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MULTIDIMENSIONAL CLASSICAL AND QUANTUM COSMOLOGY WITH PERFECT FLUID

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A cosmological model describing the evolution of n Ricci-flat spaces $(n > 1)$ in the presence of a one-component perfect-fluid and a minimally coupled scalar field is considered. When the pressures in all spaces are proportional to the density: $p_i = (1-h_i)\rho$, $h_i = \mathrm{const}$, the Einstein and Wheeler-DeWitt equations are integrated for a large variety of parameters h_i. Classical and quantum wormhole solutions are obtained for negative density $\rho < 0$. Some special classes of solutions, e.g., solutions with spontaneous and dynamical compactification, exponential and power-law inflations and nonsingular ones, are singled out. For $\rho > 0$ a third-quantized cosmological model is considered and the Planckian spectrum of created universes is obtained.

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1. Introduction

In the recent years the Kaluza-Klein ideas [1, 2] (see also [3–6]) and superstring theory [7] greatly stimulated the interest to multidimensional cosmology (see, for example, [8–58] and references therein).

Classical and quantum multidimensional cosmological models were investigated in our papers starting from [28–31] (and for the spherically symmetric case from [59, 60]). Some windows to observational effects of extra dimensions were found and analyzed, such as possible variations of the effective gravitational constant and its relations to other cosmological parameters [28, 29, 37].

One may believe that multidimensional models are most adequate in the description of the early stages of the Universe evolution where the unification of physical interactions, incorporating extra dimensions, must play an essential role.

There are also hopes that some fundamental problems of modern cosmology may be successfully solved within multidimensional models.

The treatment of classical models may be, however, only the necessary first step in analyzing the properties of the "Early Universe" and the last stages of the gravitational collapse in a multidimensional approach. In quantum multidimensional cosmology we hope to find answers to such questions as the singular state, the "creation of the Universe", the nature and value of the cosmological constant, some ideas on possible "seeds" of the observable structure of the Universe, the stability of fundamental constants etc. In the third quantization scheme the problems of topological changes may be treated thoroughly. It should be noted that multidimensional schemes may be also used in multicomponent inflationary scenarios [66–69] (see for example [56–58]).

In this paper we consider a rather general cosmological model describing the evolution of n Ricci-flat spaces $(n > 1)$ with a 1-component perfect-fluid and a minimally coupled scalar field as matter sources. The pressures in all spaces are proportional to the energy density and the proportionality factors satisfy a certain inequality (see (2.8) and (2.13)). This model incorporates many special cases of interest and is investigated in classical, quantum and third quantized cases, and the corresponding exact solutions are found. Some particular models of this type were considered previously by many authors (see, for example, [14, 19, 20, 21, 24, 30, 36]). In [47] some classes of exact solutions to Einstein and Wheeler-DeWitt equations with a multicomponent perfect fluid and a chain of Ricci-flat internal spaces were obtained. In [49] new families of classical solutions for the model [47] (including Toda-like ones) were considered.

The paper is organized as follows. In Sec.2 a general description of the model is performed. In Sec.3 we integrate the Einstein equations and analyze a class of exceptional (inflationary) solutions. (Solutions with $n = 2$ were considered recently in [58, 59]). The isotropization-like and Kasner-like asymptotical be-

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haviours of the solutions are analyzed. In Subsec.3.5 some special cases, such as isotropic (when the pressures in all spaces are equal) and curvature-like ones, are investigated. In the last case there are solutions with so-called spontaneous and dynamical compactifications. The instanton solutions (classical wormholes) with an imaginary scalar field and negative energy density are also obtained. (The interest to wormhole solutions, which play an important role in quantum gravity, is greatly stimulated by the papers [71, 72, 73]). We note that exact solutions for a 1-component perfect fluid (without a scalar field) and a chain of Ricci-flat spaces, were obtained for the first time in [30] (see also [47]). The case of a cosmological constant was considered previously in [46, 48].

In Sec.4 we consider our model at the quantum level (for pioneering papers see [74]-[79]). Here we quantize the scale factors and the scalar field but treat the perfect fluid as a classical object. Such an approach is quite consistent at least in certain special situations such as the Λ-term [46, 48] and curvature [40]-[43] cases.

In Sec.4 the Wheeler-DeWitt equation for the model is solved and quantum wormhole solutions are obtained. The notion of quantum wormholes was introduced by Hawking and Page as a quantum extension of the classical wormhole paradigm (see also [83–85] and [41, 42, 46, 48]). They proposed to regard quantum wormholes as solutions of the Wheeler-DeWitt (WDW) equation with the following boundary conditions: (i) the wave function is exponentially damped for large "spatial geometry"; (ii) the wave function is regular when the spatial geometry degenerates. The first condition expresses the fact that the space-time should be Euclidean at the spatial infinity. The second one should reflect the fact that the space-time is nonsingular when the spatial geometry degenerates. (For example, the wave function should not oscillate infinitely many times). The multidimensional quantum wormhole solutions of this paper may be considered to be a natural extension of the corresponding solutions of [41, 42] and [46, 48] for the curvature and Λ-term cases, respectively.

In Sec.5 a third quantized cosmology is investigated along the line of [40] and [54] for the curvature and cosmological constant cases, respectively. Here we are led to the theory of massless conformally coupled scalar field in a conformally flat generalized Milne universe [54]. In- and out-vacua are defined and a Planckian spectrum for the out-universes created (from an in-vacuum) is obtained using standard relations [88, 89]. The temperature is shown to depend upon the equation of state. It should be noted that recently the interest to the third quantized models was stimulated by the papers [90, 91] (see also [92]-[96] and references therein).

2. The model

We consider a cosmological model describing the evolution of n Ricci-flat spaces in the presence of a 1-component perfect-fluid matter [47] and a homogeneous massless minimally coupled scalar field. The metric of the model

$$g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)]g^{(i)}, \quad (2.1)$$

is defined on the manifold

$$M = R \times M_1 \times \ldots \times M_n, \quad (2.2)$$

where the manifold M_i with the metric $g^{(i)}$ is a Ricci-flat space of dimension N_i, $i = 1, \ldots, n$; $n \geq 2$. We take the field equations in the following form:

$$R^M_N - \frac{1}{2}\delta^M_N R = \kappa^2 T^M_N, \quad (2.3)$$

$$\Box\varphi = 0, \quad (2.4)$$

where κ^2 is the gravitational constant, $\varphi = \varphi(t)$ is scalar field, $\Box$ is the d'Alembert operator for the metric (2.1) and the energy-momentum tensor is adopted in the form

$$T^M_N = T^{M(pf)}_N + T^{M(\varphi)}_N, \quad (2.5)$$

$$(T^{M(pf)}_N) = \mathrm{diag}\,(-\rho, p_1\delta^{m_1}_{k_1}, \ldots, p_n\delta^{m_n}_{k_n}), \quad (2.6)$$

$$T^{M(\varphi)}_N = \partial^M\varphi\partial_N\varphi - \tfrac{1}{2}\delta^M_N(\partial\varphi)^2. \quad (2.7)$$

We assume the pressures of the perfect fluid in all spaces to be proportional to the density:

$$p_i(t) = (1 - \frac{u_i}{N_i})\rho(t), \quad (2.8)$$

where $u_i = \mathrm{const}$, $i = 1, \ldots, n$.

We also impose the following restriction on the vector $u = (u_i) \in R^n$:

$$< u, u >_* < 0. \quad (2.9)$$

Here the bilinear form $< .,. >_*$: $R^n \times R^n \to R$ is defined by the relation

$$< u, v >_* = G^{ij}u_i v_j, \quad (2.10)$$

$u, v \in R^n$, where

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D} \quad (2.11)$$

are components of the matrix inverse to the matrix of the minisuperspace metric [30, 31]

$$G_{ij} = N_i\delta_{ij} - N_i N_j. \quad (2.12)$$

In (2.11) $D = 1 + \sum_{i=1}^n N_i$ is the dimension of the manifold M (2.2).

Remark 1. This restriction (2.9) reads

$$<u,u>_* \equiv G^{ij}u_iu_j = \sum_{i=1}^{n}\frac{(u_i)^2}{N_i} + \frac{1}{2-D}(\sum_{i=1}^{n}u_i)^2 < 0. \tag{2.13}$$

In the notations of [30] $< u,u >_* = \acute{\Delta}(h)/(2-D)$.

3. Classical solutions

The Einstein equations (2.3) imply $\nabla_M T^M_N = 0$ and due to (2.4) $\nabla_M T^{M(pf)}_N = 0$ or equivalently

$$\dot\rho + \sum_{i=1}^{n} N_i\dot x^i(\rho + p_i) = 0. \tag{3.1}$$

From (2.8), (3.1) we get

$$\kappa^2\rho(t) = A\exp[-2N_ix^i(t) + u_ix^i(t)], \tag{3.2}$$

where $A =$ const. For positive $\rho(t)$ the constant A is also positive. We put $A \neq 0$ (the case $A = 0$ was considered thoroughly in [53]).

It is not difficult to verify that the field equations (2.3), (2.4) for the cosmological metric (2.1) in the harmonic time gauge

$$\gamma_0 \equiv \sum_{i=1}^{n} N_ix^i \tag{3.3}$$

with the energy-momentum tensor from (2.5)-(2.7) and the relations (2.8), (3.2) imposed are equivalent to the Lagrange equations for the Lagrangian

$$L = \frac{1}{2}(G_{ij}\dot x^i\dot x^j + \kappa^2\dot\varphi^2) - \kappa^2A\exp(u_kx^k) \tag{3.4}$$

with the energy constraint

$$E = \frac{1}{2}(G_{ij}\dot x^i\dot x^j + \kappa^2\dot\varphi^2) + \kappa^2A\exp(u_kx^k) = 0 \tag{3.5}$$

(for $\varphi = 0$ see [47, 56]).

Recall [30, 31] that the minisuperspace metric

$$G = G_{ij}dx^i\otimes dx^i \tag{3.6}$$

has the pseudo-Euclidean signature $(-,+,\dots,+)$, i.e. there is a linear transformation

$$z^a = e^a_ix^i, \tag{3.7}$$

diagonalizing the minisuperspace metric (3.6), (2.12)

$$G = \eta_{ab}dz^a\otimes dz^b = -dz^0\otimes dz^0 + \sum_{i=1}^{n-1}dz^i\otimes dz^i, \tag{3.8}$$

where

$$(\eta_{ab}) = (\eta^{ab}) \equiv \mathrm{diag}\,(-1,+1,\dots,+1), \tag{3.9}$$

$a,b = 0,\dots,n-1$. From (3.7)-(3.8) we get

$$\eta_{ab}e^a_ie^b_j = G_{ij} \tag{3.10}$$

and as a consequence

$$e^i_aG_{ij}e^j_b = \eta_{ab}, \qquad e^i_a = \eta_{ab}G^{ij}e^b_j, \tag{3.11}$$

where $(e^i_a) = (e^a_i)^{-1}$. As in [47], we put

$$e^0_i = u_i/(2q) \Longrightarrow z^0 = u_ix^i/(2q), \tag{3.12}$$

where here and henceforth

$$2q \equiv \sqrt{-<u,u>_*}. \tag{3.13}$$

It may be done since $< .,. >_*$ is a bilinear symmetric 2-form of the signature $(-,+,\dots,+)$ and $< u,u >_* < 0$ [47]. An example of diagonalization (3.7) satisfying (3.12) was considered in [30, 31]. From (3.11) and (3.12) we get

$$e^i_0 = -G^{ij}e^0_j = -u^i/(2q), \qquad u^i \equiv G^{ij}u_j. \tag{3.14}$$

We also denote

$$z^n = \kappa\varphi. \tag{3.15}$$

The Lagrangian (3.4) in the z variables (3.7), (3.15) (with the relation (3.12) imposed) may be rewritten as

$$L = \frac{1}{2}\eta_{AB}\dot z^A\dot z^B - \kappa^2A\exp(2qz^0), \tag{3.16}$$

where the indices $A,B = 0,\dots,n$. The energy constraint (3.5) reads:

$$E = \frac{1}{2}\eta_{AB}\dot z^A\dot z^B + \kappa^2A\exp(2qz^0) = 0. \tag{3.17}$$

The Lagrange equations for the Lagrangian (3.16)

$$-\ddot z^0 + 2qA\exp(2qz^0) = 0, \tag{3.18}$$

$$\ddot z^B = 0, \quad B = 1,\dots,n, \tag{3.19}$$

with the energy constraint (3.17) can be easily solved. From (3.19) we have

$$z^B = p^Bt + q^B, \tag{3.20}$$

where p^B and q^B are constants and $B = 1,\dots,n$. The first integral of Eq.(3.18) reads

$$-\frac{1}{2}(\dot z^0)^2 + A\exp(2qz^0) + \mathcal{E} = 0. \tag{3.21}$$

Using (3.17), (3.20) and (3.21) we get

$$\mathcal{E} = \frac{1}{2}\sum_{B=1}^{n}(p^B)^2. \tag{3.22}$$

We obtain the following solution to Eqs.(3.18), (3.21)

$$\begin{aligned}
&\exp(-2qz^0) \\
&= (A/\mathcal{E})\sinh^2(q\sqrt{2\mathcal{E}}(t-t_0)), \quad \mathcal{E}>0,\ A>0, &(3.23)\\
&= (A/|\mathcal{E}|)\sin^2(q\sqrt{2|\mathcal{E}|}(t-t_0)), \quad \mathcal{E}<0,\ A>0, &(3.24)\\
&= 2q^2A(t-t_0)^2, \quad \mathcal{E}=0,\ A>0, &(3.25)\\
&= (|A|/\mathcal{E})\cosh^2(q\sqrt{2\mathcal{E}}(t-t_0)), \quad \mathcal{E}>0,\ A<0. &(3.26)
\end{aligned}$$

Here t_0 is an arbitrary constant. For real z^B (or, equivalently, for real metric and scalar field) we get from (3.22) $\mathcal{E} \geq 0$. The case $\mathcal{E} < 0$ may take place when a pure imaginary scalar field is considered.

3.1. Kasner-like parametrization for non-exceptional solutions with a real scalar field

We first consider the real case with $\mathcal{E} > 0$. In this case the relations (3.23) and (2.26) may be written in the following form:

$$\exp(-2qz^0) = \frac{|A|}{\mathcal{E}} f_\delta^2(q\sqrt{2\mathcal{E}}(t-t_0)), \qquad (3.27)$$

where $\delta \equiv A/|A| = \pm 1$ and

$$\begin{aligned}
f_\delta(x) \equiv \frac{1}{2}(e^x - \delta e^{-x}) &= \sinh x, \quad \delta=+1, \\
&= \cosh x, \quad \delta=-1. \qquad (3.28)
\end{aligned}$$

We introduce a new time variable by the relation

$$\begin{aligned}
\tau &= \frac{T}{\sqrt{\delta}} \ln \frac{\exp[q\sqrt{2\mathcal{E}}(t-t_0)] + \sqrt{\delta}}{\exp[q\sqrt{2\mathcal{E}}(t-t_0)] - \sqrt{\delta}} &(3.29)\\
&= T \ln\coth[\frac{1}{2}q\sqrt{2\mathcal{E}}(t-t_0)], \quad \delta=+1, &(3.30)\\
&= 2T\arctan\exp[-q\sqrt{2\mathcal{E}}(t-t_0)], \quad \delta=-1, &(3.31)
\end{aligned}$$

where

$$\begin{aligned}
T &= T(u,A) \equiv (2q^2|A|)^{-1/2} \\
&= (\tfrac{1}{2}|A < u,u>_*|)^{-1/2}. \qquad (3.32)
\end{aligned}$$

For $\delta = +1$ the variable $\tau = \tau(t)$ monotonically decreases from $+\infty$ to 0 when $t - t_0$ is varying from 0 to $+\infty$. For $\delta = -1$ it monotonically decreases from πT to 0 when $t - t_0$ is varying from $-\infty$ to $+\infty$.

It is not difficult to verify that

$$\begin{aligned}
\sinh(\tau\sqrt{\delta}/T)/\sqrt{\delta} &= 1/f_\delta(q\sqrt{2\mathcal{E}}(t-t_0)), &(3.33)\\
\tanh(\tau\sqrt{\delta}/2T)/\sqrt{\delta} &= \exp[-q\sqrt{2\mathcal{E}}(t-t_0)], &(3.34)\\
d\tau &= -qT\sqrt{2\mathcal{E}}dt/f_\delta(q\sqrt{2\mathcal{E}}(t-t_0)). &(3.35)
\end{aligned}$$

To present the solutions obtained in a more familiar form we now introduce the following dimensionless "Kasner-like" parameters:

$$\begin{aligned}
\beta^i &= -e^i_{\hat{a}}p^{\hat{a}}/(q\sqrt{2\mathcal{E}}), &(3.36)\\
\beta_\varphi &= -p^n/(q\sqrt{2\mathcal{E}}). &(3.37)
\end{aligned}$$

Here and henceforth $\hat{a}, \hat{b} = 1, \ldots, n-1$. From Eqs. (3.7), (3.14), (3.20) and (3.36) we have

$$\begin{aligned}
x^i &= -(u^i/2q)z^0 + e^i_{\hat{a}}[p^{\hat{a}}(t-t_0) + \bar{q}^{\hat{a}}] \\
&= -(u^i/4q^2)(2qz^0) - q\sqrt{2\mathcal{E}}(t-t_0)\beta^i + \gamma^i, \qquad (3.38)
\end{aligned}$$

where

$$\gamma^i = e^i_{\hat{a}}\bar{q}^{\hat{a}}, \qquad \bar{q}^{\hat{a}} = q^{\hat{a}} + p^{\hat{a}}t_0. \qquad (3.39)$$

Using (3.13), (3.27), (3.33), (3.34) and (3.38), we get for the scale factors:

$$a_i = e^{x^i} = A_i[\sinh(r\tau/T)/r]^{\sigma^i}[\tanh(r\tau/2T)/r]^{\beta^i} \qquad (3.40)$$

where $r = \sqrt{\delta}$ and

$$\sigma^i = 2u^i/ < u,u>_*, \quad A_i = (\mathcal{E}/|A|)^{\sigma^i/2}e^{\gamma^i}, \qquad (3.41)$$

$i = 1, \ldots, n$. In a similar manner we obtain the expression for the scalar field (see (3.34), (3.37))

$$e^{\kappa\varphi} = e^{z^n} = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi} \qquad (3.42)$$

where $A_\varphi > 0$ is constant.

We define a bilinear symmetric form $< .,. >$: $R^n \times R^n \to R$ by the relation

$$< \alpha, \beta > = G_{ij}\alpha^i\beta^j, \qquad (3.43)$$

$\alpha = (\alpha^i)$, $\beta = (\beta^i) \in R^n$. Using the definitions (3.13), (3.22), (3.36), (3.37) and the relations (3.11), we obtain the relations between the Kasner-like parameters

$$\begin{aligned}
< \beta,\beta > + (\beta_\varphi)^2 &= G_{ij}\beta^i\beta^j + (\beta_\varphi)^2 \\
&= 1/q^2 = -4/ < u,u >_*, \qquad (3.44)
\end{aligned}$$

and (see (3.12))

$$u_i\beta^i = e^0_i e^i_{\hat{a}}P^{\hat{a}} = \delta^0_{\hat{a}}P^{\hat{a}} = 0, \qquad (3.45)$$

where $P^{\hat{a}} = -p^{\hat{a}}\sqrt{2/\mathcal{E}}$.

Similarly to (3.45), we get $u_i\gamma^i = 0$ and hence (see (3.41))

$$\prod_{i=1}^n A_i^{u_i} = \mathcal{E}/|A|. \qquad (3.46)$$

Thus the additional integral of motion $\mathcal{E}$ is a certain combination of parameters A_i and $|A|$ depending on the equation of state (2.8).

We also introduce the "quasi-volume" scale factor

$$v = \prod_{i=1}^n a_i^{u_i/2} = \exp(\tfrac{1}{2}u_ix^i). \qquad (3.47)$$

From (3.12), (3.27), (3.33), (3.46) (see also (3.40), (3.45)) we have

$$v = \frac{v_0}{r}\sinh\frac{r\tau}{T} = \sqrt{\mathcal{E}/|A|}/[f_\delta(q\sqrt{2\mathcal{E}}(t-t_0))]. \qquad (3.48)$$

Here

$$v_0 = \prod_{i=1}^{n} A_i^{u_i/2}. \tag{3.49}$$

The quasi-volume scale factor oscillates for $A < 0$ (negative energy density) and exponentially increases as $\tau \to +\infty$ for $A > 0$ (positive energy density).

From (3.3), (3.32), (3.35), (3.47), (3.48) we get

$$\begin{aligned} e^{2\gamma_0(t)} dt \otimes dt &= \Bigl(\prod_{i=1}^{n} a_i^{2N_i}\Bigr) f_\delta^2(q\sqrt{2\mathcal{E}}(t-t_0)) \frac{d\tau \otimes d\tau}{2q^2 \mathcal{E} T^2} \\ &= \Bigl(\prod_{i=1}^{n} a_i^{2N_i - u_i}\Bigr) d\tau \otimes d\tau. \end{aligned} \tag{3.50}$$

Thus we get the following solution to the field equations (2.3) and (2.4):

$$g = -\Bigl[\prod_{i=1}^{n} (a_i(\tau))^{2N_i - u_i}\Bigr] d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{3.51}$$

$$a_i(\tau) = A_i \Bigl(\frac{1}{r} \sinh \frac{r\tau}{T}\Bigr)^{2u^i/<u,u>_*} \Bigl(\frac{1}{r} \tanh \frac{r\tau}{2T}\Bigr)^{\beta^i}, \tag{3.52}$$

$$e^{\kappa\varphi(\tau)} = A_\varphi \Bigl(\frac{1}{r} \tanh \frac{r\tau}{2T}\Bigr)^{\beta_\varphi}, \tag{3.53}$$

$$\kappa^2 \rho(\tau) = A \prod_{i=1}^{n} (a_i(\tau))^{u_i - 2N_i}, \tag{3.54}$$

$i = 1, \ldots, n$; where $r = \sqrt{A/|A|}$, T is defined in (3.32), $A_i, A_\varphi > 0$ are constants, and the parameters β^i, β_φ satisfy the relations

$$\begin{aligned} \sum_{i=1}^{n} u_i \beta^i &= 0, \\ \sum_{i,j=1}^{n} G_{ij} \beta^i \beta^j + (\beta_\varphi)^2 &= -4/<u,u>_* = 1/q^2. \end{aligned} \tag{3.55}$$

Here $\tau > 0$ for $A > 0$ and $0 < \tau < \pi T$ for $A < 0$.

We note that the solution (3.51)-(3.55) without scalar field ($\beta_\varphi = 0$) was obtained previously in [47]. For $u_i = 2N_i$ (Λ-term case) the solution was considered in [48] (for $\beta_\varphi = 0$ see also [46]), where Euclidean wormholes were constructed.

For small values of τ we have the following asymptotic relations

$$a_i(\tau) \sim C_i \tau^{\bar\beta^i}, \qquad e^{\kappa\varphi(\tau)} \sim C_\varphi \tau^{\beta_\varphi} \tag{3.56}$$

as $\tau \to 0$, $i = 1, \ldots, n$, where C_i, C_φ are constants and $\bar\beta^i = \beta^i + \sigma^i$ are the new Kasner-like parameters, satisfying the relations

$$u_i \bar\beta^i = 2, \qquad G_{ij} \bar\beta^i \bar\beta^j + \beta_\varphi^2 = 0. \tag{3.57}$$

3.2. Exceptional solutions

Now we consider the exceptional real solutions corresponding to $\mathcal{E} = 0$ and $A > 0$ (see (3.25)). From $\mathcal{E} = 0$ and (3.22) we have $p^B = 0$ and hence

$$z^B = q^B \tag{3.58}$$

are constant, $B = 1, \ldots, n$. So, $\kappa\varphi = z^n = \text{const}$ in this case. From (3.7) and (3.12) we have

$$x^i = -(u^i/4q^2)(2qz^0) + \gamma^i, \qquad \gamma^i = e^i_{\hat a} q^{\hat a}, \tag{3.59}$$

($\hat a = 1, \ldots, n-1$). Using (3.13), (3.25), (3.32) and (3.59) for $t > t_0$, we get

$$a_i = e^{x^i} = [(t - t_0)/T]^{-\sigma^i} e^{\gamma^i}, \tag{3.60}$$

$i = 1, \ldots, n$.

Introducing the new time variable τ by

$$T/(t - t_0) = \exp[\pm(\tau - \tau_0)/T], \qquad t > t_0, \tag{3.61}$$

we obtain

$$a_i(\tau) = \bar A_i \exp(\pm\sigma^i \tau/T), \tag{3.62}$$

where

$$\bar A_i = \exp(\mp\sigma^i \tau_0/T) \exp(\gamma^i), \tag{3.63}$$

$i = 1, \ldots, n$.

Similarly to (3.45) we get $u_i \gamma^i = 0$ and hence (see (3.63))

$$\prod_{i=1}^{n} \bar A_i^{u_i} = \exp(\mp 2\tau_0/T). \tag{3.64}$$

For the quasi-volume from (3.62) and (3.64) we get

$$v = \prod_{i=1}^{n} a_i^{u_i/2} = \exp[\pm(\tau - \tau_0)/T]. \tag{3.65}$$

Thus for $A > 0$ we have a family of exceptional solutions with a constant real scalar field

$$g = -\Bigl(\prod_{i=1}^{n} (a_i(\tau))^{2N_i - u_i}\Bigr) d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \tag{3.66}$$

$$a_i(\tau) = \bar A_i \exp[\pm 2u^i \tau/(T < u, u >_*)], \tag{3.67}$$

$$\varphi(\tau) = \text{const}, \tag{3.68}$$

and $\rho(\tau)$ is determined by (3.54). Here $\bar A_i > 0$ ($i = 1, \ldots, n$) are constants, and T is defined in (3.32).

We note that for $A > 0$ the solution (3.67) with the + sign is an attractor for the solutions (3.52), i.e.,

$$a_i(\tau) \sim \bar A_i \exp(\sigma^i \tau/T), \qquad \varphi(\tau) \sim \text{const}, \tag{3.69}$$

$i = 1, \ldots, n$, for $\tau \to +\infty$.

Synchronous-time parametrization

The relations (3.67) imply

$$\prod_{i=1}^{n} a_i^{2N_i-u_i} = \bar{P}^2 \exp[\pm 2(\bar{\sigma}-1)\tau/T], \tag{3.70}$$

where

$$\bar{P} = \prod_{i=1}^{n} \bar{A}_i^{N_i-u_i/2}, \tag{3.71}$$

$$\bar{\sigma} = \frac{2N_i u^i}{< u,u >_*} = \frac{< u^{(\Lambda)}, u >_*}{< u,u >_*}. \tag{3.72}$$

Here and henceforth

$$u_i^{(\Lambda)} = 2N_i. \tag{3.73}$$

Now we introduce the synchronous time variable t_s satisfying the relation

$$\bar{P}^2 \exp[\pm 2(\bar{\sigma}-1)\tau/T]\, d\tau \otimes d\tau = dt_s \otimes dt_s. \tag{3.74}$$

First we consider the case

$$\bar{\sigma} \neq 1 \iff < u^{(\Lambda)} - u, u >_* \neq 0. \tag{3.75}$$

Introducing t_s by the formula

$$t_s = \frac{\bar{P}T}{|\bar{\sigma}-1|} \exp[\pm(\bar{\sigma}-1)\tau/T] > 0, \tag{3.76}$$

we get for the scale factors

$$a_i = a_i(t_s) = A_i t_s^{\nu^i}, \tag{3.77}$$

where

$$\nu^i = \sigma^i/(\bar{\sigma}-1) = 2u^i/ < u^{(\Lambda)} - u, u >_* \tag{3.78}$$

and

$$A_i = \bar{A}_i[|\bar{\sigma}-1|/(\bar{P}T)]^{\nu^i}. \tag{3.79}$$

The parameters ν^i (3.78) satisfy the relation

$$\nu^i(2N_i - u_i) = 2. \tag{3.80}$$

Eqs. (3.32), (3.71), (3.79) and (3.80) imply

$$\begin{aligned} \prod_{i=1}^{n} A_i^{u_i-2N_i} &= T^2/(\bar{\sigma}-1)^2 \\ &= -2<u,u>_*/(A<u^{(\Lambda)}-u,u>_*^2). \end{aligned} \tag{3.81}$$

From (3.54), (3.77), (3.80) and (3.81) we get the following formula for the density:

$$\kappa^2 \rho = \kappa^2 \rho(t_s) = \frac{-2 < u,u >_*}{< u^{(\Lambda)} - u, u >_*^2 \, t_s^2}. \tag{3.82}$$

The metric reads:

$$g = -dt_s \otimes dt_s + \sum_{i=1}^{n} a_i^2(t_s) g^{(i)}, \tag{3.83}$$

where the scale factors are determined by (3.77), $i = 1,\ldots,n$. Thus Eqs. (3.54), (3.77), (3.82), (3.83) and $\varphi =$ const describe the exceptional solutions for the case (3.75). We call them power-law inflationary solutions.

Now let us consider the case

$$\bar{\sigma} = 1 \iff < u^{(\Lambda)} - u, u >_* = 0. \tag{3.84}$$

From (3.54), (3.70) we obtain

$$\kappa^2 \rho = A\bar{P}^{-2} = \text{const}. \tag{3.85}$$

Introducing the synchronous time $t_s = \bar{P}\tau$, from (3.67) we get

$$a_i(t_s) = \bar{A}_i \exp[\mp \frac{u^i}{\sqrt{- < u,u >_*}} \frac{t_s}{T_0}], \tag{3.86}$$

where

$$T_0 = (2\kappa^2\rho)^{-1/2}. \tag{3.87}$$

Eqs. (3.83), (3.85)-(3.87) and $\varphi =$ const describe the exponential-type inflation for the case (3.84).

Let us consider the synchronous time parametrization for the solutions (3.51)-(3.55). The synchronous time t_s and the τ variable are related by

$$t_s = \varepsilon F(\tau), \qquad \frac{dF}{d\tau} = f(\tau), \tag{3.88}$$

where $\varepsilon = \pm 1$ and

$$\begin{aligned} f(\tau) &= \prod_{i=1}^{n} (a_i(\tau))^{N_i-u_i/2} \\ &= P[\sinh(r\tau/T)/r]^{\bar{\sigma}-1}[\tanh(r\tau/2T)/r]^{\beta^i N_i}, \end{aligned} \tag{3.89}$$

$$\begin{aligned} &\kappa^2\rho(\tau) \\ &= AP^{-2}[\sinh(r\tau/T)/r]^{2-2\bar{\sigma}}[\tanh(r\tau/2T)/r]^{-2\beta^i N_i}, \end{aligned} \tag{3.90}$$

with $P = \prod_{i=1}^{n} A_i^{N_i-u_i/2}$ and $\bar{\sigma}$ defined in (3.72). From (3.90) it follows

$$f(\tau) \sim B\tau^{p-1}, \qquad \tau \to +0, \tag{3.91}$$

where $B > 0$ is constant and

$$p = p(\beta) = \bar{\sigma} + \beta^i N_i = (\sigma^i + \beta^i)N_i. \tag{3.92}$$

Putting $\varepsilon = \text{sign}\,(p)$, from (3.88) and (3.91) we get

$$t_s \sim B_1 \tau^p, \qquad \tau \to +0, \tag{3.93}$$

with $B_1 = B/|p|$ (here the integration constant in (3.88) is properly fixed).

Proposition 1. Let $1/q^2 - (\beta_\varphi)^2 \geq 0$. Then, for all $\beta = (\beta^i)$ satisfying the relations (3.55), we have $p(\beta) \neq 0$ and

$$i)\ u^i N_i < 0 \ \Rightarrow\ p(\beta) > 0, \tag{3.94}$$

$$ii)\ u^i N_i > 0 \ \Rightarrow\ p(\beta) < 0. \tag{3.95}$$

Proposition 1 is a special case of a more general Proposition 2 proved in the Appendix:

Proposition 2. Let two vectors $u = (u_i)$, $v = (v_i) \in R^n$ satisfy the inequalities $< u, u >_* \equiv -4q^2 < 0$ and $< v, v >_* < 0$. Then $u^i v_i =< u, v >_* \neq 0$ and for all $\beta = (\beta^i)$ such as

$$u_i \beta^i = 0, \qquad G_{ij}\beta^i\beta^j \leq 1/q^2, \tag{3.96}$$

the following relation is valid:

$$\text{sign}\,(u^i v_i) = -\text{sign}\,((\sigma^i + \beta^i)v_i), \tag{3.97}$$

where $\sigma^i = 2u^i/ < u, u >_*$.

For the vector

$$v_i = N_i = \frac{1}{2} u_i^{(\Lambda)} \tag{3.98}$$

we have

$$v^i = N^i = G^{ij} N_j = \frac{1}{2-D} \tag{3.99}$$

and hence

$$< v, v >_* = N_i N^i = -\frac{D-1}{D-2} < 0. \tag{3.100}$$

Thus Eqs. (3.92), (3.100) and Proposition 2 imply the Proposition 1.

From (3.99) we get

$$u_i N^i = \frac{1}{2-D} \sum_{i=1}^{n} u_i. \tag{3.101}$$

Using (3.93), (3.101) and Proposition 1, we obtain

$$t_s \to +0 \quad \text{as} \quad \tau \to +0, \tag{3.102}$$

for

$$(A) \quad \sum_{i=1}^{n} u_i > 0, \quad p(\beta) > 0 \tag{3.103}$$

and

$$t_s \to +\infty \quad \text{as} \quad \tau \to +0, \tag{3.104}$$

for

$$(B) \quad \sum_{i=1}^{n} u_i < 0, \quad p(\beta) < 0. \tag{3.105}$$

In the limit $\tau \to +0$ we have $\tau \sim (t_s/B_1)^{1/p}$ (see (3.93)) and hence (see (3.56))

$$a_i(t_s) \sim \bar{B}_i t_s^{\alpha^i}, \quad \exp(\kappa\varphi(t_s)) \sim \bar{B}_\varphi t_s^{\alpha_\varphi} \tag{3.106}$$

as $t_s \to +0$ in the case A) (3.103) and as $t_s \to +\infty$ in the case B) (3.105). Here $\bar{B}_i, \bar{B}_\varphi$ are constants and

$$\alpha^i = (\sigma^i + \beta^i)/p(\beta), \quad \alpha_\varphi = \beta_\varphi/p(\beta), \tag{3.107}$$

Figure 1: A graphic representation of the allowed cone $< u, u >_* < 0$ and different domains in it in the plain $\xi_i = p_i/\rho$, $i = 1, 2$, for the case $n = 2$ and $N_1 = 3$, $N_2 = 6$. The hyperbola $\bar{\sigma} = 1$ corresponds to the exponential inflation (3.86).

$i = 1 \ldots n$. The parameters α^i, α_φ satisfy the Kasner-like relations

$$\sum_{i=1}^{n} N_i \alpha^i \; = \; 1, \tag{3.108}$$

$$\sum_{i=1}^{n} N_i (\alpha^i)^2 + \alpha_\varphi^2 \; = \; 1. \tag{3.109}$$

Eq.(3.108) is quite obvious, Eq.(3.109) follows from (3.108), (2.12) and the relation

$$G_{ij}\alpha^i\alpha^j + \alpha_\varphi^2 = 0, \tag{3.110}$$

which is readily verified using (3.13), (3.41), (3.55) and (3.107).

The Kasner-like asymptotical behaviour (3.106), (3.108), (3.109) for the case (A) agrees with one of the results of [56]: in the case A) the perfect fluid components with $< u, u >_* < 0$ may be neglected near the singularity $t_s \to +0$ and we are led to the Kasner-like formulas [53] (see also [39]).

Note that for the case $n = 2$ the following relation is valid:

$$\left[b_2 \frac{u_1}{N_1} - (1+s)\frac{u_2}{N_2}\right] \left[b_1 \frac{u_2}{N_2} - (1+s)\frac{u_1}{N_1}\right] = (-s^2)(1+s) < u, u >_* \tag{3.111}$$

where $b_i = 1 - 1/N_i$, $i = 1, 2$ and $s = \sqrt{1 - b_1 b_2}$. This implies the relations for the light-cone lines ($<u, u>_* =$

0):

$$l_1: \quad b_2(1-\xi_1) = (1+s)(1-\xi_2), \qquad (3.112)$$

$$l_2: \quad b_1(1-\xi_2) = (1+s)(1-\xi_1), \qquad (3.113)$$

(see Fig.1).

3.3. Isotropization-like behaviour

Here we rewrite the attractor behaviour (3.69) for the non-exceptional solutions (3.51)-(3.55) with $A > 0$ (as $\tau \to +\infty$) in terms of the synchronous time variable t_s. For the function (3.89) we have the following asymptotical behaviour

$$f(\tau) \sim P[\tfrac{1}{2}\exp(\tau/T)]^{\bar{\sigma}-1} = \bar{B}\exp[(\bar{\sigma}-1)\tau/T], \qquad (3.114)$$

when $\tau \to +\infty$ ($\bar{B} = \mathrm{const}$).

First consider the case $\bar{\sigma} = 1$ (see (3.84)). Then $f(\tau) \sim \bar{B}$ as $\tau \to +\infty$ and hence (see (3.88))

$$t_s = F(\tau) \sim \bar{B}\tau + C, \qquad (3.115)$$

as $\tau \to +\infty$. (Due to $u^i N_i < 0$ and Proposition 1, $\varepsilon = \mathrm{sign}\,(p) = +1$). The synchronous time t_s monotonically increases from 0 to $+\infty$ as τ varies from 0 to $+\infty$ (see (3.93)).

From (3.54), (3.69) and (3.115) for the case $\bar{\sigma} = 1$ we get

$$a_i(t_s) \sim \bar{A}_i \exp[-\frac{u^i}{\sqrt{-<u,u>_*}}\frac{t_s}{T_0}], \qquad (3.116)$$

$$\varphi(t_s) \sim \mathrm{const}, \qquad (3.117)$$

$$\rho(t_s) \sim \rho_0 \qquad (3.118)$$

when $t_s \to +\infty$, where $T_0 = (2\kappa^2\rho_0)^{-1/2}$.

Now consider the case $\bar{\sigma} \neq 1$ (see (3.75)). Then

$$F(\tau) \sim \frac{\bar{B}T}{(\bar{\sigma}-1)}\exp[(\bar{\sigma}-1)\tau/T] + C, \qquad (3.119)$$

where C is a constant.

Consider first the subcase $\bar{\sigma} > 0$ or, equivalently, $u_i N^i < 0$ (or $\sum_{i=1}^n u_i > 0$, see (3.101)). We have $t_s = F(\tau)$, since $p > 0$ due to (3.103) and $\varepsilon = \mathrm{sign}\,(p) = +1$). In this case t_s monotonically increases from 0 to $T_* > 0$ for $0 < \bar{\sigma} < 1$ and to $+\infty$ for $\bar{\sigma} > 1$ as τ varies from 0 to $+\infty$ (see (3.93)). Using (3.54), (3.69) we get

$$a_i(t_s) \sim A_i(T_* - t_s)^{\nu^i}, \qquad (3.120)$$

$$\varphi(t_s) \sim \mathrm{const}, \qquad (3.121)$$

$$\kappa^2\rho(t_s) \sim \frac{-2<u,u>_*}{<u^{(\Lambda)}-u,u>_*^2 (T_*-t_s)^2}. \qquad (3.122)$$

as $t_s \to T_* - 0$, for $\bar{\sigma} < 1$. For $\bar{\sigma} > 1$ we have an asymptotic behaviour in the limit $t_s \to +\infty$ described by the relations

$$a_i(t_s) \sim A_i t_s^{\nu^i}, \qquad (3.123)$$

$$\varphi(t_s) \sim \mathrm{const}, \qquad (3.124)$$

$$\kappa^2\rho(t_s) \sim \frac{-2<u,u>_*}{<u^{(\Lambda)}-u,u>_*^2 \, t_s^2}. \qquad (3.125)$$

as $t_s \to +\infty$, where ν^i is defined in (3.78).

Now consider the subcase $\bar{\sigma} < 0$ or, equivalently, $u_i N^i > 0$ (or $\sum_{i=1}^n u_i < 0$, see (3.101)). Recall that $p < 0$ due to (3.105) and $\varepsilon = \mathrm{sign}\,(p) = -1$. Then $t_s = -F(\tau)$ (we put $C = 0$ in (3.119)) and t_s monotonically decreases from $+\infty$ to 0 as τ varies from 0 to $+\infty$ (see (3.93)). In this subcase we obtain the asymptotic behaviour in the limit $t_s \to +0$ described by (3.123)-(3.125).

3.4. Solutions with a pure imaginary scalar field

Here we consider solutions to the field equations with a complex scalar field and a real metric. In this case $\mathcal{E}, p^1, \ldots, p^{n-1}$ are real and hence (see (3.21), (3.22)) p^n is either real or pure imaginary. The case of real p^n was considered above.

For pure imaginary p^n we have three subcases: (a) $\mathcal{E} > 0$, (b) $\mathcal{E} = 0$, and (c) $\mathcal{E} < 0$. In the first case (a) after the reparametrization (3.29)-(3.32) we get the solutions (3.51)-(3.55) with an imaginary value of β_φ. The cases (b) and (c) take place only for $A > 0$, i.e., positive energy density (see (3.24), (3.25)).

In the case $\mathcal{E} < 0$ we have (see (3.36), (3.37)) imaginary β^k:

$$\beta^k = i\hat{\beta}^k, \qquad k = 1, \ldots, n, \qquad (3.126)$$

and real β_φ. The solution is obtained from (3.51)-(3.55) by substituting (3.126) and $\tau/T \mapsto \tau/T + i\frac{\pi}{2}$:

$$g = -\Big[\prod_{i=1}^n (a_i(\tau))^{2N_i - u_i}\Big]\, d\tau \otimes d\tau + \sum_{i=1}^n a_i^2(\tau) g^{(i)}, \qquad (3.127)$$

$$a_i(\tau) = \hat{A}_i[\cosh(\tau/T)]^{\sigma^i}[f(\tau/2T)]^{\hat{\beta}^i}, \qquad (3.128)$$

$$\varphi(\tau) = c + 2i\beta_\varphi \arctan\exp(-\tau/T) \qquad (3.129)$$

where $c, \hat{A}_i \neq 0$ are constants, $i = 1, \ldots, n$, T is defined in (3.32), σ^i are given in (3.41), $A > 0$ and the real parameters $\hat{\beta}^i, \beta_\varphi$ satisfy the relations

$$\sum_{i=1}^n u_i\hat{\beta}^i = 0,$$

$$-\sum_{i,j=1}^n G_{ij}\hat{\beta}^i\hat{\beta}^j + (\beta_\varphi)^2 = -\frac{4}{<u,u>_*} = \frac{1}{q^2} \qquad (3.130)$$

Here, as in [48],

$$f(x) \equiv [\tanh(x + \frac{i\pi}{4})]^i = \exp(-2\arctan e^{-2x}) \qquad (3.131)$$

is a smooth monotonically increasing function bounded by its asymptotics: $e^{-\pi} < f(x) < 1$; $f(x) \to 1$ as $x \to +\infty$ and $f(x) \to e^{-\pi}$ as $x \to -\infty$. The solution (3.127)-(3.130) (with ρ from (3.54)) may be also obtained from formulas (3.20), (3.24). The relation between the harmonic time and the τ variable (3.33) for $\mathcal{E} < 0$ is modified:

$$\cosh(\tau/T) = 1/\sin(q\sqrt{2|\mathcal{E}|}(t-t_0)). \qquad (3.132)$$

For the quasi-volume scale factor we have

$$v = \prod_{i=1}^{n} a_i^{u_i/2} = \Bigl(\prod_{i=1}^{n} \hat{A}_i^{u_i/2}\Bigr)\cosh(\tau/T). \qquad (3.133)$$

The scalar field $\varphi(t)$ varies from $c + i\pi\beta_\varphi$ to c as τ varies from $-\infty$ to $+\infty$. The solution (3.127)-(3.130) for $\tau \in (-\infty, +\infty)$ is nonsingular. Any scale factor $a_i(\tau)$ for some τ_{0i} has a minimum and

$$a_i(\tau) \sim A_i^{\pm} \exp(\sigma^i|\tau|/T), \qquad (3.134)$$

for $\tau \to \pm\infty$.

The above "Lorentzian" solutions have "Euclidean" analogues for $A < 0$ as well:

$$g = \Bigl(\prod_{i=1}^{n}(a_i(\tau))^{2N_i - u_i}\Bigr) d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \qquad (3.135)$$

$$a_i(\tau) = \hat{A}_i[\cosh(\tau/T)]^{\sigma^i}[f(\tau/2T)]^{\hat{\beta}^i}, \qquad (3.136)$$

$$\varphi(\tau) = c + 2i\beta_\varphi \arctan\exp(-\tau/T), \qquad (3.137)$$

with the parameters $\hat{\beta}^i, \beta_\varphi$ satisfying the relations (3.130). When all spaces $(M_i, g^{(i)})$ are Riemannian, this solution may be interpreted as a classical Euclidean wormhole (instanton) solution. Such solutions play a crucial role in quantum gravity.

An interesting special case of the solution (3.135)-(3.137) occurs for $\hat{\beta}^i = 0$, $i = 1,\ldots,n$, (this corresponds to $p^{\hat{a}} = 0$):

$$a_i(\tau) = \hat{A}_i[\cosh(\tau/T)]^{\sigma^i}, \qquad (3.138)$$

$$\varphi(\tau) = c \pm 2iq^{-1}\arctan\exp(-\tau/T). \qquad (3.139)$$

All scale factors (3.138) have a minimum at $\tau = 0$ and are symmetric with respect to time reversion: $\tau \mapsto -\tau$. It is necessary to stress that here, as in [48], wormhole solutions exist only in the presence of an imaginary scalar field.

3.5. Some examples

In this subsection we consider some applications of the above formulas, valid for different equations of state in different factor spaces.

3.5.1. The isotropic case

Consider the isotropic case:

$$u_i = hN_i \Longleftrightarrow u = \frac{h}{2}u^{(\Lambda)}, \qquad p_i = (1-h)\rho, \qquad (3.140)$$

where $h \neq 0$ is constant. From (3.98)-(3.100) and (3.140) it follows

$$u^i = \frac{h}{2-D}, \qquad < u, u >_* = -h^2\frac{D-1}{D-2} < 0, \qquad (3.141)$$

and hence

$$\sigma^i = 2u^i/ < u, u >_* = \frac{2}{h(D-1)} = \sigma(h) = \sigma. \qquad (3.142)$$

The solution (3.51)-(3.55) reads:

$$g = -\Bigl(\prod_{i=1}^{n}(a_i(\tau))^{(2-h)N_i}\Bigr) d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)}, \qquad (3.143)$$

$$a_i(\tau) = A_i[\sinh(r\tau/T)/r]^{\sigma(h)}[\tanh(r\tau/2T)/r]^{\beta^i}, \qquad (3.144)$$

$$e^{\kappa\varphi(\tau)} = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi}, \qquad (3.145)$$

$$\kappa^2\rho(\tau) = A\prod_{i=1}^{n}(a_i(\tau))^{(h-2)N_i} = A\Bigl(\prod_{i=1}^{n} A_i^{(h-2)N_i}\Bigr)[\sinh(r\tau/T)/r]^{2(h-2)/h}, \qquad (3.146)$$

$i = 1,\ldots,n$, where $r = \sqrt{A/|A|}$ and

$$T = |h|^{-1}\left[\frac{|A|(D-1)}{2(D-2)}\right]^{-1/2}, \qquad (3.147)$$

$A_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\sum_{i=1}^{n} N_i\beta^i = 0, \qquad \sum_{i=1}^{n} N_i(\beta^i)^2 + (\beta_\varphi)^2 = \frac{4(D-2)}{h^2(D-1)}. \qquad (3.148)$$

A special case of this solution with $h = 2$ (the Λ-term case) was considered in [48].

Consider now the exceptional solutions for $A > 0$. From (3.72) and (3.141) we have

$$\bar{\sigma} = \sigma^i N_i = 2/h, \qquad < u^{(\Lambda)} - u, u >_* = h(h-2)\frac{D-1}{D-2}. \qquad (3.149)$$

From (3.149) we get: $< u^{(\Lambda)} - u, u >_* = 0 \Longleftrightarrow h = 2$ ($h \neq 0$). The matter in this case corresponds to a cosmological constant: $\Lambda = \kappa^2\rho > 0$. Eqs. (3.86), (3.141) imply the solution of [46, 48] with metric (3.83) and

$$a_i(t_s) = \bar{A}_i \exp\left[\pm\frac{t_s\sqrt{2\Lambda}}{\sqrt{(D-1)(D-2)}}\right] \qquad (3.150)$$

($\varphi = \text{const}$), i.e., we here obtain a case of exponential inflation.

For $h \neq 2$ ($\iff$ $<u^{(\Lambda)} - u, u>_* \neq 0$)

$$\nu^i = \frac{2u^i}{<u^{(\Lambda)} - u, u>_*} = \frac{2}{(2-h)(D-1)} = \nu(h) = \nu. \tag{3.151}$$

From (3.77), (3.82), (3.141), (3.149) and (3.151) we obtain the relations for scale factors and the density:

$$a_i(t_s) = A_i t_s^{\nu(h)}, \tag{3.152}$$

$$\kappa^2 \rho(t_s) = \frac{2(D-2)}{(h-2)^2 (D-1) t_s^2}, \tag{3.153}$$

i.e., power law inflation. For $h < 2$ (or $p > -\rho$) we have an isotropic expansion of all scale factors and for $h > 2$ (or $p < -\rho$) an isotropic contraction (see Fig.1).

Kasner-like behaviour.

In the above case $\sum_{i=1}^n u_i = h(D-1)$ and hence (see (3.102)-(3.105)) a Kasner-like behaviour (3.106), (3.108), (3.109) takes place as (A) $t_s \to +0$, for $h > 0$ (or $p < \rho$), and (B) $t_s \to +\infty$, for $h < 0$ (or $p > \rho$).

Isotropization-like behaviour

Using the results of Subsec.3.3 and (3.149), we are led to the following attractor behaviour:

$$a_i(t_s) \sim A_i t_s^{\nu(h)}, \tag{3.154}$$

$$\kappa^2 \rho(t_s) \sim \frac{2(D-2)}{(h-2)^2 (D-1) t_s^2}. \tag{3.155}$$

in the limits $t_s \to +\infty$, for $0 < h < 2$ (or $-\rho < p < \rho$) and $t_s \to +0$, for $h < 0$ (or $p > \rho$).

Remark 2. For the dust matter case $h = 1$ ($p = 0$), $\rho > 0$, the solution (3.143)-(3.148) has the sinchronous-time representation

$$g = -dt_s \otimes dt_s + \sum_{i=1}^n a_i^2(t_s) g^{(i)},$$

$$a_i(t_s) = \bar{A}_i t_s^{1/(D-1)+\beta^i/2} (t_s + T_1)^{1/(D-1)-\beta^i/2}, \tag{3.156}$$

$$e^{2\kappa\varphi(t_s)} = A_\varphi [t_s/(t_s + T_1)]^{\beta_\varphi}, \tag{3.157}$$

$$\kappa^2 \rho(t_s) = 2(D-2)/[(D-1) t_s (t_s + T_1)], \tag{3.158}$$

$i = 1, \ldots, n$; where $0 < t_s < +\infty$, $T_1 > 0$, $\bar{A}_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\sum_{i=1}^n N_i \beta^i = 0, \qquad \sum_{i=1}^n N_i (\beta^i)^2 + (\beta_\varphi)^2 = \frac{4(D-2)}{(D-1)}. \tag{3.159}$$

A special case of this solution with $\beta_\varphi = 0$ was considered previously in [30] (for $n = 2$ and $N_1 = \ldots = N_n = 1$ see [17] and [18] respectively.)

3.5.2. A curvature-like fluid component

Consider the perfect fluid matter with

$$u_i = 2h(-\delta_i^1 + N_i) = h u_i^{(1)} \tag{3.160}$$

where $h \neq 0$ is constant and $N_1 > 1$. For $h = 1$ this component corresponds to a nonzero curvature term in the first space [31] (see below). A calculation gives

$$u^i = -\frac{2h}{N_1} \delta_1^i, \qquad < u, u >_* = -4h^2 b_1 < 0, \tag{3.161}$$

where $b_1 = 1 - \frac{1}{N_1}$ and

$$< u, u^{(\Lambda)} >_* = 2u^i N_i = -4h,$$

$$< u, u^{(\Lambda)} - u >_* = 4h(-1 + h b_1),$$

$$\sigma^i = \frac{\delta_1^i}{h(N_1 - 1)}. \tag{3.162}$$

Using (3.160) and (3.162) we get from (3.51)-(3.55):

$$g = -(a_1(\tau))^{2h} \Big(\prod_{i=1}^n (a_i(\tau))^{2N_i(1-h)} \Big) d\tau \otimes d\tau + \sum_{i=1}^n a_i^2(\tau) g^{(i)}, \tag{3.163}$$

$$a_1(\tau) = A_1 [\sinh(r\tau/T)/r]^{\frac{1}{h(N_1-1)}} [\tanh(r\tau/2T)/r]^{\beta^1}, \tag{3.164}$$

$$a_i(\tau) = A_i [\tanh(r\tau/2T)/r]^{\beta^i}, \quad i > 1, \tag{3.165}$$

$$e^{\kappa\varphi(\tau)} = A_\varphi [\tanh(r\tau/2T)/r]^{\beta_\varphi}, \tag{3.166}$$

$$\kappa^2 \rho(\tau) = A (a_1(\tau))^{-2h} \prod_{i=1}^n (a_i(\tau))^{2N_i(h-1)}, \tag{3.167}$$

$i = 1, \ldots, n$, where $r = \sqrt{A/|A|}$, $T = |h|^{-1} (2|A| b_1)^{-\frac{1}{2}}$, $A_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\beta^1 = \frac{1}{1 - N_1} \sum_{i=2}^n N_i \beta^i,$$

$$\Big(\sum_{i=2}^n N_i \beta^i \Big)^2 + (N_1 - 1) \Big[\sum_{i=2}^n N_i (\beta^i)^2 + (\beta_\varphi)^2 \Big] = N_1 h^{-2}. \tag{3.168}$$

For $h \neq h_0 \equiv b_1^{-1} = N_1/(N_1 - 1) > 1$ we have from (3.162) $< u, u^{(\Lambda)} - u >_* \neq 0$ and (see (3.78))

$$\nu^i = \delta_1^i \nu(h), \qquad \nu(h) = [N_1 + h(1 - N_1)]^{-1}. \tag{3.169}$$

The power-law inflationary solution for this case is

$$g = -dt_s \otimes dt_s + A_1^2 t_s^{2\nu(h)} g^{(1)} + \sum_{i=2}^n A_i^2 g^{(i)}, \tag{3.170}$$

$$\varphi = \text{const}, \tag{3.171}$$

$$\kappa^2 \rho(t_s) = \frac{b_1}{2(-1 + h b_1)^2 t_s^2}. \tag{3.172}$$

The internal space scale factors in this solution are constant (the so-called "spontaneous compactification"). It is easily shown that the constancy of internal scale factors leads to the equation of state (3.160).

Using the relation $\bar{\sigma} = h_0/h$ and the analysis carried out in subsection 3.3 we obtain that the solution (3.169)-(3.172) is an attractor for non-exceptional solutions with $\rho > 0$ as $t_s \to T_* - 0$, for $h > h_0$; $t_s \to +\infty$, for $0 < h < h_0$ and $t_s \to +0$, for $h < 0$. Thus we have also obtained solutions with the "dynamical compactification".

The 1-curvature case.

Here we apply the relations obtained to a cosmological model described by the action

$$S = \int d^D x \sqrt{|g|} \{R[g] - \partial_M \varphi \partial_N \varphi g^{MN}\} \qquad (3.173)$$

with a scalar field $\varphi = \varphi(t)$ and metric (2.1) defined on the manifold (2.2), where $(M_i, g^{(i)})$, $i = 2, \ldots, n$, are Ricci-flat spaces and $(M_1, g^{(1)})$ is an Einstein space of nonzero curvature, i.e. $R_{mn}[g^{(1)}] = \lambda^1 g^{(1)}_{mn}, \lambda^1 \neq 0$. Here $n \geq 2$ and $N_i = \dim M_i$. This "1-curvature model" is equivalent to a special case of the above model (3.160) with $h = 1$ and $A = -\frac{1}{2}\lambda^1 N_1$ (see [39]). The solution (3.163)-(3.168) reads for this case:

$$g = (a_1(\tau))^2 [-d\tau \otimes d\tau + g^{(1)}] + \sum_{i=2}^{n} a_i^2(\tau) g^{(i)}, \qquad (3.174)$$

$$a_1(\tau) = A_1 [\sinh(r\tau/T)/r]^{\frac{1}{(N_1-1)}} [\tanh(r\tau/2T)/r]^{\beta^1}, \qquad (3.175)$$

$$a_i(\tau) = A_i [\tanh(r\tau/2T)/r]^{\beta^i}, \ i > 1, \qquad (3.176)$$

$$e^{\kappa\varphi(\tau)} = A_\varphi [\tanh(r\tau/2T)/r]^{\beta_\varphi}, \qquad (3.177)$$

$$\kappa^2 \rho(\tau) = A (a_1(\tau))^{-2} \qquad (3.178)$$

$i = 1, \ldots, n$, where $r = \sqrt{-\lambda^1/|\lambda^1|}$, $T = [|\lambda^1|(N_1 - 1)]^{-1/2}$, $A_i, A_\varphi > 0$ are constants and the parameters β^i, β_φ satisfy the relations

$$\beta^1 = \frac{1}{1 - N_1} \sum_{i=2}^{n} N_i \beta^i,$$

$$\frac{1}{N_1 - 1} \Big(\sum_{i=2}^{n} N_i \beta^i \Big)^2 + \sum_{i=2}^{n} N_i (\beta^i)^2 + (\beta_\varphi)^2 = \frac{N_1}{N_1 - 1}. \qquad (3.179)$$

The power-law inflationary solution for the negative curvature case $\lambda^1 < 0$ reads:

$$g = -dt_s \otimes dt_s + A_1^2 t_s^2 g^{(1)} + \sum_{i=2}^{n} A_i^2 g^{(i)}, \qquad (3.180)$$

$$\varphi = \mathrm{const}, \qquad (3.181)$$

where $A_1^2 = |\lambda^1|/(N_1 - 1)$ (see (3.81), (3.172)). We are led here to the Milne-type solution recently considered in [51].

There is another parametrization of the solution (3.174)- (3.179) in terms of an R variable related to τ-variable as

$$F = F(R) = 1 - \Big(\frac{R_0}{R}\Big)^{N_1 - 1} = \tanh^2 \frac{\tau}{2T}, \quad \lambda^1 < 0, \qquad (3.182)$$

$$= \Big(\frac{R_0}{R}\Big)^{N_1 - 1} - 1 = \tan^2 \frac{\tau}{2T}, \quad \lambda^1 > 0. \qquad (3.183)$$

Here $R > R_0$ for $\lambda^1 < 0$ and $R < R_0$ for $\lambda^1 > 0$; $R_0 = A_1 2^{1/(N_1-1)} \sqrt{(N_1 - 1)/|\lambda^1|}$. In new variables the metric and the scalar field may be written as

$$g = -F^{b-1} dR \otimes dR + F^b R^2 A_1^2 g^{(1)} + \sum_{i=2}^{n} F^{\beta^i} A_i^2 g^{(i)}, \qquad (3.184)$$

$$e^{2\kappa\varphi} = A_\varphi^2 F^{\beta_\varphi}, \qquad (3.185)$$

$A_1^2 = |\lambda^1|/(N_1 - 1)$, $A_i, A_\varphi > 0$ are constants and

$$b = (1 - \sum_{i=2}^{n} N_i \beta^i)/(N_1 - 1) = (N_1 - 1)^{-1} + \beta^1, \qquad (3.186)$$

and the parameters $\beta^i (i > 1), \beta_\varphi$ satisfy the relations (3.179). A special case of the solution (3.179), (3.184)-(3.186) with $\beta_\varphi = 0$ (a constant scalar field) was obtained earlier in [39].

Remark 3. As a special case of the above solution, we get a scalar-vacuum analog of the spherically-symmetric Tangherlini solution [81] with n Ricci-flat internal spaces:

$$g = -f^a dt \otimes dt + f^{b-1} dR \otimes dR + f^b R^2 d\Omega_d^2 + \sum_{i=1}^{n} f^{a_i} B_i g^{(i)}, \qquad (3.187)$$

$$e^{2\kappa\varphi} = B_\varphi f^{a_\varphi}, \qquad (3.188)$$

where $d\Omega_d^2$ is the canonical metric on a d-dimensional sphere S^d ($d \geq 2$), $f = f(R) = 1 - BR^{1-d}$; $B_\varphi, B_i > 0, B$ are constants and the parameters $a, a_1, \ldots, a_n$ satisfy the relations

$$b = (1 - a - \sum_{i=1}^{n} a_i N_i)/(d - 1), \qquad (3.189)$$

$$(a + \sum_{i=1}^{n} a_i N_i)^2 + (d - 1)(a^2 + a_\varphi^2 + \sum_{i=1}^{n} a_i^2 N_i) = d. \qquad (3.190)$$

For $a_\varphi = 0$ see also [61, 65]. In the parametrization of the harmonic-type variable this solution was presented earlier in [63, 65].

Thus, using the above transformations, we can obtain spherically symmetric solutions from cosmological ones.

4. Wheeler-DeWitt equation

Now, having studied the classical multidimensional solutions, we start an investigation of their quantum analogs. As usual, quantization of the zero-energy constraint (3.17) leads to the Wheeler-DeWitt (WDW) equation in the harmonic time gauge (3.3) [31, 47, 48]

$$2\hat{H}\Psi \equiv \left[\frac{\partial}{\partial z^0}\frac{\partial}{\partial z^0} - \sum_{i=1}^{n}\frac{\partial}{\partial z^i}\frac{\partial}{\partial z^i} + 2A\mathrm{e}^{2qz^0}\right]\Psi = 0. \tag{4.1}$$

We are seeking a solution to (4.1) in the form

$$\Psi(z) = \exp(i\vec{p}\vec{z})\Phi(z^0), \tag{4.2}$$

where $\vec{p} = (p^1,\ldots,p^n)$ is a constant vector (generally from C^n), $\vec{z} = (z^1,\ldots,z^{n-1},\ z^n = \kappa\varphi)$, $\vec{p}\vec{z} \equiv \sum_{i=1}^{n} p_i z^i$. Substitution of (4.2) into (4.1) gives

$$\left[-\frac{1}{2}(\frac{\partial}{\partial z^0})^2 + V_0(z^0)\right]\Phi = \mathcal{E}\Phi, \tag{4.3}$$

where $\mathcal{E} = \frac{1}{2}\vec{p}\vec{p}$ and $V_0(z^0) = -A\mathrm{e}^{2qz^0}$. Solving (4.3), we get

$$\Phi(z^0) = B_{i\sqrt{2\mathcal{E}}/q}(\sqrt{-2A}q^{-1}\mathrm{e}^{qz^0}) \tag{4.4}$$

where $i\sqrt{2\mathcal{E}}/q = i|\vec{p}|/q$, and $B = I, K$ are modified Bessel functions. Note that

$$v = \exp(qz^0) = \prod_{i=1}^{n} a_i^{u_i/2} \tag{4.5}$$

is the "quasivolume" (3.47) (see (3.12)).

The general solution of Eq.(4.1) has the following form:

$$\Psi(z) \ = \sum_{B=I,K}\int d^n\vec{p}\; C_B(\vec{p})e^{i\vec{p}\vec{z}}B_{i|\vec{p}|/q}(\frac{\sqrt{-2A}}{q}\mathrm{e}^{qz^0}) \tag{4.6}$$

where the functions C_B ($B = I, K$) belong to an appropriate class. For the Λ-term case this solution was considered in [46, 48] and for the two-component model ($n = 2$) and $\Lambda > 0$ in [95].

In the ground state we put all momenta $p^a (a = 1,\ldots,n)$ equal to zero, and the ground state wave function reads:

$$\Psi_0 = B_0\left(\sqrt{-2A}q^{-1}\mathrm{e}^{qz^0}\right). \tag{4.7}$$

It is to be stressed that the function Ψ_0 is invariant with respect to the rotation group O(n).

Remark 4. Applying the arguments of [40, 48], one can show that the ground state wave function

$$\Psi_0^{(HH)} \ = I_0\left(\frac{\sqrt{2|A|}}{q}\exp(qz^0)\right), \qquad A < 0, \tag{4.8}$$

$$= J_0\left(\frac{\sqrt{2A}}{q}\exp(qz^0)\right), \qquad A > 0, \tag{4.9}$$

satisfies the Hartle-Hawking boundary condition [97]. Special cases of this formula were considered in Refs. [40] (the 1-curvature case) and [48] (the Λ-term case).

From (4.3) it follows that in the case $A < 0$ (negative energy density) a Lorentzian domain exists as well as a Euclidean one for $\mathcal{E} > 0$. In the case $A > 0$ only the Lorentzian domain occurs for $\mathcal{E} \geq 0$ but for $\mathcal{E} < 0$ both domains exist. The wave functions (4.2), (4.4) with $A > 0$ and $\mathcal{E} < 0$ describe transitions between the Euclidean and Lorentzian domains, i.e. tunneling universes.

4.1. Quantum wormholes

We consider only real values of p_i. In this case we have $\mathcal{E} \geq 0$.

If $A > 0$, the wave function Ψ (4.2) is not exponentially damped when $v \to \infty$, i.e. the condition (i) for quantum wormholes (see the Introduction) is not satisfied. The wave function oscillates and may be interpreted as corresponding to the classical Lorentzian solution.

For $A < 0$, the wave function (4.2) is exponentially damped for large v only, when $B = K$ in (4.4). (Recall that

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \qquad K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z},$$

for $z \to \infty$). However, in this case the function Φ oscillates infinitely many times when $v \to 0$. Thus the condition (ii) is not satisfied. The wave function describes a transition between the Lorentzian and Euclidean domains.

The functions

$$\Psi_{\vec{p}}(z) = e^{i\vec{p}\vec{z}}K_{i|\vec{p}|/q}(\sqrt{-2A}q^{-1}e^{qz^0}), \tag{4.10}$$

may be used for constructing quantum wormhole solutions. As in [84, 85, 46, 48], we consider superpositions of the singular solutions

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dk\Psi_{qk\vec{n}}(z)e^{-ik\lambda}, \tag{4.11}$$

where $\lambda \in R$ and $\vec{n} \in S^{n-1}$ is a unit vector ($\vec{n}^2 = 1$). A calculation gives

$$\hat{\Psi}_{\lambda,\vec{n}}(z) = \exp\Big[-\frac{\sqrt{-2A}}{q}e^{qz^0}\cosh(\lambda - q\vec{z}\vec{n})\Big]. \tag{4.12}$$

It is easy to verify that Eq.(4.12) leads to solutions of the WDW equation (4.1) satisfying the quantum wormholes boundary conditions.

Note that the functions

$$\Psi_{m,\vec{n}} = H_m(x^0)H_m(x^1)\exp[-\frac{(x^0)^2 + (x^1)^2}{2}], \tag{4.13}$$

where

$$x^0 = (2/q)^{1/2}(-2A)^{1/4}\exp(qz^0/2)\cosh(\tfrac{1}{2}q\vec{z}\vec{n}), \quad (4.14)$$
$$x^1 = (2/q)^{1/2}(-2A)^{1/4}\exp(qz^0/2)\sinh(\tfrac{1}{2}q\vec{z}\vec{n}), \quad (4.15)$$

$m = 0, 1, \ldots,$ are also solutions to the WDW equation with the quantum wormhole boundary conditions. Solutions of such type were previously considered in [83, 41, 42, 46, 48]. They are called discrete spectrum quantum wormholes (see [85]) (and may form a basis in the Hilbert space of the system [86]).

Thus in the case considered quantum wormhole solutions (with respect to quasi-volume (4.5)) exist for matter with a negative density (3.2) ($A < 0$).

5. A third-quantized model

Another step in studying topological changes in quantum cosmology can be made in the so-called third quantization approach, where it is meant that the WDW equation is considered within a second-quantized scheme.

Here we put $A > 0$, i.e. the matter density is positive. Consider the case of a real Ψ-field as in [54] for simplicity. The WDW equation (4.1) corresponds to the action

$$S = \frac{1}{2}\int d^{n+1}z \Psi \hat{H} \Psi. \quad (5.1)$$

Consider two bases of solutions to the WDW equation, $\{\Psi_{\rm in}(\vec{p}), \Psi^*_{\rm in}(\vec{p})\}$ and $\{\Psi_{\rm out}(\vec{p}), \Psi^*_{\rm out}(\vec{p})\}$

$$\Psi_{\rm in}(\vec{p}) = \Psi_{\rm in}(\vec{p}, z) = \left[\frac{\pi}{2q\sinh(\pi|\vec{p}|/q)}\right]^{1/2} J_{-i|\vec{p}|/q}\left(\frac{\sqrt{2A}}{q}e^{qz^0}\right)\frac{e^{i\vec{p}\vec{z}}}{(2\pi)^{n/2}}; \quad (5.2)$$

$$\Psi_{\rm out}(\vec{p}) = \Psi_{\rm out}(\vec{p}, z) = \frac{1}{2}\left(\frac{\pi}{q}\right)^{1/2} H^{(2)}_{i|\vec{p}|/q}\left(\frac{\sqrt{2A}}{q}e^{qz^0}\right)\frac{e^{i\vec{p}\vec{z}}}{(2\pi)^{n/2}}. \quad (5.3)$$

where J_ν and $H^{(2)}_\nu$ are the Bessel and Hankel functions respectively. These solutions are normalized by the following conditions

$$(\Psi_{\rm in}(\vec{p}), \Psi_{\rm in}(\vec{p}\,')) = (\Psi_{\rm out}(\vec{p}), \Psi_{\rm out}(\vec{p}\,')) = \delta\left(\vec{p}-\vec{p}\,'\right) \quad (5.4)$$

where

$$(\Psi_1, \Psi_2) = i\int d^n\vec{z}\left(\Psi^*_1 \overleftrightarrow{\partial_0} \Psi_2\right) \quad (5.5)$$

is the charge form (indefinite scalar product). Here $\Psi_1 \overleftrightarrow{\partial} \Psi_2 = \Psi_1 \partial\Psi_2 - (\partial\Psi_1)\Psi_2$. Due to the asymptotic behaviour

$$\Psi_{\rm in}(\vec{p}, z) \sim c_{\rm in}(|\vec{p}|)\exp(i\vec{p}\vec{z} - i|\vec{p}|z^0), \quad v \to 0, \quad (5.6)$$

$$\Psi_{\rm out}(\vec{p}, z) \sim \frac{c_{\rm out}(|\vec{p}|)}{\sqrt{v}}\exp(i\vec{p}\vec{z} - i\frac{\sqrt{2A}}{q}v), \quad v \to +\infty. \quad (5.7)$$

where $\Psi_{\rm in}(\vec{p}, z)$ and $\Psi_{\rm out}(\vec{p}, z)$ are negative-frequency modes of "Kasner"- and "Milne"- types respectively.

The standard quantization procedure [88, 89] gives us

$$\Psi(z) = \int d^n\vec{p}\left[a^+_{\rm in}(\vec{p})\Psi^*_{\rm in}(\vec{p}, z) + a_{\rm in}(\vec{p})\Psi_{\rm in}(\vec{p}, z)\right] = \int d^n\vec{p}\left[a^+_{\rm out}(\vec{p})\Psi^*_{\rm out}(\vec{p}, z) + a_{\rm out}(\vec{p})\Psi_{\rm out}(\vec{p}, z)\right], \quad (5.8)$$

where the non-trivial commutators are

$$[a_{\rm in}(\vec{p}), a^+_{\rm in}(\vec{p}\,')] = [a_{\rm out}(\vec{p}), a^+_{\rm out}(\vec{p}\,')] = \delta\left(\vec{p} - \vec{p}\,'\right). \quad (5.9)$$

The "in" and "out" vacuum states satisfy the relations

$$a_{\rm in}(\vec{p})|0, {\rm in}>= a_{\rm out}(\vec{p})|0, {\rm out}>= 0. \quad (5.10)$$

The modes (5.2) and (5.3) are related by the Bogoliubov transformation

$$\Psi_{\rm in}(\vec{p}) = \alpha(\vec{p})\Psi_{\rm out}(\vec{p}) + \beta(\vec{p})\Psi^*_{\rm out}(\vec{p}), \quad (5.11)$$

$$\alpha(\vec{p}) = \left[\frac{\exp(\pi|\vec{p}|/q)}{2\sinh(\pi|\vec{p}|/q)}\right]^{1/2},$$
$$\beta(\vec{p}) = \left[\frac{\exp(-\pi|\vec{p}|/q)}{2\sinh(\pi|\vec{p}|/q)}\right]^{1/2}. \quad (5.12)$$

The vacua $|0, {\rm in}>$ and $|0, {\rm out}>$ are unitarily inequivalent. A standard calculation [88, 89] gives for the number density of "out-Universes" (of "Milne type") contained in the "in-vacuum" ("Kasner-type" vacuum)

$$n(\vec{p}) = |\beta(\vec{p})|^2 = \left(\exp(2\pi|\vec{p}|/q) - 1\right)^{-1}. \quad (5.13)$$

Thus we obtain a Planck distribution of created universes with the temperature

$$T_{\rm Pl} = q/2\pi = \sqrt{- < u, u >_*}/4\pi. \quad (5.14)$$

The temperature (5.14) depends on the vector $u = (u_i)$ (i.e., on the equation of state of the matter content of the Universe): $T_{\rm Pl} = T_{\rm Pl}(u)$. For example, we get $T_{\rm Pl}(u^{(\Lambda)}) = 2T_{\rm Pl}(u^{({\rm dust})})$. In the Zeldovich matter limit $u \to 0$ we have $T_{\rm Pl} \to +0$.

Remark 5. In [98] a regularization of propagators (in quantum field theory) was introduced using the complex signature matrix

$$(\eta_{ab}(w)) = {\rm diag}\,(w, 1, \ldots, 1), \quad (5.15)$$

where $w \in C \setminus (-\infty, 0]$ is a complex parameter (Wick parameter). Path integrals are originally defined (in covariant manner) for $w > 0$ (i.e. in Euclidean-like region) and then analytically continued to negative w. The Minkowsky space limit corresponds to $w = -1 + i0$ (in notations of [98] $w^{-1} = -a$). The prescription [98] is a natural realization of Wick's rotation. In [99] analogs of the Bogoliubov-Parasiuk theorems [87] for a wide class of propagators regularized by the complex metric (5.15) were proved. This formalism may be applied for third-quantized models of the multidimensional cosmology. In this case the corresponding path integrals should be analytically continued from the interval $1 < D < 2$ (D is the dimension), where the minisuperspace metric (2.12) is Euclidean, to $D = D_0 - i0$, $D_0 = 1 + \sum_{i=1}^{n} N_i$. We note also that recently J.Greensite proposed the idea of treating the space-time signature as a dynamical degree of freedom [100] (see also [101, 102]).

6. Appendix

Proof of Proposition 2. We introduce the new "diagonalized" variables

$$\beta^a = e^a_i \beta^i, \qquad u_a = e^i_a u_i, \qquad v_a = e^i_a v_i \tag{6.1}$$

where matrices (e^i_a), (e^a_i) satisfy the relations (3.11)-(3.12) and (3.14). From (3.12)-(3.14) we have

$$(u_a) = (2q, \vec{0}) \qquad (\sigma^a) = (\sigma^i e^a_i) = (q^{-1}, \vec{0}) \tag{6.2}$$

and consequently (see (3.96))

$$0 = \beta^i u_i = \beta^a u_a = 2q\beta^0 \ \Rightarrow \ (\beta^a) = (0, \vec{\beta}). \tag{6.3}$$

From the second relation in (3.96) we get

$$G_{ij}\beta^i\beta^j = \eta_{ab}\beta^a\beta^b = \vec{\beta}^2 \leq 1/q^2. \tag{6.4}$$

For the vector $(v_a) = (v_0, \vec{v})$ we have $-v_0^2 + \vec{v}^2 = <v, v>_* < 0$ and hence

$$|v_0| > |\vec{v}|, \qquad v_0 \neq 0. \tag{6.5}$$

We also obtain from (6.2) and (6.5)

$$< u, v >_* = -u_0 v_0 = -2qv_0 \neq 0. \tag{6.6}$$

Using relations (6.2), (6.3) and (6.5) we get

$$\begin{aligned}(\sigma^i + \beta^i)v_i &= (\sigma^a + \beta^a)v_a \\ &= q^{-1}v_0 + \vec{\beta}\vec{v} = q^{-1}v_0(1 + \frac{q}{v_0}\vec{\beta}\vec{v}).\end{aligned} \tag{6.7}$$

Eqs. (6.4), (6.5) imply the inequality

$$|\frac{q}{v_0}\vec{\beta}\vec{v}| \leq \frac{|\vec{v}|}{v_0} q|\vec{\beta}| \leq \frac{|\vec{v}|}{v_0} < 1. \tag{6.8}$$

From (6.6)-(6.8) (and $q > 0$) we obtain the proposed identity (3.97). The proposition is proved.

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