Gravitation & Cosmology, Vol. 1 (1995), No. 2, pp. 155–159
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VACUUM STATIC, AXIALLY SYMMETRIC FIELDS IN D-DIMENSIONAL GRAVITY

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Received 20 April 1995

Vacuum static, axially symmetric space-times in D-dimensional general relativity with a Ricci-flat internal space are discussed. It is shown, in particular, that some of the monopole-type solutions are free of curvature singularities and their source can be a disk membrane bounded by a ring with a string or branching type singularity. Another possibility is a wormhole configuration where a particle can penetrate to another spatial infinity by passing through a ring with a string or branching type singularity. The results apply, in particular, to vacuum and scalar-vacuum configurations in conventional general relativity.

1. Introduction

Spherically symmetric static solutions of multidimensional gravity have been considered by many authors with a goal to study possible observational windows to extra dimensions [1–3]. Among such windows one can name possible variations of fundamental physical constants [4], deviations from Newton's and Coulomb's laws and modified properties of black holes and gravitational radiation as compared with the conventional theory.

Another class of multidimensional models, to be discussed in this paper, is the class of axially symmetric models, including spherically symmetric ones as a special case.

Although static, axially symmetric (SAS) configurations are a less popular object of gravitational studies than stationary ones (used for describing fields due to rotating bodies), their properties are of much interest as well. In many papers such solutions are sought and studied, see, for instance, [5, 6, 7] and references therein. We will study monopole SAS vacuum configurations in multidimensional gravity and find some features of interest, in particular, membrane and string type sources of fields possessing no curvature singularities.

We consider D-dimensional general relativity and start from the action

$$S=\int d^Dx\sqrt{{}^Dg}({}^DR+L_m) \tag{1}$$

where L_m is a matter Lagrangian, in a space with the metric

$$ds_D^2=g_{\mu\nu}dx^\mu dx^\nu+\mathrm{e}^{2\beta_1}ds_1^2 \tag{2}$$

where Greek indices range from 0 to 3 and $\beta_1(x^\mu)$ is a scale factor of an internal N-dimensional space with a Ricci-flat ds_1^2 independent of x^μ.

In a 4-dimensional formulation

$$S=\int d^4x\sqrt{{}^4g}\mathrm{e}^\sigma\Big[R-\Big(\frac{1}{N}-1\Big)\sigma^\alpha\sigma_\alpha+L_m\Big] \tag{3}$$

where $\sigma=N\beta_1$ and R is the 4-curvature corresponding to $g_{\mu\nu}$.

There are other 4-dimensional formulations of the theory, connected with (3) by conformal mappings (conformal gauges). The gauge (3) corresponds to the original theory. The so-called Einstein gauge, obtained from (3) by the conformal mapping

$$\overline{g}_{\mu\nu}=\mathrm{e}^\sigma g_{\mu\nu}, \tag{4}$$

is more convenient for solving the field equations since the curvature enters into the Lagrangian with a constant factor:

$$S=\int d^4x\sqrt{{}^4\overline{g}}\,[\overline{R}+\alpha_0\overline{g}^{\alpha\beta}\sigma_\alpha\sigma_\beta+\mathrm{e}^{-\sigma}L_m],\qquad \alpha_0=1/2+1/N. \tag{5}$$

where $\overline{R}$ is the scalar curvature corresponding to $\overline{g}_{\mu\nu}$. Another important gauge, the so-called atomic one, in which a test particle moves along geodesics, is defined by

$$g^*_{\mu\nu}=\mathrm{e}^{\sigma/2}g_{\mu\nu} \tag{6}$$

and is most suitable for interpretaion of measurements, e.g., in the Solar system. However, for studies of singularities and topology the original metric $g_{\mu\nu}$ must

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be used. For more detailed discussion of the notion of systems of measurement, closely connected with that of conformal gauges, see Ref.[4] and, as applied to multidimensional theory, Refs.[8, 9, 10].

In what follows we use the Einstein gauge to find the metric (4) for vacuum SAS configurations. So we start from the equations due to (5) with $L_m = 0$:

$$\overline{R}_{\mu\nu} = -\alpha_0 \sigma_{,\mu}\sigma_{,\nu}, \qquad (7)$$

$$\Box\sigma = 0 \qquad (8)$$

where $\overline{R}_{\mu\nu}$ and $\Box$ are the Ricci tensor and the D'Alembert operator corresponding to $\overline{g}_{\mu\nu}$.

Vacuum D-dimensional equations are thus reduced to scalar-vacuum ones in 4 dimensions. Although such SAS configurations were repeatedly considered [6, 18], it makes sense to return to them to reveal some new features, in particular, those connected with higher dimensions.

2. Field equations for axial symmetry

The SAS 4-metric in the Einstein gauge (4) may be written in the Weyl canonical form [7]

$$d\overline{s}^2 = e^{2\nu}dt^2 - e^{-2\nu}[e^{2\beta}(d\rho^2 + dz^2) + \rho^2 d\phi^2] \qquad (9)$$

The field equations then can be written as

$$\Delta\sigma = 0, \qquad (10)$$

$$\Delta\nu = 0, \qquad (11)$$

$$\beta_z = \rho(2\nu_\rho\nu_z + \alpha_0\sigma_\rho\sigma_z) \qquad (12)$$

$$\beta_\rho = \rho[\nu_\rho^2 - \nu_z^2 + \tfrac{1}{2}\alpha_0(\sigma_\rho^2 - \sigma_z^2)] \qquad (13)$$

where the indices z and ρ denote the partial derivatives ∂_ρ and ∂_z, respectively, and Δ is the "flat" Laplace operator in the cylindrical coordinates:

$$\Delta = \rho^{-1}\partial_\rho(\rho\partial_\rho) + \partial_z\partial_z.$$

The integrability condition for (12) and (13) is satisfied automatically.

Following the example of [6], let us seek solutions in the new coordinates (x, y), connected with ρ and z by

$$\rho^2 = L^2(x^2+\varepsilon)(1-y^2), \qquad z = Lxy \qquad (14)$$

where L is a fixed positive constant and $\varepsilon = 0, \pm 1$, so that x and y are spherical ($\varepsilon = 0$), prolate spheroidal ($\varepsilon = -1$), or oblate spheroidal ($\varepsilon = +1$) coordinates, respectively. The Laplace operator Δ acquires the form

$$\Delta = \partial_x(x^2+\varepsilon)\partial_x + \partial_y(1-y^2)\partial_y. \qquad (15)$$

Separating the variables in Eq.(11), i.e., putting $\nu(x,y) = \chi(x)\psi(y)$, one obtains

$$[(x^2+\varepsilon)\chi_x]_x + \lambda\chi = 0, \qquad (16)$$

$$[(1-y^2)\psi_y]_y - \lambda\psi = 0 \qquad (17)$$

where λ is the separation constant. Solutions to (17) finite on the symmetry axis $\rho = 0$ are the Legendre polynomials $P_l(y)$, while $\lambda = l(l+1)$ with $l = 0, 1, 2, \ldots$. The corresponding solutions to (16) are combinations of Legendre functions of the first and second kinds.

Eq.(10) is solved in a similar way.

This is the way to obtain solutions of arbitrary multipolarity l or even superpositions of different multipolarities: after writing out the solutions to the linear equations (11) and (10), Eqs. (12) and (13) are integrable by quadratures. In what follows, however, we restrict ourselves to $l = 0$ (monopole solutions).

3. Monopole solutions

The monopole solution to Eq.(17) may without loss of generality be written in the form

$$e^\psi = [(1+y)/(1-y)]^{c_1}, \quad c_1 = \text{const}. \qquad (18)$$

Regularity at $y = \pm 1$ then requires $c_1 = 0$, so that $\nu = \nu(x)$. Eq.(16) takes the form $(x^2+\varepsilon)d\chi/dx = \text{const}$. Its integration leads to the following expressions for $\nu(x)$ satisfying the asymptotic flatness condition:

$$\nu = \begin{cases} -\frac{1}{2}b\ln\frac{x+1}{x-1}, & \varepsilon = -1, \\ -b/x, & \varepsilon = 0, \\ -b\cot^{-1}x, & \varepsilon = +1. \end{cases} \qquad (19)$$

In a similar way $\sigma(x)$ is found:

$$\sigma = \begin{cases} -\frac{1}{2}s\ln\frac{x+1}{x-1}, & \varepsilon = -1, \\ -s/x, & \varepsilon = 0, \\ -s\cot^{-1}x, & \varepsilon = +1. \end{cases} \qquad (20)$$

Integrating (12) and (13), one obtains the expressions for $\beta(x,y)$ satisfying the asymptotic flatness condition $\beta(\infty, y) = 0$

$$e^{2\beta} = \begin{cases} (x^2-1)^K(x^2-y^2)^{-K}, & \varepsilon = -1, \\ \exp[-K(1-y^2)/x^2], & \varepsilon = 0, \\ (x^2+y^2)^K(x^2+1)^{-K}, & \varepsilon = +1 \end{cases} \qquad (21)$$

with $K = \frac{1}{2}(2b^2 + \alpha_0 s^2) \geq 0$.

4. General properties of the solutions

The solutions have been found under the boundary condition providing regularity (local euclidity) at the symmetry axis $\rho = 0$, or $y = \pm 1$.

At spatial infinity the solutions are asymptotically spherically symmetric. Indeed, assuming $y = \cos\theta$ where θ is the conventional polar angle, the SAS line element (9) transformed by (14), is spherically symmetric under the condition

$$e^{2\beta} = (x^2+\varepsilon)/(x^2+\varepsilon y^2). \qquad (22)$$

The condition (22) holds for all the solutions in the limit $x \to \infty$ where they have Schwarzschild asymptotics. A particular expression for the Schwarzschild mass in terms of the integration constants is conformal gauge-dependent. Recalling that the mass is most meaningfully defined in the atomic gauge (6), one can write:

$$g^*_{tt} \approx 1 - 2GM/r, \qquad r \approx Lx;$$
$$GM = (b - s/4)L \tag{23}$$

As for the whole space, the condition (22) is fulfilled under the additional requirement

$$K\varepsilon = \tfrac{1}{2}(2b^2 + \alpha_0 s^2)\varepsilon = -1. \tag{24}$$

As b and s are real, this condition can hold only for $\varepsilon = -1$. Quite naturally, the solution with $\varepsilon = -1$ constrained by (24) coincides with the well-known generalized Schwarzschild solution [11] with the $(4+N)$-dimensional metric (2)

$$\begin{aligned} ds_D^2 \;=\; & \Big(1 - \frac{2k}{R}\Big)^{a_0} \\ & - \Big(1 - \frac{2k}{R}\Big)^{-a_0 - Na_1}\Big[dR^2 + R^2\Big(1 - \frac{2k}{R}\Big)d\Omega^2\Big] \\ & + \Big(1 - \frac{2k}{R}\Big)^{a_1} ds_1^2, \\ & Na_1^2 + a_0^2 + (a_0 + Na_1)^2 = 2 \end{aligned} \tag{25}$$

where the variable R and the integration constants are connected with ours in the following way:

$$x + 1 = R/k; \; Na_1 = -s; \; a_0 = b + s/2; \; K = L. \tag{26}$$

In [2] (see also references therein) solutions with a chain of Ricci-flat internal spaces, generalizing [11], are given; still more general spherical solutions with massless gauge and dilaton fields are discussed, e.g., in [9–13].

The general solution with $\varepsilon = -1$ has a naked singularity at $x = 1$ in all cases, except the spherically symmetric one when, in addition, the scalar field σ is constant (or the extra dimensions are frozen), in agreement with [2]. The singularity at $x = 1$ is anisotropic in all cases except (25): the metric coefficients behave in different ways when the singularity is approached from different directions. For some sets of integration constants the path to the singularity $y = \text{const}$, $\phi = \text{const}$, $x \to 1$ has an infinite length; however, the explicit conditions of such a behavior are conformal gauge-dependent.

In the case $\varepsilon = 0$ the solution generalizes the well-known Curzon vacuum solution of general relativity [14], extensively studied in a number of papers, see, e.g., Ref.[15] and references therein. The metric can be written in the form (9) with

$$\nu = -b/x, \; 2\beta = -K\rho^2/(L^2x^4), \; Lx = \sqrt{\rho^2 + z^2}. \tag{27}$$

In the special case $s = 0$ our solution coincides with the Curzon one up to a re-definition of constants.

The solution is singular at $x = 0$ in all cases except $s = b = 0$ when it reduces to flat space-time. The singularity is anisotropic, such that even the finiteness or infiniteness of some metric coefficients can depend on the direction of approach. As shown in [15], in the Curzon case the true nature of the singularity is revealed in some new coordinates, allowing one to penetrate beyond $x = 0$ (in our notation). It turns out that curvature singularity $x = 0$ has the shape of a ring and some spatial geodesics can pass through it to reach a second spatial infinity on the other side of the ring.

This quasi-wormhole structure is preserved for the present, more general solution, although the exact conditions when this is the case or, on the contrary, $x = 0$ is just a singular center, is conformal gauge-dependent.

A further study of this solution, despite its possible interest, is beyond the scope of this paper. We will instead pay more attention to the solution with $\varepsilon = +1$, which has no curvature singularity and therefore seems more promising; and although a preferred conformal gauge does exist (the one in which the original D-dimensional theory is formulated), it is remarkable that the most important features of the configuration to be discussed do not depend on conformal factors of the form $\exp(\text{const} \times \sigma)$.

5. Membranes, strings and wormholes

The non-existence of a curvature singularity for $\varepsilon = +1$ does not necessarily mean that the space-time is globally regular. Let us study the limit $x \to 0$ in some detail.

The functions ν, σ and e^β are finite at $x = 0$.

The curve $x = 0$, $y = 0$ as viewed in the coordinates (ρ, z) lies in the plane $z = 0$ and forms a ring $\rho = L$ of finite length (Fig.1). In the original conformal gauge (2) the ring radius is $r_0 = L\exp(b\pi/2 + s\pi/4)$.

The surface $x = 0$, $y > 0$ is a disk bounded by the above ring and parametrized by the coordinates y and ϕ. Its 2-dimensional metric is

$$dl^2_{\text{disk}} = L^2 e^{-2\nu-\sigma}[(1-\xi^2)^K d\xi^2 + \xi^2 d\phi^2] \tag{28}$$

where $\xi = \sqrt{1-y^2}$. This metric is flat if and only if $K = 0$, i.e., when the solution is trivial. Otherwise the disk is curved but has a regular center at $y = 1$ (the upper small black circle in Fig.1). The limit $x \to 0$ corresponds to approaching the disk from the half-space $z > 0$.

Another similar disk, the lower half-space one, corresponds to $y < 0$. The two disks are naturally identified when our oblate spheroidal coordinates are used in flat space (obtained here in the case $K = 0$).

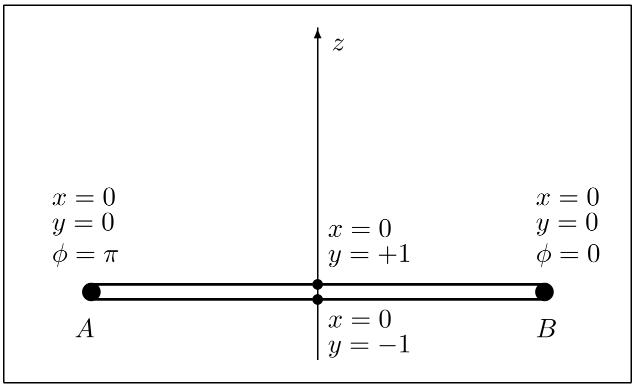

Figure 1: Axial section of the neighborhood of the ring $x = y = 0$. The points A and B, marked by big black circles, belong to the ring, the thick lines connecting them show the upper and lower disks $x = 0,\ y \gtrless 0$.

A possible identification of points $(x = 0,\ y = y_0,\ \phi = \phi_0)$ and $(x = 0,\ y = -y_0,\ \phi = \phi_0)$, where ϕ_0 is arbitrary and $0 < y_0 \le 1$, leads to a finite discontinuity of the extrinsic curvature of the surfaces identified, or, physically, to a finite discontinuity of forces acting on test particles. Such a behavior corresponds to a membrane-like matter distribution. Thus a source of the global vacuum (or scalar-vacuum) gravitational field may be a membrane bounded by the ring $x = y = 0$.

There is another possibility, with no field discontinuity across the surface $x = 0$. Namely, one can continue the (x, y) coordinates to negative x by just replacing in (19) and (20) the function $\cot^{-1} x$ (undefined for $x < 0$) by $\pi/2 - \tan^{-1} x$, coinciding with the former at $x > 0$. This results in the appearance of another "copy" of the 3-space, so that a particle crossing the regular disk $x = 0$ along a trajectory with fixed y, threads a path through the ring and can ultimately get to another flat spatial infinity, with a different asymptotic value of ν and σ:

$$\begin{aligned} &\nu(+\infty) = 0, \quad \nu(-\infty) = -\pi b, \\ &\sigma(+\infty) = 0, \quad \sigma(-\infty) = -\pi s, \end{aligned} \tag{29}$$

The function β is even with respect to x and hence coincides at both asymptotics. We obtain a wormhole configuration, nonsymmetric with respect to its "neck" $x = 0$, having no curvature discontinuities, except maybe the ring $x = y = 0$.

It now remains to study the geometry near the ring. To this end let us consider a section of the ring by an (x, y) surface at fixed ϕ and small x and y. Its 2-dimensional metric near the point $x = y = 0$ is

$$dl^2_{(x,y)} = (x^2 + y^2)^{K+1}(dx^2 + dy^2). \tag{30}$$

This metric is flat, as is directly verified by the following transformation: introduce the polar coordinates r and ψ ($x = r\cos\psi$, $y = r\sin\psi$) and further transform them to ξ and η by the formulas

$$r = [(K + 2)\xi]^{1/(K+2)}, \ \psi = \eta/(K + 2). \tag{31}$$

The result is

$$dl^2_{(x,y)} = r^{2K+2}(dr^2 + r^2 d\psi^2) = d\xi^2 + \xi^2 d\eta^2 \tag{32}$$

Thus we have above all assured that the metric near the ring x=y=0 is locally flat. However, it is locally flat on the ring itself only if the proper radius-circumference relation near the origin (the point A or B in Fig.1) in (32) holds, i.e., if η is defined on a segment of length 2π. Let us find out the η range.

Given $x > 0$, the polar angle ψ is defined on the segment $[-\pi/2,\ \pi/2]$, hence $\eta \in [-\pi - K\pi/2,\ \pi + K\pi/2]$. Consequently, the local flatness condition is fulfilled on the ring only in the trivial case $K = 0$. Identifying the points $\eta = \pm\pi$ and returning to the (x, y) coordinates, we then obtain flat space-time provided with oblate spheroidal coordinates with the single parameter L.

For $x > 0$, $K > 0$ there is an excess polar angle, the situation opposite to a top-of-a-cone singularity. Such singularities are conventionally interpreted as cosmic strings, although in those objects a deficient rather than excessive polar angle range is considered. One can conclude that a possible source of the vacuum or scalar-vacuum gravitational field is a disk membrane bounded by a special kind of string.

In the wormhole case x can have either sign, hence

$$\psi \in [-\pi,\ \pi] \ \Rightarrow \eta \in [-(2 + K)\pi,\ (2 + K)\pi]. \tag{33}$$

Thus the axially symmetric wormhole solution contains a string-like ring singularity with a polar angle excess greater than 2π.

The excessive polar angle can have another mathematical meaning. Namely, if the excess is a multiple of 2π, the singularity behaves like a branching point in a Riemannian surface of an analytic function of a properly defined complex variable. In our case the variable is $\zeta = \xi + i\eta$ and the analytic function is $\zeta^{1/(K+2)}$. Conformal mappings with analytic functions represent a natural way of regularizing metrics like (30); this method was indeed used in similar situations in [9, 10, 16, 17] where the relevant analytic function was logarithmic and the branching multiplicity was potentially (without additional identifications) infinite.

If one postulates that the "string" should behave as a branching point, the integrality condition (K = integer for (33)) is a quantization-type condition for the parameters of the solution. For instance, in the case $s = 0$, i.e., a purely vacuum configuration (with maybe trivial extra dimensions), the mass is determined by $GM = bL$ and $K = b^2$, so that, given L is a fixed length, the spectrum of masses has the form $GM = L\sqrt{K}$ where K is a positive integer.

6. Concluding remarks

The results described appear from solving the field equations for pure vacuum or scalar vacuum in conventional general relativity as well as multidimensional gravity. One can conclude that SAS configurations can have nontrivial structures; those free of curvature singularities are in our view of greatest physical interest. Notably the singularities in SAS solutions are naked, except special spherically symmetric cases (for the vacuum case see (25) for $a_1 = 0$). For general relativity this is a manifestation of the well-known uniqueness or no-hair theorems; it would be, however, of interest to analyze the situation in dilaton gravity for which spherically symmetric black-hole and non-black-hole solutions are known (see, e.g., [9]) and SAS ones are either known [10], or can be readily obtained, for instance, in D dimensions with Ricci-flat internal spaces.

The above solutions can be of interest for describing late stages of gravitational collapse and/or cosmological dark matter. Their monopole nature probably means that they cannot decay through gravitational-wave emission.

Other generalizations of the present solutions, which are either known or easily obtainable by the known methods and are yet to be investigated in detail, are those with pure imaginary, nonminimally coupled and multiple scalar fields and/or multiple internal spaces.

Acknowledgement

This work was supported in part by the Russian Ministry of Science.

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