diff}\,(M)$ be the group of diffeomorphisms of M. Then the superspace of geometries of signature s is defined as

$$\mathrm{Geom}^s(M) := \mathrm{Met}^s(M)/\mathrm{Diff}\,(M). \tag{2.1}$$

As the action of $\mathrm{Diff}\,(M)$ on $\mathrm{Met}^s(M)$ is not free, the quotient topology does not yield a manifold structure on $\mathrm{Geom}^s(M)$. At symmetrical metrics the latter quotient has singularities. In [5] its minimal (non-singular) resolution has been constructed for $s=0$. Though, the extension of this construction to $s>1$, most importantly the Lorentzian case $s=1$, is not straightforward, because then the orthogonal group $\mathrm{O}(s, n-s)$ is noncompact. In the following we will drop the index s under the assumption that the following construction can be performed (the manifold $\mathrm{Met}^{\,s}$ admitting slicings transversal to the orbits of $\mathrm{O}(s,n-s)$ and $\mathrm{Diff}\,(M)$).

Let $\mathrm{GL}(M)$ be the frame bundle of M. $F(M) := \mathrm{Met}(M)\times \mathrm{GL}(M)$ is the Fischer manifold of M (see [11]). The reduction $O(M)\subset F(M)$, according to a reduction of $\mathrm{GL}(M)$ to the (s-)orthonormal frame bundle $\mathrm{O}(M,g)\subset \mathrm{GL}(M)$, is given by

$$O(M) := \{(g,u)\in \mathrm{Met}\,(M)\times \mathrm{GL}(M) : u \in \mathrm{O}(M,g)\}. \tag{2.2}$$

While $\mathrm{O}(M,g)$ depends on the metric, $O(M)$ depends only on M as a smooth manifold. There are 2 different fibrations of $O(M)$,

$$\mathrm{O}(s,n-s)\hookrightarrow O(M)\longrightarrow \mathrm{Met}\,(M)\times M \tag{2.3}$$

and

$$\mathrm{Diff}\,(M)\hookrightarrow O(M)\longrightarrow \mathrm{Geom}_0(M), \tag{2.4}$$

where $\mathrm{Geom}_0(M)$ defines the minimal resolution of $\mathrm{Geom}(M)$. Similarily there exists also a fibration $\mathrm{Diff}\,(M)\hookrightarrow F(M)\longrightarrow \mathrm{Geom}_F(M)$, where $\mathrm{Geom}_F(M)$ defines the so-called Fischer resolution [11] of $\mathrm{Geom}(M)$. But this resolution is not a minimal one (it exists an embedding $\mathrm{Geom}_0\hookrightarrow \mathrm{Geom}_F(M)$).

The resolution $\mathrm{Geom}_0(M)\to \mathrm{Geom}(M)$ is in remarkable analogy to resolutions of simple singularities of Cartan type ADE, i.e. singularities generated by a finite subgroup $\mathrm{F}\subset \mathrm{SU}(2)$ (see [12]).

In another, more formal attempt to circumvent the question of $\mathrm{Diff}\,(M)$-equivalence one introduces appropriate supercoordinates and assumes general supercoordinate invariance to define a reduced version of superspace. In [4] it was shown how the space of Riemannian metrics $\mathrm{Met}(M)$ can be equipped with a metric G. Let us choose some supercoordinates χ^A indexed by A within an appropriate index set. Then we set

$$G = G_{AB}d\chi^A d\chi^B \tag{2.5}$$

and define it via

$$G_{AB} := G_{ijkl}h^{ij}_A h^{kl}_B, \tag{2.6}$$

where

$$G_{ijkl} := g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}. \tag{2.7}$$

Note the similarity of Eq. (2.7) to the usual first Christoffel symbols. Both G_{ijkl} and h^{ij}_A are covariant (4- or 2-tensor) components with respect to usual coordinate transformations. Therefore G_{AB} is independent of coordinates on M. The components h^{ij}_A define a generalized soldering form $\theta := h^{ij}_A e_{ij}\otimes d\chi^A$, where the 2-tensors $e_{ij} = h^A_{ij}(\partial/\partial\chi^A)$ contain the components h^A_{ij} dual to h^{ij}_A.

Eq.(2.7) singles out a special class of metrics and, together with general superspace covariance, yields a reduced superspace, which we will call midi-superspace. The index set of the midi-superspace coordinates is still infinite.

A further reduction of superspace to finite dimensions is well defined for a class of metrics of multidimensional type. Here the geometry is described on a (pseudo-) Riemannian manifold

$$\begin{aligned} M &= \mathrm{I\!R}\times M_1\times\ldots\times M_n,\\ D: &= \dim M = 1 + d_1 + \ldots + d_n,\\ g\equiv ds^2 &= -\mathrm{e}^{2\gamma}dt\otimes dt + \sum_{i=1}^{n} a_i^2\, ds_i^2, \end{aligned} \tag{2.8}$$

where $a_i = e^{\beta^i}$ is the scale factor of the d_i-dimensional space M_i with the first fundamental form

$$ds_i^2 = g^{(i)}_{kl}\, dx^k_{(i)}\otimes dx^l_{(i)}.$$

The scale factors e^{β^A}, $A=1,\ldots,n$, yield (reduced) supercoordinates

$$\chi^A := e^{2\beta^A}, \qquad A=1,\ldots,n. \tag{2.9}$$

The mini-superspace $\mathcal{MS}(M)$ over M is then defined by mini-superspace coordinates $\beta^1,\ldots,\beta^n$ subject to the principle of general covariance w.r.t. mini-superspace coordinate transformations.

3. Conformal Transformations on Three Structural Levels

Generally we will have to distinguish between (1) conformal coordinate transformations in D-dimensional geometry, (2) conformal transformations of D-dimensional geometry, and (3) conformal transformations of n-dimensional minisuperspace geometry.

(1) Conformal transformation to new coordinates

We fix the geometry g and transform the coordinates conformally, i.e. so that

$$dx'^i = e^{-f(x)} dx^i \quad \text{and} \quad g_{i'j'} = e^{2f(x)} g_{kl}. \qquad (3.10)$$

With $ds^2 = g_{i'j'} dx^{i'} dx^{j'} = g_{ij} dx^i dx^j$ the intrinsic geometry remains invariant, though looking different in different coordinate frames.

(2) Conformal transformations of ordinary (D-dimensional) geometry

We consider a differentiable manifold M. On a geometry g on M, we represent conformal transformations as Weyl transformations $g \mapsto e^{2f} g$ with $f \in C^\infty(M)$.

With the metric components g_{ij} and scalar fields $(\phi^1, \ldots, \phi^k)$ on M, a Lagrangian model is given by a Lagrangian variation principle $\delta S = 0$ with $S = \int_M \sqrt{|g|} L d^D x$ for a second order Lagrangian

$$L = L(g_{ij},\ \phi^1, \ldots, \phi^k;\ g_{ij,l}, \phi^1_{,l}, \ldots, \phi^k_{,l};\ g_{ij,lm}).$$

Conformal transformation of the Lagrangian model keeps M fixed as a differentiable manifold, but varies its additional structures, including (pseudo-)Riemannian geometry, conformally

$$(g_{ij}, \phi^1, \ldots, \phi^k) \to (\hat{g}_{ij}, \hat{\phi}^1, \ldots, \hat{\phi}^k). \qquad (3.11)$$

A corresponding new variational principle is obtained via

$$\sqrt{|g|} L \overset{!}{=} \sqrt{|\hat{g}|} \hat{L}, \qquad (3.12)$$

for the new (transformed) Lagrangian

$$\hat{L} = \hat{L}(\hat{g}_{ij}, \hat{\phi}^1, \ldots, \hat{\phi}^k; \hat{g}_{ij,l}, \hat{\phi}^1_{,l}, \ldots, \hat{\phi}^k_{,l}; \hat{g}_{ij,lm}). \qquad (3.13)$$

Therefore conformal transformation of (Lagrangian) models for ordinary geometry (plus eventual scalar fields) are performed in practice on a fixed coordinate patch x^i of M.

Invariance of (2.6) under conformal transformations (2) with $g \mapsto e^{2f} g$ yields

$$h^{ij}_A \mapsto e^{-2f} h^{ij}_A. \qquad (3.14)$$

Forgetting about its origin, (3.14) can be interpreted, by analogy to (3.10) in (1), as induced by supercoordinate transformations

$$\chi^A \mapsto e^{2f} \chi^A, \qquad (3.15)$$

leaving (2.5) invariant. The conformal weight differs from that of an analogous ordinary coordinate transformation (3.10) by a factor -2 (corresponding to contragredience and the fact that h^{ij}_A relates (super-)vectors to 2-tensors).

For a minisuperspace $\mathcal{M} = \mathcal{MS}(\mathcal{M})$ like (2.8) the supercoordinate transformations (3.15) correspond via (2.9) just to translations of the minisuperspace coordinates

$$\beta^i \to \beta^i + f. \qquad (3.16)$$

(3) Conformal transformations of (reduced) superspace geometry

A conformal transformation of a midi-superspace geometry G (as in (2.5) and (2.6)) is a transformation $G \mapsto e^{2F} G$ with a function F on the midi-superspace.

On a mini-superspace we have (by analogy to (2.5))

$$G = G_{ij} d\beta^i \otimes d\beta^j. \qquad (3.17)$$

But since $\dim \mathcal{M} < \infty$ here a Weyl transformation

$$G \mapsto {}^f G := e^{2f} G \qquad (3.18)$$

is well defined with $f \in C^\infty(\mathcal{M})$.

Similarly to (2) and (1), (3) and (2) must be also carefully kept apart from each other. Applications of invariance under transformations (2) and (3) will be given later.

4. Multidimensional Lagrangian Models

If we assume within a multidimensional geometry that the M_i are Einstein spaces of constant curvature, then the Ricci scalar curvature of M is

$$R = e^{-2\gamma} \left\{ \left[\sum_{i=1}^{n} (d_i \dot{\beta}^i) \right]^2 + \sum_{i=1}^{n} d_i [(\dot{\beta}^i)^2 - 2\dot{\gamma}\dot{\beta}^i + 2\ddot{\beta}^i] \right\} + \sum_{i=1}^{n} R^{(i)} e^{-2\beta^i}. \qquad (4.1)$$

Let us now consider a variation principle with the action

$$S = S_{EH} + S_{GH} + S_M, \qquad (4.2)$$

where

$$S_{EH} = \frac{1}{2\kappa^2} \int_M \sqrt{|g|} R \, dx$$

is the Einstein-Hilbert action, S_{GH} is the Gibbons-Hawking boundary term [13], and S_M the action of matter.

Let us consider the matter given by a minimally coupled scalar field Φ with potential $U(\Phi)$. Then the variational principle of (4.2) is equivalent to a Lagrangian variational principle over the minisuperspace

$\mathcal{M}$, which is spanned by the β^i, and the scalar field Φ,

$$\begin{aligned} S &= \int L\,dt, \\ L &= \frac{1}{2}\mu \exp\Big\{-\gamma + \sum_{i=1}^{n} d_i\beta^i\Big\}\Big\{\sum_{i=1}^{n} d_i(\dot\beta^i)^2 \\ &\quad - \Big[\sum_{i=1}^{n} d_i\dot\beta^i\Big]^2 + \kappa^2\dot\Phi^2\Big\} - V(\beta^i,\Phi) \end{aligned} \tag{4.3}$$

with

$$V(\beta^i,\Phi) = \mu\exp\Big\{\gamma+\sum_{i=1}^{n} d_i\beta^i\Big\}\Big[-\frac{1}{2}R^{(i)}\mathrm{e}^{-2\beta^1}+\kappa^2 U(\Phi)\Big]$$

where

$$\mu := \kappa^{-2}\prod_{i=1}^{n}\sqrt{|\det g^{(i)}|}. \tag{4.4}$$

It is a convenient procedure of cosmologists, to extend the minisuperspace $\mathcal{M}$ of pure geometry directly by an additional dimension from the scalar field Φ as further minisuperspace coordinate, yielding an enlarged minisuperspace $\mathcal{MS}$.

Let us define a metric on $\mathcal{MS}$, given in coordinates β^i, $i = 1,\ldots,n+1$ with $\beta^{n+1} := \kappa\Phi$. We set

$$G_{n+1\,i} = G_{i\,n+1} := \delta_{i\,n+1}, \quad G_{kl} := d_k\delta_{kl} - d_k d_l \tag{4.5}$$

for $i = 1,\ldots,n+1$ and $k,l = 1,\ldots,n$, thus defining the components G_{ij} of the minisuperspace metric

$$G = G_{ij}d\beta^i \otimes d\beta^j. \tag{4.6}$$

Then in the time harmonic gauge

$$N := \exp\Big\{\gamma - \sum_{i=1}^{n} d_i\beta^i\Big\} \overset{!}{=} 1 \tag{4.7}$$

(see e.g. [14]) we get the Lagrangian

$$L = \frac{\mu}{2}G_{ij}\dot\beta^i\dot\beta^j - V(\beta^i) \tag{4.8}$$

with the energy constraint

$$\frac{\mu}{2}G_{ij}\dot\beta^i\dot\beta^j + V(\beta^i) = 0. \tag{4.9}$$

Independent global conformal transformations of the spaces $M^{(i)}$ yield just translations in the functions β^i.

Note that the signature of $\mathcal{M}$ is Lorentzian for $n > 1$, and $G_{11} < 0$ for $d_1 > 1$ implies that the signature of $\mathcal{MS}$ is Lorentzian not only for $n > 1$ but also for $n = 1$ if $d_1 > 1$. If there is at least one (e.g. compact "internal") extra factor space, i.e. $n > 1$, then $\mathcal{M}$ has the Lorentzian signature $(- + \ldots +)$.

After diagonalization of (4.5) by a minisuperspace coordinate transformation $\beta^i \to \alpha^i$ $(i = 1,\ldots,n)$, there is just one new coordinate, say α^1, which corresponds to the unique negative eigenvalue of G. With a further (sign preserving) coordinate rescaling, G is equivalent to the Minkowsky metric. Hence $\mathcal{M}$ is conformally flat. (Actually for the present metric it is also flat; however under conformal transformation on $\mathcal{M}$ flatness is not an invariant property.)

While $\beta^i \to \alpha^i$ is only a coordinate transformation on $\mathcal{M}$ or $\mathcal{MS}$, it transforms a multidimensional geometry (2.8) with scale exponents β^i to another geometry of the same multidimensional type (2.8), i.e. with the same d_i and ds_i^2, but new scale exponents α^i of the factor spaces M_i. We can always perform the diagonalization of (4.5) such that α^1 and hence M_1 belongs to the unique negative eigenvalue of G. This M_1 is identified as the "external" space. The scale factors of the "internal" spaces $M_2,\ldots,M_n$ and Φ contribute only to positive eigenvalues of $\mathcal{M}$ and accordingly $\mathcal{MS}$. (For $n = 1$ there are no "internal" spaces, but $G_{11} < 0$ for $d_1 > 1$ still provides a negative eigenvalue that is distinguished at least against the additional positive eigenvalue from Φ in $\mathcal{MS}$.) α^1 assumes in $\mathcal{M}$ or $\mathcal{MS}$ the role played by time in usual geometry and quantum mechanics. In this way the "external" space is distinguished against the "internal" ones, since its scale factor provides a natural "time" coordinate on $\mathcal{M}$. If in the multidimensional geometry (3.14) M_1 with α^1 is strictly expanding w.r.t. time t, then the "minisuperspace time" α^1 can be considered in the geometry g as a time equivalent to t. So the Lorentzian structure of $\mathcal{M}$ finally provides the expanding M_1 with a natural "arrow of time" [15].

5. Conformal Quantization on Minisuperspace

Canonical quantization has been considered e.g. in [6, 16]. It essentially consists in replacing the constraint equation (4.9) by the WDW equation

$$\left(-\frac{1}{2}\left[\Delta + aR\right] + V\right)\Psi = 0 \tag{5.1}$$

where Ψ is a wave function from a distribution space $\mathcal{S}^*$, which is the dual of the test function space $\mathcal{S} \subset \mathcal{H}$, dense in the Hilbert space $\mathcal{H} = \mathcal{H}(\mathcal{M})$. (Often one might think of $\mathcal{S}$ as the Schwartz space and correspondingly of $\mathcal{S}^*$ as the space of tempered distributions over $\mathcal{S}$. Note however that the proper choice of $\mathcal{S}$ depends on the Hamiltonian H, and more specifically on the shape of the potential V.)

The Lagrangian (4.8) is invariant under arbitrary time reparametrization $h \in \mathrm{Diff}\,\mathbb{R}$ acting via

$$h(\beta^i)(t) := \beta^i(h(t)), \qquad h(N)(t) := N(h(t))\frac{dh}{dt}$$

on minisuperspace coordinates β^i and N defined as in (4.7).

We set in the following

$$N =: \mathrm{e}^{-2f} \tag{5.2}$$

and admit $f \in C^\infty(\mathcal{M})$ to be an arbitrary smooth function on $\mathcal{M}$.

In the time gauge given by f the Lagrangian is

$$L^f := \frac{\mu}{2} {}^f G_{ij}(\beta)\dot\beta^i\dot\beta^j - V^f(\beta) \tag{5.3}$$

and the energy constraint is

$$E^f := \frac{\mu}{2} {}^f G_{ij}(\beta)\dot\beta^i\dot\beta^j + V^f(\beta) = 0, \tag{5.4}$$

where

$${}^f G = e^{2f} G \text{ and } V^f = e^{-2f} V.$$

With the canonical momenta

$$\pi_i = \frac{\partial L^f}{\partial \dot\beta^i} = \mu G^f_{ij}\dot\beta^j \tag{5.5}$$

this is equivalent to a Hamiltonian system given by

$$H^f = \frac{1}{2\mu}({}^f G)^{ij}\pi_i\pi_j + V^f \tag{5.6}$$

and the energy constraint

$$H^f = 0. \tag{5.7}$$

The inverse of the minisuperspace metric is given by ${}^f G^{-1} = e^{-2f} G^{-1}$, where for the system with Eq. (4.5) the components of G^{-1} are

$$G^{ij} = \frac{\delta_{ij}}{d_i} + \frac{1}{1 - \sum_{i=1}^n d_i}. \tag{5.8}$$

At the quantum level H^f has to be replaced by an operator $\hat H^f$, acting by the energy constraint

$$\hat H^f \Psi^f = 0 \tag{5.9}$$

on $\Psi^f \in \mathcal{S}^{*f}$, where $\mathcal{S}^{*f}$ is given by the action of a represention D_b of $C^\infty(\mathcal{M})$ with conformal weight b on $\Psi \in \mathcal{S}^*$, i.e. for $f \in C^\infty(\mathcal{M})$

$$\Psi^f = D^b(f)(\Psi) = e^{bf}\Psi. \tag{5.10}$$

Note that correspondingly a test function $\varphi \in \mathcal{S}$ has to transform to $\varphi^f = e^{-bf}\varphi \in \mathcal{S}^f$ in order to keep $\Psi[\varphi]$ conformally invariant. Generally on a dual space the weight should be the negative of the weight on the original space. In our application to the quantization of H^f from (5.6), the condition

$$\hat H^f = e^{-2f} e^{bf} \hat H \, e^{-bf} \tag{5.11}$$

implies that

$$\hat H^f = -\frac{1}{2\mu}\left[\Delta^f - \xi_c R^f\right] + V^f \tag{5.12}$$

on wave functions $\Psi^f = e^{bf}\Psi \in \mathcal{S}^{*f}$. The WDW equation (5.9) is conformally equivariant if and only if Eq. (5.9) for any f is equivalent to

$$\hat H \Psi = 0 \tag{5.13}$$

where

$$\hat H = \hat H^f \mid_{f=0} \text{ and } \Psi = \Psi^f \mid_{f=0}$$

are the Hamilton operator and the wave function in the harmonic time gauge.

6. The Conformal Laplace Operator

In this section we search for a linear combination

$$\Delta_a = \Delta + aR \tag{6.1}$$

of the Laplace-Beltrami operator $\Delta = \Delta[G]$ and the Ricci scalar curvature $R = R[G]$ of an n-dimensional manifold $\mathcal{M}$, such that Δ_a is not only a generally covariant but also a conformal operator of weight -2, which furthermore transforms according to the conjugate representation D_b of Weyl transformations of weight b on the Hilbert space $\mathcal{H}(\mathcal{M})$. With $f \in C^\infty(M)$, the latter transform $\mathcal{H} := \mathcal{H}(\mathcal{M})$ to $\mathcal{H}^f := e^{bf}\mathcal{H}$. Then the conformal operator on $\mathcal{H}^f$ is

$$\Delta^f_a = e^{(b-2)f}\Delta_a e^{-bf} \tag{6.2}$$

where $\Delta^f_a = \Delta^f + aR^f$ with

$$\Delta^f = {}^f G^{ij}\nabla^f_i\nabla^f_j. \tag{6.3}$$

Here the covariant derivative ∇^f is determined by the connection Γ^f corresponding to the metric G^f. Since the components of the inverse metric are

$${}^f G^{ij} = \mathrm{e}^{-2f} G^{ij}, \tag{6.4}$$

the connection coefficients are

$$\begin{aligned} {}^f\Gamma^k_{ij} &= \frac{1}{2} {}^f G^{kl}\left\{{}^f G_{li,j} + {}^f G_{li,j} - {}^f G_{ij,l}\right\} \\ &= \Gamma^k_{ij} + \left\{\delta^k_i f_{,j} + \delta^k_j f_{,i} - G_{ij} f^{,k}\right\} \end{aligned} \tag{6.5}$$

and the Ricci scalar for ${}^f G$ is

$$\begin{aligned} {}^f R &= \mathrm{e}^{-2f} G^{cd}\Big[R_{cd} - 2(n{-}1) f_{,cd} \\ &\quad - (n{-}1)(n{-}2) f_{,c} f_{,d} + 2(n{-}1)\Gamma^e_{cd} f_{,e}\Big] \\ &\mathrm{e}^{-2f}\left\{R - 2(n{-}1)\Delta f - (n{-}1)(n{-}2) f^{,k} f_{,k}\right\}. \end{aligned} \tag{6.6}$$

On $\mathcal{H}^f$ we find

$$\begin{aligned} \Delta^f &= {}^f G^{ij} \, {}^f\nabla_i \, \partial_j \\ &= \mathrm{e}^{-2f}\Delta - \mathrm{e}^{-2f} G^{cd}\{f_{,i}\partial_j + f_{,j}\partial_i - \Gamma_{ij} f^k \partial_k\} \end{aligned} \tag{6.7}$$

in terms of the original metric G and its Laplacian Δ on acting on $\mathcal{H}$. Thus we obtain

$$\begin{aligned} \Delta^f\Psi^f = e^{(b-2)f}\big\{&\Delta\Psi + [2(b-1) + n] f^{,k}\Psi_{,k} \\ &+ [b\Delta f + (b + n - 2) b f^k f_k]\Psi\big\} \end{aligned} \tag{6.8}$$

and together with Eq. (6.6) it is

$$\begin{aligned}\Delta_a^f \Psi^f \ &= \ \mathrm{e}^{(b-2)f}\{\Delta_a \Psi + [2(b-1)+n] f^{,k}\Psi_{,k} \\ &\quad + [A\Delta f + B f^k f_k]\Psi\} \qquad (6.9)\end{aligned}$$

with the coefficients

$$A = b - 2(n-1)a, \quad B = (b+n-2)b - (n-1)(n-2)a. \qquad (6.10)$$

Vanishing of the $f^{,k}\Psi_{,k}$ term in Eq. (6.9) requires

$$b = 1 - n/2 = -(n-2)/2, \qquad (6.11)$$

which then yields the coefficients

$$A = -2(n-1)a - (n-2)/2 \qquad (6.12)$$

and

$$B = -(n-2)(n-1)a - (n-2)^2/4 \qquad (6.13)$$

both proportional to $4(n-1)a + (n-2)$. Then for $n \neq 1$ their vanishing requires

$$a = -\frac{n-2}{4(n-1)} \equiv -\xi_c. \qquad (6.14)$$

For $n = 1$ the condition (6.11) implies $b = \frac{1}{2}$ where $A \neq 0 \neq B$, whereas the vanishing of A and B, according to Eq. (6.10), implies $b = 0$ where condition (6.11) is violated. Thus for $n = 1$ there is no conformal operator (6.1) for any value of a. This is because every 1-dimensional manifold is intrinsically flat and hence $R \equiv 0$, while the representation D_b is nontrivial for $b = \frac{1}{2}$.

Note that the condition (6.11) excludes the trivial representation $b = 0$ in all dimensions except $n = 2$, where $a = b = 0$ and $\Psi^f = \Psi$ for all f, since any 2-dimensional manifold is conformally flat. We find that the operator (6.1) is conformally invariant if and only if $n \neq 1$ and the values of a and b are given by Eq. (6.11) and (6.14).

7. The Number ξ_c in Different Dimensions

In Sections 5 and 6 we saw that among all possible parameters a the conformal coupling constant ξ_c is distinguished. In this section we examine the dependence of the number $\xi_c = (D-2)/[4(D-1)]$ on the dimension D. Therefore we consider the prime factorization of ξ_c. Table 1 lists $\xi_c =: \frac{r}{s}$ with trivial greatest common divisor of r and s, i.e. $\gcd(r,s) = 1$, the maximal prime factor p_m contained in either r or s and the least common multiple $\mathrm{lcm}(\xi) := \mathrm{lcm}(r,s)$, for dimensions $D = 3 \dots 30$.

Table 1: p_m and lcm of ξ_c for $D = 3, \dots, 30$.

D :	3	4	5	6	7	8	9	10	11
$\xi_c = \frac{r}{s}$:	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{16}$	$\frac{1}{5}$	$\frac{5}{24}$	$\frac{3}{14}$	$\frac{7}{32}$	$\frac{2}{9}$	$\frac{9}{40}$
p_m :	2	3	3	5	5	7	7	3	5
lcm :	8	6	48	5	120	42	224	18	360

D	12	13	14	15	16	17	18	19	20	21
ξ_c	$\frac{5}{22}$	$\frac{11}{48}$	$\frac{3}{13}$	$\frac{13}{56}$	$\frac{7}{30}$	$\frac{15}{64}$	$\frac{4}{17}$	$\frac{17}{72}$	$\frac{9}{38}$	$\frac{19}{80}$
p_m	11	11	13	13	7	5	17	17	19	19
lcm	110	528	39	728	210	960	68	1224	342	1520

D	22	23	24	25	26	27	28	29	30
ξ_c	$\frac{5}{21}$	$\frac{21}{88}$	$\frac{11}{46}$	$\frac{23}{96}$	$\frac{6}{25}$	$\frac{25}{104}$	$\frac{13}{54}$	$\frac{27}{112}$	$\frac{7}{29}$
p_m	7	11	23	23	5	13	13	7	29
lcm	105	1848	506	2208	150	2600	702	3024	203

Assume that a system is described by a Lagrangian

$$L(q,\dot{q}) := L_1(q,\dot{q}) + \xi_c L_2(q,\dot{q}), \qquad (7.1)$$

where q denotes the configuration variables. Thus for the ground state $|0>$ we obtain (in natural units with $\hbar = 1$)

$$<0|H|0> = \omega_1 + \xi_c \omega_2 \qquad (7.2)$$

with frequencies $\omega_{1,2} =< 0|H_{1,2}|0>$ respectively. The latter are the better in resonance the smaller the lcm of ξ_c and the 'simpler' the fraction ξ_c. From the above table we find the best resonance for $D = 6$, followed by $D = 4$ and $D = 3$. In these dimensions ξ_c^{-1} is just an integer. Besides $D = 6, 4, 3$ the next best is $D = 10$ with $\xi_c = \frac{2}{9}$. Note that all dimensions of the form $D = 4i + 2, i \in \mathbb{N}$ are more resonant than their neighboring dimensions. If we admit for the rational composition only the first 3 prime numbers, then in the range $10 < D < 81$ the next best choice is $D = 26$.

In the following section we will see that ξ_c is also a critical coupling in a class of Lagrangian models of gravity plus scalar field. Then in the coupling limit $\xi \to \xi_c$ the resonance might become relevant asymptotically.

8. Conformally Equivalent Lagrangian Models

Now we want to study the effect of transformations (2) in more detail. One application of special interest is the transformation from a Lagrangian model with minimally coupled scalar field (MCM) to a conformally equivalent one with not minimally coupled scalar field (CCM) and vice versa. Let us follow [17] and consider an action of the kind

$$S = \int d^D x \sqrt{|g|}(F(\phi, R) - \frac{\epsilon}{2}(\nabla\phi)^2), \qquad (8.1)$$

specialized to the R-linear case

$$F(\phi, R) = f(\phi)R - V(\phi). \qquad (8.2)$$

A conformal transformation to the MCM yields a metric

$$\hat{g}_{\mu\nu} = \mathrm{e}^{2\omega} g_{\mu\nu}, \tag{8.3}$$

where

$$\omega := \frac{1}{D-2} \ln\left(2\kappa^2 \left|\frac{\partial F}{\partial R}\right|\right) + C, \tag{8.4}$$

and a scalar field

$$\Phi = (2\kappa)^{-1} \int d\phi \left\{ \frac{2\epsilon f(\phi) + \xi_c^{-1}(f'(\phi))^2}{f^2(\phi)} \right\}^{1/2} \tag{8.5}$$

where, as before,

$$\xi_c := \frac{D-2}{4(D-1)} \tag{8.6}$$

is the conformal coupling constant.

With the MCM potential

$$U(\Phi) = (\operatorname{sign} f(\phi)) \left[2\kappa^2 |f(\phi)|\right]^{-D/(D-2)} V(\phi) \tag{8.7}$$

($\operatorname{sign} x$ being $+1$ for $x \geq 0$ and -1 for $x < 0$) the corresponding MCM action is

$$S = \operatorname{sign} f \int d^D x \sqrt{|\hat{g}|} \Big(-\frac{1}{2} [(\hat{\nabla}\Phi)^2 - \frac{1}{\kappa^2} \hat{R}] - U(\Phi) \Big). \tag{8.8}$$

In the following we concentrate on the example

$$f(\phi) = \frac{1}{2}(1 - \xi\phi^2), \tag{8.9}$$

$$V(\phi) = \Lambda. \tag{8.10}$$

The constant $V = \Lambda$ corresponds to a non-constant potential

$$U(\Phi) = \pm\Lambda |\kappa^2 (1 - \xi\phi^2)|^{-D/(D-2)} \tag{8.11}$$

respectively for $\phi^2 < \xi^{-1}$ or $\phi^2 > \xi^{-1}$.

Let us set in the following $\epsilon = 1$. Then we obtain

$$\Phi = \kappa^{-1} \int d\phi \left\{ \frac{1 + c\xi\phi^2}{(1 - \xi\phi^2)^2} \right\}^{1/2}, \tag{8.12}$$

where

$$c := \frac{\xi}{\xi_c} - 1. \tag{8.13}$$

For $\xi = 0$ it is $\Phi = \kappa^{-1}\phi + A$, i.e. the coupling remains minimal. To calculate this integral for $\xi \neq 0$, we substitute $u := \xi\phi^2$. To assure a solution of (8.12) to be real, let us assume $\xi \geq \xi_c$ which yields $c \geq 0$. Then we obtain

$$\begin{aligned} \Phi &= \frac{\operatorname{sign}\phi}{2\kappa\sqrt{\xi}} \int \frac{\sqrt{u^{-1} + c}}{|1 - u|} du + C_{\substack{<\\>}} \\ &= \frac{\operatorname{sign}((1 - \xi\phi^2)\phi)}{2\kappa\sqrt{\xi}} \\ &\times \ln \frac{[2\sqrt{1+c}\sqrt{1+c\xi\,\phi^2}\sqrt{\xi}|\phi| + (2c+1)\xi\,\phi^2 + 1]^{\sqrt{1+c}}}{[2\sqrt{c\xi(1+c\xi\phi^2)}|\phi| + 2c\xi\phi^2 + 1]^{\sqrt{c}} \cdot |1 - \xi\phi^2|^{\sqrt{1+c}}} \\ &+ C_{\substack{<\\>}}\,. \end{aligned} \tag{8.14}$$

The integration constants $C_{\substack{<\\>}}$ for $\phi^2 < \xi^{-1}$ and $\phi^2 > \xi^{-1}$ respectively may be arbitrary functions of ξ and the dimension D. The singularities of the transform $\phi \rightarrow \Phi$ are located at $\phi^2 = \xi^{-1}$.

If the coupling is conformal $\xi = \xi_c$, i.e. $c = 0$, the expression (8.14) simplifies to

$$\kappa\Phi = \frac{1}{\sqrt{\xi_c}} [(\operatorname{artanh} \sqrt{\xi_c}\phi) + c_<] \tag{8.15}$$

for $\phi^2 < \xi_c^{-1}$ and to

$$\kappa\Phi = \frac{1}{\sqrt{\xi_c}} [(\operatorname{arcoth} \sqrt{\xi_c}\phi) + c_>] \tag{8.16}$$

for $\phi^2 > \xi_c^{-1}$. This result agrees with [18]. For $D = 4$ it has been obtained earlier in [9, 10, 19]. In [19] it has been shown for $D = 4$, that while the MCM shows a curvature singularity, the CCM with ϕ from Eq. (8.15) has no such singularity. The quantity

$$\omega = \frac{1}{D-2} \ln(\kappa^2 |1 - \xi_c\phi^2|) + C. \tag{8.17}$$

determines the conformal factor. Hence the conformal transformation has a singularity at $\phi^2 = \xi_c^{-1}$, separating different regions in ϕ where conformal equivalence between the MCM and CCM holds. Eqs.(8.15) and (8.16) illustrate a qualitatively different behavior in the two regions. In [1] this qualitative difference has been also found in multidimensional solutions of the corresponding models.

9. Application to Multidimensional Cosmologies

In what follows we want to pursue the comparison of the MCM and the CCM on the level of their classes of solutions for a multidimensional geometrical model of cosmology. Let us specify the geometry for the MCM to be of multidimensional type (2.8), with all M_i, $i = 1, \ldots, n$ being Ricci-flat and the minimally coupled scalar field to have zero potential $U \equiv 0$. In the harmonic time gauge (4.7) with the harmonic time

$$\tau \equiv t_h^{(m)}, \tag{9.1}$$

we demand this model to be a solution for (4.8) with vanishing $R^{(1)}$ and $U(\Phi)$ with $\beta^{n+1} = \kappa\Phi$. This solution is a multidimensional Kasner like universe, given by

$$\hat\beta^i = b^i\tau + c^i, \quad \hat\gamma = \sum_{i=1}^n d_i\hat\beta^i = \Sigma_b\tau + \Sigma_c;$$

$$\Sigma_b := \sum_{i=1}^n d_i b^i, \quad \Sigma_c := \sum_{i=1}^n d_i c^i, \tag{9.2}$$

with $i = 1,\ldots,n{+}1$, where with $V \equiv 0$ the constraint (4.9) simply reads

$$G_{ij}b^ib^j + (b^{n+1})^2 = 0. \tag{9.3}$$

With Eq.(8.17) the scaling powers of the universe given by Eqs.(9.2) with $i = 1,\ldots,n$ transform to the corresponding scale factors of the CCM universe

$$\begin{aligned}\beta^i &= \hat\beta^i - \omega\\ &= b^i\tau + \frac{1}{2{-}D}\ln|1{-}\xi_c(\phi)^2| + c^i + \frac{2}{2{-}D}\ln\kappa - C\end{aligned} \tag{9.4}$$

and

$$\begin{aligned}\gamma = \sum_{i=1}^n d_i\beta^i \;&=\; \Sigma_b\tau + \frac{1}{2{-}D}\ln|1{-}\xi_c(\phi)^2|\\ &+\; \Sigma_c + \frac{2}{2{-}D}\ln\kappa - C.\end{aligned} \tag{9.5}$$

Let us take for simplicity

$$C = \frac{2}{2{-}D}\ln\kappa, \tag{9.6}$$

which yields the lapse function

$$\mathrm{e}^\gamma = \mathrm{e}^{\Sigma_b\tau+\Sigma_c}|1-\xi_c(\phi)^2|^{1/(2-D)} \tag{9.7}$$

and for $i = 1,\ldots,n$ the scale factors

$$\mathrm{e}^{\beta^i} = \mathrm{e}^{b^i\tau+c^i}|1-\xi_c(\phi)^2|^{1/(2-D)}. \tag{9.8}$$

Let us further set for simplicity

$$c_< = c_> = \sqrt{\xi_c}c^{n+1}. \tag{9.9}$$

The transformation of the scalar field from the solution (8.17) of the MCM

$$\kappa\Phi(\tau) = b^{n+1}\tau + c^{n+1} \tag{9.10}$$

to the scalar field of the CCM by Eqs.(8.15) and (8.16) and the substitution of the latter into Eqs.(9.7) and (9.8) yield, with $i = 1,\ldots,n$, the nonsingular scale factors

$$\begin{aligned}\mathrm{e}^{\beta^i} &= \mathrm{e}^{b^i\tau+c^i}\cosh^{2/(D-2)}(\sqrt{\xi_c}b^{n+1}\tau)\\ \mathrm{e}^{\gamma} &= \mathrm{e}^{\Sigma_b\tau+\Sigma_c}\cosh^{2/(D-2)}(\sqrt{\xi_c}b^{n+1}\tau)\end{aligned} \tag{9.11}$$

and the singular scale factors

$$\begin{aligned}\mathrm{e}^{\beta^i} &= \mathrm{e}^{b^i\tau+c^i}|\sinh^{2/(D-2)}(\sqrt{\xi_c}b^{n+1}\tau)|\\ \mathrm{e}^{\gamma} &= \mathrm{e}^{\Sigma_b\tau+\Sigma_c}|\sinh^{2/(D-2)}(\sqrt{\xi_c}b^{n+1}\tau)|\end{aligned} \tag{9.12}$$

for the CCM. The scale factor singularity of the MCM for $\tau \to -\infty$ vanishes in the CCM of Eqs. (9.11) for a scalar field ϕ bounded according to (8.15). For $D = 4$ this result had already been indicated by [19].

On the other hand, in the CCM of Eqs.(9.12), with ϕ according to (8.16), though the scale factor singularity of the MCM for $\tau \to -\infty$ has also disappeared, instead there is another new scale factor singularity at finite (harmonic) time $\tau = 0$.

Let us consider a special case of the nonsingular solution with $\phi^2 < \xi_c^{-1}$, where we assume the internal spaces to be static in the MCM, i.e., $b^i = 0$ for $i = 2,\ldots,n$. Then in the CCM, the internal spaces are no longer static. Their scale factors 9.11) with $i > 2$ have a minimum at $\tau = 0$. Recall that for solution (9.2) all spaces M_i, internal and external, $i = 1,\ldots,n$ have been assumed to be flat. From Eq.(9.3) with $G_{11} = d_1(1-d_1)$ we find that the scalar field is given by

$$(b^{n+1})^2 = d_1(d_1-1)(b^1)^2. \tag{9.13}$$

With real b^1 then also

$$b^{n+1} = \pm\sqrt{d_1(d_1-1)}b^1 \tag{9.14}$$

is real and by Eq.(9.11) the scale a_1 of M_1 has a minimum at

$$\tau_0 = (\sqrt{\xi_c}b^{n+1})^{-1}\mathrm{artanh}\left(\frac{(2-D)}{2\sqrt{\xi_c}}\frac{b^1}{b^{n+1}}\right), \tag{9.15}$$

with $\tau_0 > 0$ for $b^1 < 0$ and $\tau_0 < 0$ for $b^1 > 0$.

The points $\tau = \tau_0$ and $\tau = 0$ are the turning points at the minimum for the factor spaces M_1 and $M_2,\ldots,M_n$, respectively. It is possible to explain the creation of our Lorentzian universe by a "birth from nothing" [20], i.e., quantum tunneling from an Euclidean region (see [1]), and (according to [12]) this interpretation has a direct topological correspondence in a projective blow-up of a singularity.

Remarkably, the multidimensional geometries with $\tau < \tau_0$ and $\tau > \tau_0$ are τ-asymmetric to each other. Taking one as contracting, the other as expanding with respect to M_1, the two are distinguished by a qualitatively different behavior of internal spaces M_k, $k \geq 2$.

In contrast to models with only one (external) space factor M_1, the additional internal spaces $M_2,\ldots,M_n$ yield an asymmetry of M in (harmonic) time τ for $\tau_0 \neq 0$, which is according to Eq.(9.15) the case exactly when $D \neq 2$ and the external space is nonstatic, i.e., $b_1 \neq 0$.

The extremal hypersurfaces of the external and internal spaces are located at different times $\tau = \tau_0$ and $\tau = 0$. Let M_1 be the external space with $b_1 > 0$ and hence $\tau_0 < 0$. Let us start with an Euclidean region of complex geometry given by the scale factors

$$a_k = \mathrm{e}^{-ib^k\tau+\tilde c^k}|\sin(\sqrt{\xi_c}b^{n+1}\tau)|^{2/(D-2)}. \tag{9.16}$$

Figure 1: Quantum birth with compact Ricci-flat spaces and birth time τ_1 of external Lorentzian space M_1. The birth of internal factor space M_2 is delayed by the interval $\Delta\tau = |\tau_2 - \tau_1|$. The internal space M_k remains for ever in the (unobservable) classically forbidden region ($\tau_k \to \infty$).

Then we can perform an analytic continuation to the Lorentzian region with $\tau \to i\tau + \pi/(2\sqrt{\xi_c} b^{n+1})$, and we require $c^k = \tilde{c}^k - i\pi b^k/(2\sqrt{\xi_c} b^{n+1})$ to be a real constant of the real geometry (9.8).

First the factor space M_1 comes into real existence and after a time interval $\Delta\tau = |\tau_0|$ the internal factor spaces $M_2, \ldots, M_n$ appear in the Lorentzian region. Since $\Delta\tau$ is arbitrarily large, there is in principle an alternative explanation of the unobservable extra dimensions, independent of the concepts of compactification and shrinking to a fundamental length in symmetry breaking. Here, they may have been up to now still in the Euclidean region and hence unobservable. This view is also compatible with the interpretation [12] of the internal symmetries as complex resolutions of simple singularities of Cartan series ADE.

Now let us perform a transition from the Lorentzian time τ to the Euclidean time $i\tau$. Then with a simultaneous transition from b^k to $-ib^k$ for $k = 1, \ldots, n$ the geometry remains real, since $\hat{\beta}^k = b^k\tau + c^k$ is unchanged, but the analog of Eq.(9.14) for the Euclidean region then becomes

$$b^{n+1} = \mp i\sqrt{d_1(d_1-1)}\, b^1. \tag{9.17}$$

Hence the scalar field is purely imaginary. This solution corresponds to a classical (instanton) wormhole. The sizes of the wormhole throats in the factor spaces $M_2, \ldots, M_n$ coincide with the sizes of static spaces in the MCM, i.e. $\hat{a}_2(0), \ldots, \hat{a}_n(0)$, respectively.

With Eq.(9.6) replaced by (9.17), Eq.(9.15) remains unchanged in the transition to the Euclidean region, and the minimum of the scale a_1 (unchanged geometry !) now corresponds to the throat of the wormhole.

The quantum creation (via tunneling) of different factor spaces M_i with nonvanishing constants b^i, c^i takes place at different times τ_k (see Fig. 1).

10. Conclusion

We have emphasized that conformal coordinate transformations (1) have to be distinguished sharply from conformal transformations of geometrical Lagrangian models (2). Similarly conformal equivalence transformations (2) of the classical Lagrangian models and minisuperspace conformal transformations (3) are conceptually very different procedures, which have to be kept apart very carefully. For multidimensional geometries (2.8) global conformal transformations of the factor spaces lead only to coordinate translations in the corresponding minisuperspace $\mathcal{M}$.

Canonical quantization of conformally equivalent models (2) should not be expected to yield minisuperspace conformal WDW equations equivalent under (3). This is especially evident when the minisuperspace contains also data beyond pure geometry, e.g., a scalar field. While a scalar field coupled to D-dimensional geometry transforms to the scalar field of the equivalent model by a complicated integral transform (see e.g. Eq.(8.14)), on the minisuperspace $\mathcal{MS}$ it is described just as an additional coordinate on equal footing with coordinates from geometry.

In Sec.7 we have seen that, besides playing a distinguished role for invariance under both (2) and (3), the conformal coupling constant ξ_c number-theoretically indicates different stability for different dimensions, with distinguished dimensions $3, 4, 6, 10$ and 26.

In Sec.9 the equivalent Lagrangian models of Sec. 3 have been compared on the level of multidimensional solutions. For the case of a massless ($U(\Phi) = 0$) minimally coupled scalar field Φ we found a multidimensional generalization of the classical Kasner solution. A conformal transformation of the Kasner solution for the MCM with flat internal spaces M_i yields the nonsingular solution (9.11) for $\phi^2 < \xi_c^{-1}$ and the singular solution (9.12) for $\phi^2 > \xi_c^{-1}$. This resolution of the scale factor singularity of the Kasner solution for a proper CCM solution (9.11) confirms for arbitrary dimension D what has been indicated in Ref.[19] for $D = 4$. At $\phi^2 = \xi_c^{-1}$ there is a singularity of the conformal transformation. The conformal equivalence holds only separately in the ranges $\phi^2 < \xi_c^{-1}$ and $\phi^2 > \xi_c^{-1}$.

In the special case of static internal spaces, we find a minimal scale $a_1(\tau_0)$ at (harmonic) time τ_0 where the birth of the universe M is happening. An analytic continuation of this solution to the Euclidean time region (preserving real geometry) yields a purely imaginary scalar field. This solution corresponds to an (instanton) wormhole, where the scale $a_1(i\tau_0)$ now indicates the throat of the wormhole.

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