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ON SINGULAR SOLUTIONS IN MULTIDIMENSIONAL GRAVITY

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It is proved that the Riemann tensor squared is divergent as $\tau \to 0$ for a wide class of cosmological metrics with non-exceptional Kasner-like behaviour of scale factors as $\tau \to 0$, where τ is the synchronous time. Using this result, it is shown that any nontrivial generalization of the spherically-symmetric Tangherlini solution to the case of n Ricci-flat internal spaces has a divergent Riemann tensor squared as $R \to R_0$, where R_0 is a length parameter of the solution.

1. Introduction

In the recent decade there has been a great interest in multidimensional models of classical and quantum gravity (see, for example, [1]-[3] and references therein). This interest was stimulated mainly by studies in supergravity and superstring theories [4], i.e., the ideas of unification of interactions gave a "new life" to the ideas of Kaluza and Klein.

As shown in papers [5]-[8], devoted to multidimensional cosmological models with perfect fluid, a large variety of exact solutions have a Kasner-like asymptotical behaviour for small values of the synchronous time parameter, $\tau \to 0$. Here we do not consider exceptional solutions such as exponential and power-law inflationary ones [8]. Solutions with oscillatory (or stochastic behaviour) as $\tau \to 0$ are also omitted in the presented scheme.

In this paper we use the Riemann tensor squared as an indicator of a singular behaviour of cosmological solutions as $\tau \to 0$. In Sec. 2 we present an explicit formula for the quadratic invariant (Riemann tensor squared) of the metric defined on the product of spaces with the scale factors depending on the points of the first space. In Sec. 3 the quadratic invariant is presented for a wide class of cosmological metrics describing the evolution of n spaces of arbitrary dimensions. It is proved (see Proposition 2) that when all spaces are one-dimensional, the Riemann tensor squared is positive and divergent as $t = \tau \to 0$ for all non-trivial (non-Milne-like) configurations. In Subsection 3.2 this result is generalized to the case of Ricci-flat spaces. In Subsec. 3.3 the main theorem concerning the divergence of the Riemann tensor squared for a wide class of cosmological metrics with non-exceptional Kasner-like behaviour of scale factors as $\tau \to 0$ is proved. In Sec. 4 we apply the obtained result to the generalization of the spherically-symmetric Tangherlini solution [10] to the case of n Ricci-flat internal spaces [13]. In Subsec. 4.1 the multitemporal generalization of the Tangherlini solution is considered [14]. This solution is shown to describe the (multitemporal) naked singularity when the parameters are non-exceptional.

2. General formalism

Let (M, g) be a manifold M with the metric g. The squared Riemann tensor

$$I[g] \equiv R_{MNPQ}[g] R^{MNPQ}[g] \qquad (2.1)$$

is a smooth real-valued function on M. For a smooth function $\phi : M \longrightarrow R$ we also define two smooth functions on M

$$U[g, \phi] \equiv g^{MN} (\partial_M \phi) \partial_N \phi, \qquad (2.2)$$

$$\begin{aligned} V[g, \phi] \equiv\ & g^{M_1 N_1} g^{M_2 N_2} [\nabla_{M_1} (\partial_{M_2} \phi) + (\partial_{M_1} \phi) \partial_{M_2} \phi] \\ & \times [\nabla_{N_1} (\partial_{N_2} \phi) + (\partial_{N_1} \phi) \partial_{N_2} \phi], \end{aligned} \qquad (2.3)$$

where $\nabla = \nabla[g]$ is a covariant derivative with respect to g. The scalar invariants (2.1)-(2.3) play an important role in what follows. Now, we consider the manifold

$$M = M_0 \times M_1 \times \ldots \times M_n \qquad (2.4)$$

with the metric

$$g = g^{(0)} + \sum_{i=1}^{n} \exp[2\phi^i(x)] g^{(i)} \qquad (2.5)$$

where $g^{(0)} = g^{(0)}_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ is the metric on M_0, $g^{(i)}$ is the metric on M_i and $\phi^i : M_0 \longrightarrow \mathbf{R}$ is a smooth function, $i = 1, \ldots, n$.

Proposition 1. The Riemann tensor squared for the metric (2.5) has the following form

$$\begin{aligned} I[g] =\ & I[g^{(0)}] \\ & + \sum_{i=1}^{n} \{ \mathrm{e}^{-4\phi^i} I[g^i] - 4 \mathrm{e}^{-2\phi^i} U[g^{(0)}, \phi^i] R[g^{(i)}] \\ & - 2 N_i U^2[g^{(0)}, \phi^i] + 4 N_i V[g^{(0)}, \phi^i] \} \\ & + \sum_{i,j=1}^{n} 2 N_i N_j [g^{(0),\mu\nu} (\partial_\mu \phi^i) \partial_\nu \phi^j]^2, \end{aligned} \qquad (2.6)$$

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where the U- and V-invariants are defined in (2.2) and (2.3), $R[g^{(i)}]$ is the scalar curvature of $g^{(i)}$ and $N_i = \mathrm{dim} M_i$, $i = 1, \ldots, n$.

Sketch of proof. For $n = 1$ the relation (2.6) may be verified by a straightforward calculation. For $n > 1$ the same relation may be proved by induction (in n) using the following decomposition formulas:

$$U[g^{(0)} + \mathrm{e}^{2\phi^1(x)} g^{(1)}, \phi(x)] = U[g^{(0)}, \phi(x)], \tag{2.7}$$

$$V[g^{(0)} + \mathrm{e}^{2\phi^1(x)} g^{(1)}, \phi(x)] = V[g^{(0)}, \phi(x)] + N_1 [g^{(0),\mu\nu} (\partial_\mu \phi^1) \partial_\nu \phi]^2 \tag{2.8}$$

($x \in M_0$). •

For the scalar curvature of the metric (2.5) we get

$$\begin{aligned} R[g] = {} & R[g^{(0)}] + \sum_{i=1}^{n} \mathrm{e}^{-2\phi^i} R[g^{(i)}] \\ & - \sum_{i,j=1}^{n} (N_i \delta_{ij} + N_i N_j) g^{(0),\mu\nu} (\partial_\mu \phi^i) \partial_\nu \phi^j \\ & - 2 \sum_{i=1}^{n} N_i \triangle[g^{(0)}] \phi^i \end{aligned} \tag{2.9}$$

where $\triangle[g^{(0)}]$ is the Laplace-Beltrami operator corresponding to $g^{(0)}$ (see also [15]).

Remark 1. In (2.6) and in what follows we use the following condensed notations: $I[g] = I[g](x)$, $I[g^{(\nu)}] = I[g^{(\nu)}](x_\nu)$, for $x \in M$, $x_\nu \in M_\nu$, $\nu = 0, \ldots, n$ and analogously for scalar curvatures.

3. Multidimensional cosmology

Here we are interested in the special case of (2.4), (2.5) with $M_0 = (t_1, t_2)$, $t_1 < t_2$. We thus consider the metric

$$g_c = -B(t) dt \otimes dt + \sum_{i=1}^{n} A_i(t) g^{(i)}, \tag{3.1}$$

defined on the manifold

$$M = (t_1, t_2) \times M_1 \times \ldots \times M_n. \tag{3.2}$$

Here, just as in (2.4) and (2.5), $g^{(i)}$ is a metric on M_i and $B(t), A_i(t) \neq 0$, $i = 1, \ldots, n$.

From Proposition 1 we obtain the Riemann tensor squared for the metric (3.1) (see also [14])

$$\begin{aligned} I[g_c] \; = \; & \sum_{i=1}^{n} \Big\{ A_i^{-2} I[g^{(i)}] + A_i^{-3} B^{-1} \dot{A}_i^2 R[g^{(i)}] \\ & - \frac{1}{8} N_i B^{-2} A_i^{-4} \dot{A}_i^4 \\ & + \frac{1}{4} N_i B^{-2} (2 A_i^{-1} \ddot{A}_i - B^{-1} \dot{B} A_i^{-1} \dot{A}_i - A_i^{-2} \dot{A}_i^2)^2 \Big\} \\ & + \frac{1}{8} B^{-2} [\sum_{i=1}^{n} N_i (A_i^{-1} \dot{A}_i)^2]^2. \end{aligned} \tag{3.3}$$

(Recall that $\dim M_i = N_i$, $i = 1, \ldots, n$.)

For the scalar curvature of (3.1) we get from (2.9)

$$\begin{aligned} R[g_c] = & \sum_{i=1}^{n} \Big\{ \mathrm{e}^{-2x^i} R[g^{(i)}] \\ & + \mathrm{e}^{-2\gamma} N_i \Big[2\ddot{x}^i + \dot{x}^i \Big(\sum_{j=1}^{n} N_j \dot{x}^j - 2\dot{\gamma} \Big) + (\dot{x}^i)^2 \Big] \Big\} \end{aligned} \tag{3.4}$$

where $B = \mathrm{e}^{2\gamma}$ and $A_i = \mathrm{e}^{2x^i}$, $i = 1, \ldots, n$.

3.1. $(n+1)$-dimensional Kasner solution

Let us consider the metric on $\mathbf{R}_+ \times \mathbf{R}^n$

$$g = -dt \otimes dt + \sum_{i=1}^{n} t^{2\alpha_i} dx^i \otimes dx^i, \tag{3.5}$$

where $t > 0$, $-\infty < x^i < \infty$ and α_i are constants, $i = 1, \ldots, n$. From (3.3) we get

$$I[g] = 2F(\alpha) t^{-4}, \tag{3.6}$$

where

$$F(\alpha) = \sum_{i=1}^{n} [2\alpha_i^2 (\alpha_i - 1)^2 - \alpha_i^4] + \Big[\sum_{i=1}^{n} \alpha_i^2 \Big]^2. \tag{3.7}$$

Now we impose the following restrictions upon the parameters α_i:

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \alpha_i^2 = 1. \tag{3.8}$$

The metric (3.5) with the restrictions (3.8) imposed satisfies the vacuum Einstein equations (or, equivalently, $R_{MN}[g] = 0$). It is a trivial generalization of the well-known Kasner solution. In this case

$$F(\alpha) = \Phi(\alpha) = \Phi_n(\alpha) \equiv \sum_{i=1}^{n} [\alpha_i^4 - 4\alpha_i^3] + 3. \tag{3.9}$$

We define the Milne set as

$$\mathcal{M} = \mathcal{M}_n = \{(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\} \subset \mathcal{E} \tag{3.10}$$

where

$$\mathcal{E} = \mathcal{E}_n \equiv \Big\{ \alpha = (\alpha_1, \ldots \alpha_n) \in \mathbf{R}^n \Big| \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \alpha_i^2 = 1 \Big\}. \tag{3.11}$$

Notice that $\mathcal{E}$ is an $(n-2)$-dimensional ellipsoid for $n > 2$ ($\mathcal{E} \simeq \mathbf{S}^{n-2}$).

For $n = 1$, $\mathcal{M} = \mathcal{E} = \{(1)\}$ and we are led to the well-known Milne solution

$$g_M = -dt \otimes dt + t^2 dx^1 \otimes dx^1. \tag{3.12}$$

We recall that the coordinate transformation $y^0 = t \cosh x^1$, $y^1 = t \sinh x^1$ reduces (3.12) to the Minkowski metric $\eta = -dy^0 \otimes dy^0 + dy^1 \otimes dy^1$ in the upper light cone $y^0 > |y^1|$.

For $\alpha = (\ldots, 0, 1_i, 0, \ldots) \in \mathcal{M}$ we get a trivial extension of the Milne metric:

$$g_m = -dt \otimes dt + t^2 dx^i \otimes dx^i + \sum_{j \neq i}^{n} dx^j \otimes dx^j, \tag{3.13}$$

$i = 1, \ldots, n$; $n > 1$.

Proposition 2. Let $\alpha = (\alpha_1, \dots \alpha_n) \in \mathcal{E}$. Then $\Phi(\alpha) \geq 0$ and $\Phi(\alpha) = 0$ if and only if $\alpha \in \mathcal{M}$.

Proof. For $n = 1, 2$ the proposition is trivial. So we consider the case $n > 2$. Let

$$\Phi_| = \Phi_{|\mathcal{E}} : \mathcal{E} \longrightarrow \mathbf{R} \qquad (3.14)$$

be a restriction on the function Φ (3.9) on $\mathcal{E}$ (3.11). Since $\mathcal{E}$ is a smooth submanifold in $\mathbf{R}^n$ (see (3.8)), the function $\Phi_|$ is also smooth ($\Phi_| = \Phi \circ i$, where $i : \mathcal{E} \longrightarrow \mathbf{R}^n$ is a canonical embedding). The manifold $\mathcal{E}$ is compact (it is isomorphic to $\mathbf{S}^{n-2}$). Let Min $=$ Min$(\Phi_|)$ be the set of points of (absolute) minimum of $\Phi_|$ and Ext $=$ Ext$(\Phi_|)$ is the set of extremum points of $\Phi_|$. The set Min is non-empty: Min $\neq \emptyset$, since $\Phi_|$ is a continuous real-valued function defined on the compact topological space $\mathcal{E}$. It is clear that Min $\subset$ Ext.

First we find Ext using the standard conditional extremum scheme. Consider the function

$$\hat{\Phi}(\alpha, \lambda, \mu) = \Phi(\alpha) + \mu\Big(\sum_{i=1}^{n} \alpha_i^2 - 1\Big) + \lambda\Big(\sum_{i=1}^{n} \alpha_i - 1\Big) \qquad (3.15)$$

where $\mu, \lambda \in \mathbf{R}$. The point α belongs to Ext if and only if there are $\lambda, \mu \in \mathbf{R}$ such that (α, μ, λ) is an extremum point of the function (3.15), i.e. the relations (3.8) and

$$\frac{\partial \hat{\Phi}}{\partial \alpha_i} = 4\alpha_i^3 - 12\alpha_i^2 + 2\mu\alpha_i + \lambda = 0, \qquad (3.16)$$

$i = 1, \dots, n$, are satisfied. From (3.8) and (3.16) we obtain

$$4\sum_{i=1}^{n} \alpha_i^3 - 12 + 2\mu + \lambda n = 0, \qquad (3.17)$$

$$4\sum_{i=1}^{n} \alpha_i^4 - 12\sum_{i=1}^{n} \alpha_i^3 + 2\mu + \lambda = 0, \qquad (3.18)$$

and hence

$$4\Phi(\alpha) = \lambda(n-1). \qquad (3.19)$$

Let us consider the cubic equation

$$4y^3 - 12y^2 + 2\mu y + \lambda = 4(y - y_1)(y - y_2)(y - y_3) = 0. \qquad (3.20)$$

We prove that for given μ and λ the cubic equation (3.20) has three different real roots. Indeed, there must be at least two different real roots since otherwise $\alpha_1 = \dots = \alpha_n$, but this is impossible due to the Kasner constraints (3.8). The third root should be also real and we are led to the following three possibilities for the roots: (i) $y_1 = y_2 < y_3$; (ii) $y_1 < y_2 = y_3$; (iii) $y_1 < y_2 < y_3$. It follows from (3.20) that

$$y_1 + y_2 + y_3 = 3 \qquad (3.21)$$

and hence $y_3 > 1$. But due to (3.8) $\alpha_i \leq 1$ and as a consequence the possibilities (i) and (ii) cannot occur (otherwise $\alpha_i = y_1$ for all i). Thus $y_1 < y_2 < y_3$ and by (3.16) and $y_3 > 1$ we obtain

$$\{x | x = \alpha_i, i = 1, \dots, n\} = \{y_1, y_2\}. \qquad (3.22)$$

Solving (3.8) for α satisfying (3.22), we get

$$y_1 = y_1(m_1, m_2) = \frac{m_1 - \sqrt{\Delta}}{m_1 n}, \qquad (3.23)$$

$$y_2 = y_2(m_1, m_2) = \frac{m_2 + \sqrt{\Delta}}{m_2 n}, \qquad (3.24)$$

$$\Delta = m_1 m_2 (n-1). \qquad (3.25)$$

Here m_a is the number of α_i equal to y_a, $a = 1, 2$. Evidently $m_1 + m_2 = n$ and $m_1, m_2 \geq 1$. It follows from (3.19) and (3.20) that

$$\lambda = 4\Phi(\alpha)/(n-1) = -4 y_1 y_2 y_3, \qquad (3.26)$$

$$\mu = 2(y_1 y_2 + y_2 y_3 + y_3 y_1). \qquad (3.27)$$

We actually find an expression for the set of extrema

$$\text{Ext} = E_1 \sqcup \dots \sqcup E_n, \qquad (3.28)$$

where

$$E_k = \{(\alpha_1 = y_2, \dots, \alpha_k = y_2, \alpha_{k+1} = y_1, \dots, \alpha_n = y_1) \text{ and all permutations}\}, \qquad (3.29)$$

$k = 1, \dots, n-1$ and $y_1 = y_1(n-k, k)$, $y_2 = y_2(n-k, k)$ (see (3.23), (3.24)). It is clear that the number of elements in Ext is $2^n - 2$.

For $\alpha \in E_k$, $k > 1$, we have

$$\Phi(\alpha) > 0. \qquad (3.30)$$

This can be readily verified using the inequalities $y_1(n-k, k) < 0$, $0 < y_2(n-k, k) < y_3(n-k, k)$, $k > 1$, and the relation (3.26). For $\alpha \in E_1$

$$\Phi(\alpha) = 0. \qquad (3.31)$$

Using the inclusion Min $\subset$ Ext and the relations (3.28), (3.30) and (3.31), we obtain

$$\text{Min} = \text{Min}(\Phi_|) = E_1 = \mathcal{M}. \qquad (3.32)$$

Proposition 2 follows from the relations (3.31) and (3.32). •

Remark 2. It will be proved in a separate publication that $\Phi_|$ (see (3.14)) is a Morse function.

From (3.6), (3.9) and Proposition 2 we get for the Kasner metric (3.5) with $\alpha \in \mathcal{E} \setminus \mathcal{M}$

$$I[g](t, \vec{x}(t)) \to +\infty, \qquad \text{as } t \to +0. \qquad (3.33)$$

for an arbitrary curve $\vec{x}(t)$.

3.2. Kasner-like solutions with Ricci-flat spaces

Here we consider the following metric [9]:

$$g = -dt \otimes dt + \sum_{i=1}^{n} t^{2\alpha_i} c_i g^{(i)}, \qquad (3.34)$$

defined on the manifold (3.2) with $(t_1, t_2) = (0, +\infty) = \mathbf{R}_+$, where $(g^{(i)}, M_i)$ are Ricci-flat internal spaces, i.e. $R_{m_i n_i}[g^{(i)}] = 0$, and $c_i \neq 0$ are constants, $i =$

$1, \ldots, n$, $n \geq 2$. The parameters α_i satisfy the relations

$$\sum_{i=1}^{n} N_i \alpha_i = \sum_{i=1}^{n} N_i \alpha_i^2 = 1. \tag{3.35}$$

The metric (3.34) under the restrictions (3.35) satisfies the vacuum Einstein equations.

For the metric (3.34), (3.35) we get from (3.3)

$$I[g] = \sum_{i=1}^{n} t^{-4\alpha_i} c_i^{-2} I[g^{(i)}] + 2\Phi_*(\alpha) t^{-4}, \tag{3.36}$$

where

$$\Phi_*(\alpha) \equiv \sum_{i=1}^{n} N_i [\alpha_i^4 - 4\alpha_i^3] + 3. \tag{3.37}$$

Similarly to (3.10), we introduce the Milne set

$$\mathcal{M}_* = \{\alpha | \alpha = (\ldots, 0, 1_i, 0, \ldots), N_i = 1\} \subset \mathcal{E}_*, \tag{3.38}$$

where

$$\mathcal{E}_* \equiv \Big\{ \alpha = (\alpha_1, \ldots \alpha_n) \in \mathbf{R}^n \Big| \sum_{i=1}^{n} N_i \alpha_i = \sum_{i=1}^{n} N_i \alpha_i^2 = 1 \Big\}. \tag{3.39}$$

For $n > 2$ $\mathcal{E}_*$ is an $(n-2)$-dimensional ellipsoid.

Example 1. The set (3.38) is empty: $\mathcal{M}_* = \emptyset$, if and only if $N_i > 1$ for all i.

Example 2. For $N_1 = \ldots = N_n = 1$ we have $\mathcal{M}_* = \mathcal{M}$ (see (3.10)).

Proposition 3. Let $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathcal{E}_*$. Then $\Phi_*(\alpha) \geq 0$ and $\Phi_*(\alpha) = 0$ if and only if $\alpha \in \mathcal{M}_*$.

Proof. Consider the function $\Phi = \Phi_N$ (3.9) corresponding to $N = \sum_{i=1}^{n} N_i$. For $\alpha \in \mathcal{E}_*$ we have

$$\Phi_*(\alpha) = \Phi_N(\beta(\alpha)), \tag{3.40}$$

where the set $\beta(\alpha) = \beta = (\beta_1, \ldots, \beta_N)$ is defined by the relations

$$\begin{aligned} &\beta_1 = \ldots = \beta_{N_1} = \alpha_1, \\ &\ldots \\ &\beta_{N-N_n+1} = \ldots = \beta_N = \alpha_n. \end{aligned} \tag{3.41}$$

It is evident that $\beta \in \mathcal{E}_N$ (see (3.11)). Proposition 3 follows from Proposition 2, relation (3.40) and the equivalence

$$\beta(\alpha) \in \mathcal{M} \Longleftrightarrow \alpha \in \mathcal{M}_*. \tag{3.42}$$

Here $\mathcal{M} = \mathcal{M}_N$ is the Milne set corresponding to N. •

Proposition 4. Let g be the metric (3.34) with the set $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathcal{E}_* \setminus \mathcal{M}_*$ and

$$I[g^{(i)}] \geq 0 \tag{3.43}$$

for all $i = 1, \ldots, n$. Then

$$I[g](t, f(t)) \to +\infty, \qquad \text{as } t \to +0, \tag{3.44}$$

for any function

$$f : \mathbf{R}_+ \longrightarrow M_1 \times \ldots \times M_n. \tag{3.45}$$

When the condition (3.43) is not imposed, the relation (3.44) takes place if $f(t) \to f_0 \in M_1 \times \ldots \times M_n$ as $t \to +0$.

Proof. The first part of the proposition follows from (3.36), (3.43) and the inequality $\Phi_*(\alpha) > 0$ for $\alpha \notin \mathcal{M}_*$ (see Proposition 3). Its second part follows from the continuity of the functions $I[g^{(i)}]$ on M_i, the inequalities $\alpha_i < 1$, $i = 1, \ldots, n$, and the relation (3.36). Indeed, the functions $I[g^{(i)}](f^i(t))$, $i = 1, \ldots, n$, have limits as $t \to +0$ and hence are bounded. Here $f(t) = (f^1(t), \ldots, f^n(t))$. Thus the second term on the right hand side of (3.36) is dominating in the limit $t \to +0$ and we are led to (3.44). •

3.3. Asymptotically Kasner solutions

Here we consider the metric

$$g = -w d\tau \otimes d\tau + \sum_{i=1}^{n} A_i(\tau) g^{(i)}, \tag{3.46}$$

defined on the manifold

$$(0, T) \times M_1 \times \ldots \times M_n, \tag{3.47}$$

where $T > 0$ and $w = \pm 1$. We assume that the metric $g^{(i)}$ on the manifold M_i satisfies the following conditions: the functions $I[g^{(i)}]$ and $wR[g^{(i)}]$, are bounded from below, i.e.,

$$I[g^{(i)}](x_i) \geq C_i \tag{3.48}$$

for all $x_i \in M_i$, and

$$wR[g^{(i)}](x_i) \geq D_i \tag{3.49}$$

for all $x_i \in M_i$, $i = 1, \ldots, n$.

Remark 3. Clearly the conditions (3.48) and (3.49) are satisfied for a compact manifold M_i. The first condition is also satisfied when the metric $g^{(i)}$ has the Euclidean signature: in this case $C_i = 0$.

We also assume that the scale factors $A_i : (0, T) \longrightarrow \mathbf{R}$ are smooth functions ($A_i(\tau) \neq 0$), satisfying the following asymptotic relations:

$$A_i(\tau) = c_i \tau^{2\alpha_i} [1 + o(1)], \tag{3.50}$$

$$\dot{A}_i(\tau) = c_i \tau^{2\alpha_i - 1} [2\alpha_i + o(1)], \tag{3.51}$$

$$\ddot{A}_i(\tau) = c_i \tau^{2\alpha_i - 2} [2\alpha_i(2\alpha_i - 1) + o(1)], \tag{3.52}$$

as $\tau \to +0$, where $c_i \neq 0$ and α_i are constants and $i = 1, \ldots, n$. Recall that the notation $\varphi(\tau) = o(1)$ as $\tau \to +0$ means that $\varphi(\tau) \to 0$ as $\tau \to +0$.

Remark 4. The relations (3.51) and (3.52) should not obviously follow from (3.50). A simple counterexample is

$$A_i(\tau) = 1 + \tau \sin \frac{1}{\tau^2}. \tag{3.53}$$

But if

$$A_i(\tau) = c_i \tau^{2\alpha_i}[1 + \varphi_i(\tau)], \tag{3.54}$$

where $c_i \neq 0$ and

$$\varphi_i(\tau) = o(1), \quad \tau\dot{\varphi}_i(\tau) = o(1), \quad \tau^2\ddot{\varphi}_i(\tau) = o(1) \tag{3.55}$$

as $\tau \to +0$, $i = 1, \dots, n$, then the relations (3.50)-(3.52) are satisfied.

Theorem. Let g be the metric (3.46) defined on the manifold (3.47), where the metrics $g^{(i)}$, $i = 1, \dots, n$, satisfy the relations (3.48) and (3.49). Let the scale factors $A_i(\tau)$, $i = 1, \dots, n$, satisfy the relations (3.50)-(3.52), where the set of parameters $\alpha = (\alpha_i)$ satisfies the Kasner-like relations (3.35) ($N_i = \dim M_i$) and is non-exceptional, i.e. $\alpha \notin \mathcal{M}_*$ ($\mathcal{M}_*$ is defined in (3.38)). Then

$$I[g](\tau, x) \to +\infty \qquad \text{as } \tau \to +0 \tag{3.56}$$

uniformly on $x \in M_1 \times \dots \times M_n$.

Proof. From (3.3) we get

$$I[g] = I_1[g] + I_2[g] + I_3[g], \tag{3.57}$$

where

$$I_1[g] = \sum_{i=1}^{n} A_i^{-2} I[g^{(i)}], \tag{3.58}$$

$$I_2[g] = \sum_{i=1}^{n} A_i^{-3} \dot{A}_i^2 w R[g^{(i)}], \tag{3.59}$$

$$I_3[g] = \sum_{i=1}^{n} \{ -\frac{1}{8} N_i A_i^{-4} \dot{A}_i^4 + \frac{1}{4} N_i (2A_i^{-1}\ddot{A}_i - A_i^{-2}\dot{A}_i^2)^2 \}$$
$$+ \frac{1}{8}[\sum_{i=1}^{n} N_i (A_i^{-1}\dot{A}_i)^2]^2. \tag{3.60}$$

From (3.50)-(3.52) and (3.60) we obtain

$$I_3[g] = [2\Phi_*(\alpha) + o(1)]\tau^{-4}, \tag{3.61}$$

as $\tau \to +0$, where $\Phi_*(\alpha)$ is defined in (3.37). Note that due to $\alpha \notin \mathcal{M}_*$ and Proposition 3

$$\Phi_*(\alpha) > 0. \tag{3.62}$$

From (3.50) and (3.51) we get

$$\frac{\dot{A}_i^2}{A_i^3} = c_i^{-1} \tau^{-2-2\alpha_i}[4\alpha_i^2 + o(1)], \tag{3.63}$$

as $\tau \to +0$. Note also that due to $\alpha \notin \mathcal{M}_*$

$$\alpha_i < 1, \qquad i = 1, \dots, n. \tag{3.64}$$

Let

$$\delta = \min_i (1 - \alpha_i) > 0. \tag{3.65}$$

Then it follows from (3.63) that there exists $\tau_1 > 0$ such that

$$0 \leq \frac{\dot{A}_i^2}{A_i^3} < \tau^{-4+\delta}, \tag{3.66}$$

for all $\tau < \tau_1$, $i = 1, \dots, n$. From (3.49) and (3.66) we get

$$\frac{\dot{A}_i^2}{A_i^3} w R[g^{(i)}](x_i) \geq -|D_i| \tau^{-4+\delta} \tag{3.67}$$

for all $x_i \in M_i$, $\tau < \tau_1$, $i = 1, \dots, n$, and hence

$$I_2[g](\tau, x) \geq -A\tau^{-4+\delta}, \tag{3.68}$$

for all $\tau < \tau_1$ and $x \in M_1 \times \dots \times M_n$ ($A = \sum_{i=1}^n |D_i|$). We get in a similar way from (3.50)

$$A_i^{-2} = c_i^{-2} \tau^{-4\alpha_i}[1 + o(1)], \tag{3.69}$$

and consequently there exists $\tau_2 > 0$ such that

$$A_i^{-2} < \tau^{-4+\delta} \tag{3.70}$$

for all $\tau < \tau_2$, $i = 1, \dots, n$. Using (3.48) and (3.70), we obtain (similarly to (3.68))

$$I_1[g](\tau, x) \geq -B\tau^{-4+\delta} \tag{3.71}$$

for all $\tau < \tau_2$ and $x \in M_1 \times \dots \times M_n$ ($B = \sum_{i=1}^n |C_i|$). It follows from (3.61), (3.68), (3.71) that

$$I[g](\tau, x) \geq \tau^{-4}[2\Phi_*(\alpha) - G\tau^{\delta} + o(1)] \to +\infty \tag{3.72}$$

as $\tau \to +0$ ($G = A + B$). This implies the relation (3.56). The theorem is proved. •

Remark 5. When the relations (3.48) and (3.49) are not imposed, the relation (3.56) is valid (at least) for any (fixed) $x \in M_1 \times \dots \times M_n$.

4. Spherically symmetric solutions with Ricci-flat internal spaces

Now we apply the above results to the following scalar vacuum solution [8]:

$$g = -f^a dt \otimes dt + f^{b-1} dR \otimes dR$$
$$+ f^b R^2 d\Omega_d^2 + \sum_{i=1}^{n} f^{a_i} B_i g^{(i)}, \tag{4.1}$$

$$\exp(2\varphi) = B_\varphi f^{a_\varphi}, \tag{4.2}$$

defined on the manifold

$$M = (R_0, +\infty) \times \mathbf{R} \times \mathbf{S}^d \times M_1 \times \dots \times M_n, \tag{4.3}$$

where $(M_i, g^{(i)})$ are Ricci-flat internal spaces, $\dim M_i = N_i$, $i = 1, \dots, n$, $d\Omega_d^2$ is the canonical metric on a d-dimensional sphere $\mathbf{S}^d$ ($d \geq 2$) and $f = f(R) = 1 - (R_0/R)^{d-1}$. Here R_0, B_φ, $B_i > 0$ are constants and the parameters $b, a, a_1, \dots, a_n$ satisfy the relations

$$b = (1 - a - \sum_{i=1}^{n} a_i N_i)/(d-1), \tag{4.4}$$

$$(a + \sum_{i=1}^{n} a_i N_i)^2 + (d-1)(a^2 + a_\varphi^2 + \sum_{i=1}^{n} a_i^2 N_i) = d. \tag{4.5}$$

The solution (4.1)-(4.3) is a scalar-vacuum multispace generalization of the Tangherlini solution [10]. In the parametrization of the harmonic-type variable this solution was presented earlier in [17, 14]. For $a_{\varphi} = 0$ see also [13, 14]. Some special cases were considered earlier in [12] (for $d = 2$, $a_{\varphi} = 0$) and [11] ($n = 1$ and $d = 2$).

The metric and scalar field from (4.1), (4.2) satisfy the field equations

$$R_{MN}[g] = \partial_M \varphi \partial_N \varphi, \tag{4.6}$$
$$\triangle[g]\varphi = 0, \tag{4.7}$$

corresponding to the action

$$S = \int d^D x \sqrt{|g|} \{ R[g] - \partial_M \varphi \partial_N \varphi g^{MN} \}. \tag{4.8}$$

Now, we introduce the new variable

$$\tau = \tau(R) = \int_{R_0}^{R} dx [f(x)]^{(b-1)/2}. \tag{4.9}$$

The integral in (4.9) is convergent since due (4.4) and (4.5)

$$b > -1. \tag{4.10}$$

The map (4.9) defines a diffeomorphism from $(R_0, +\infty)$ to $\mathbf{R}_+$. We consider the diffeomorphism

$$\sigma : M' \longrightarrow M, \tag{4.11}$$

created by (4.9): $\sigma(\tau, t, \ldots) = (R(\tau), t, \ldots)$, where

$$M' = \mathbf{R}_+ \times \mathbf{R} \times \mathbf{S}^d \times M_1 \times \ldots \times M_n. \tag{4.12}$$

The substitution of (4.9) into (4.1) gives a metric on the manifold (4.12) (the dragging of (4.1) by the map (4.11))

$$\begin{aligned} \sigma^* g = {} & d\tau \otimes d\tau + A_0(\tau) g^{(0)} \\ & + \sum_{i=1}^{n} A_i(\tau) g^{(i)} - A_{-1}(\tau) dt \otimes dt, \end{aligned} \tag{4.13}$$

where $g^{(0)} = d\Omega_d^2$ and

$$A_i(\tau) = [f(R(\tau))]^{a_i}, \tag{4.14}$$
$$A_0(\tau) = R^2(\tau)[f(R(\tau))]^b, \tag{4.15}$$

$i = -1, 1, \ldots, n$; $a_{-1} = a$. From (4.9) and the asymptotical behaviour

$$f(R) \sim \frac{(d-1)}{R_0}(R - R_0), \qquad \text{as } R \to R_0 \tag{4.16}$$

we get

$$R - R_0 \sim (c_* \tau)^{2/(b+1)}, \quad f(R(\tau)) \sim c_. \tau^{2/(b+1)} \tag{4.17}$$

as $\tau \to +0$, where $c_*, c_.$ are constants. From (4.17) we get for the scale factors (4.14), (4.15) and the scalar field the following asymptotical relations

$$A_i(\tau) \sim c_i \tau^{2\alpha_i}, \tag{4.18}$$
$$A_0(\tau) \sim c_0 R_0^2 \tau^{2\alpha_0}, \tag{4.19}$$
$$\exp(2\varphi(\tau)) \sim c_{\varphi} \tau^{2\alpha_{\varphi}}, \tag{4.20}$$

as $\tau \to +0$, where c_i, c_0, c_{φ} are constants, and

$$\alpha_i = a_i/(b+1), \qquad \alpha_0 = b/(b+1), \tag{4.21}$$
$$\alpha_{\varphi} = a_{\varphi}/(b+1), \tag{4.22}$$

$i = -1, 1, \ldots, n$. The parameters (4.21) and (4.22) are correctly defined due to (4.10) and satisfy the Kasner-like relations

$$\sum_{\nu=-1}^{n} N_{\nu} \alpha_{\nu} = 1, \tag{4.23}$$

$$\sum_{\nu=-1}^{n} N_{\nu} \alpha_{\nu}^2 + \alpha_{\varphi}^2 = 1. \tag{4.24}$$

Here $N_{-1} = 1$ and $N_0 = d$.

Now we consider the case $\alpha_{\varphi} = 0$ (or equivalently $a_{\varphi} = 0$). Let

$$\begin{aligned} \mathcal{M}_1 = \{ & \alpha = (\alpha_{-1}, \ldots, \alpha_n) \\ & = (\ldots, 0, 1_{\nu}, 0, \ldots), N_{\nu} = 1 \} \subset \mathbf{R}^{n+2}, \end{aligned} \tag{4.25}$$
$$\begin{aligned} \mathcal{T} = \{ & a_. = (a_{-1}, a_1, \ldots, a_n) \\ & = (\ldots, 0, 1_{\nu}, 0, \ldots), N_{\nu} = 1 \} \subset \mathbf{R}^{n+1}. \end{aligned} \tag{4.26}$$

Evidently, $\mathcal{M}_1 \subset \mathcal{E}_2$ and $\mathcal{T} \subset \mathcal{E}_1$, where $\mathcal{E}_1 \subset \mathbf{R}^{n+1}$ and $\mathcal{E}_2 \subset \mathbf{R}^{n+2}$ are n-dimensional ellipsoids defined by relations (4.5) and (4.23), (4.24), respectively. It is not difficult to verify that the function $\alpha = \alpha(a_.)$ from (4.21) defines the diffeomorphism $\mathcal{E}_1 \to \mathcal{E}_2$ and

$$\alpha(a_.) \in \mathcal{M}_1 \Longleftrightarrow a_. \in \mathcal{T}. \tag{4.27}$$

Proposition 5. Let $\sigma^* g$ be the metric (4.13)-(4.15) with the parameters (4.21), (4.22) satisfying the relations (4.23), (4.24) and obeying the restrictions: $\alpha_{\varphi} = 0$, $\alpha = (\alpha_{-1}, \ldots \alpha_n) \notin \mathcal{M}_1$. Let the Ricci-flat internal spaces $(M_i, g^{(i)})$, $i = 1, \ldots, n$, satisfy the self-boundness conditions (3.48). Then

$$I[\sigma^* g](\tau, y) \to +\infty, \text{ as } \tau \to +0, \tag{4.28}$$

uniformly on $y \in \mathbf{R} \times \mathbf{S}^d \times M_1 \times \ldots \times M_n$.

Proof. We denote $(M_{-1}, g^{(-1)}) = (\mathbf{R}, -dt \otimes dt)$ and $(M_0, g^{(0)}) = (\mathbf{S}^d, d\Omega_d^2)$. Due to the assumption of the proposition, the flatness of (-1)-space and the relations

$$I[g^{(0)}] = 2d(d-1) = 2R[g^{(0)}], \tag{4.29}$$

the conditions of the Theorem ((3.48), (3.49)) are satisfied for all spaces $(M_{\nu}, g^{(\nu)})$, $\nu = -1, \ldots, n$. All scale factors $A_{\nu}(\tau)$, $\nu = -1, \ldots, n$, and their first and second derivatives satisfy the Kasner-like asymptotical conditions of the theorem (see (3.50)-(3.52)). This may be proved using the asymptotic relations (4.18), (4.19) and (4.9). Thus Proposition 5 follows from the Theorem. •

Using the equivalence (4.27) and the relation $I[\sigma^* g](\tau, y) = I[g](R(\tau), y)$, we can reformulate Proposition 5 for the metric (4.1).

Proposition 6. Let g be the metric (4.1) with the parameters satisfying (4.4), (4.5) and $a_\varphi = 0$, $(a, a_1, \dots a_n) \notin \mathcal{T}$ (see (4.26)). Let the Ricci-flat internal spaces $(M_i, g^{(i)})$, $i = 1, \dots, n$, satisfy the self-boundedness conditions (3.48). Then

$$I[g](R, y) \to +\infty, \text{ as } R \to R_0, \tag{4.30}$$

uniformly on $y \in \mathbf{R} \times \mathbf{S}^d \times M_1 \times \ldots \times M_n$.

Remark 6. Due to (3.3), (4.29) and the flatness of the t-space, I[g] does not depend on $y_j \in M_j$, $j = -1, 0$.

Remark 7. From (4.6) we obtain

$$R[g] = g^{MN} \partial_M \varphi \partial_N \varphi = a_\varphi^2 f^{-1-b} (f')^2 \tag{4.31}$$

where $f = f(r) = 1 - (R_0/r)^{d-1}$. Using (4.31), (4.10) and the relation

$$f' = (d-1) R_0^{d-1} r^{-d}, \tag{4.32}$$

we obtain for $a_\varphi \neq 0$

$$R[g](r, y) \to +\infty, \text{ as } r \to R_0. \tag{4.33}$$

4.1. Multitemporal generalization of the Tangherlini solution

Now we consider a special case of the solution (4.1)-(4.3) with $n-1$ one-dimensional internal spaces (extra times). This solution, defined on the manifold

$$M = (R_0, +\infty) \times \mathbf{R}^n \times \mathbf{S}^d, \tag{4.34}$$

reads:

$$g = -\sum_{i=1}^{n} f^{a_i} B_i dt^i \otimes dt^i + f^{b-1} dR \otimes dR + f^b R^2 d\Omega_d^2, \tag{4.35}$$

$$\exp(2\varphi) = B_\varphi f^{a_\varphi}, \tag{4.36}$$

where $f = f(R) = 1 - (R_0/R)^{d-1}$ and $R_0, B_\varphi, B_i > 0$ are constants; the parameters $b, a_1, \dots, a_n$ satisfy the relations

$$b = (1 - \sum_{i=1}^{n} a_i)/(d-1), \tag{4.37}$$

$$(\sum_{i=1}^{n} a_i)^2 + (d-1)(a_\varphi^2 + \sum_{i=1}^{n} a_i^2) = d. \tag{4.38}$$

Let us consider the case $a_\varphi = 0$ [14]. The "Tangherlini set" in this case (see (4.26))

$$\mathcal{T} = \{(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\} \subset \mathbf{R}^n. \tag{4.39}$$

consists of n points. As shown in [14], a multitemporal horizon takes place only for $(a_1, \dots, a_n) \in \mathcal{T}$. For $(a_1, \dots, a_n) \notin \mathcal{T}$ we get from Proposition 6

$$I[g](R) \to +\infty, \text{ as } R \to R_0, \tag{4.40}$$

i.e., the solution (4.35) describes a multitemporal naked singularity. (The relation (4.40) was stated previously in [14].) This singular solution is unstable under monopole perturbations: this follows from the recent (more general) result of Bronnikov et al. [18, 17, 16].

When $a_\varphi \neq 0$, we get from (4.33)

$$R[g](r) \to +\infty, \text{ as } r \to R_0. \tag{4.41}$$

It can be shown that in this case we also have a multitemporal naked singularity. It should be also noted that for the case $n = 2$ the multitemporal solutions under consideration were recently generalized in [18] to a more complicated model.

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References

[1] Yu.S. Vladimirov, "Physical Space-Time Dimension and Unification of Interactions", Moscow Univ. Press, 1987 (in Russian).

[2] Yu.S. Vladimirov, "Space-Time: Explicit and Hidden Symmetries", Nauka, Moscow, 1989 (in Russian).

[3] V.N. Melnikov, in: "Itogi Nauki i Tekhniki. Classical Field Theory and Theory of Gravity", vol. 1: Gravitation and Cosmology. Moscow, VINITI, 1991, p. 49 (in Russian); preprint CBPF-NF-051/93, Rio de Janeiro, Brazil, 1993; in: "Cosmology and Gravitation", ed. M. Novello, Edition Frontieres, Singapore, 1994, p. 147.

[4] M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory", Cambridge University Press., Cambridge, 1987.

[5] V.D. Ivashchuk and V.N. Melnikov, Int. J. Mod. Phys. D3 (1994), 795.

[6] U. Bleyer, V.D. Ivashchuk, V.N. Melnikov and A.I. Zhuk, Nucl. Phys. B 429 (1994), 177.

[7] V.D. Ivashchuk and V.N. Melnikov, Class. Quantum Grav. 12 (1995), 809.

[8] V.D. Ivashchuk and V.N. Melnikov, Gravitation and Cosmology 1 (1995), No.2, 133.

[9] V.D. Ivashchuk, Phys. Lett. A170 (1992), 16.

[10] F.R. Tangherlini, Nuovo Cim. 27 (1963), 636.

[11] K.A. Bronnikov and V.D. Ivashchuk, Abstr. 7th Soviet Grav. Conf, Erevan, EGU, 1988, p. 156 (in Russian).

[12] K.A. Bronnikov, V.D. Ivashchuk and V.N.Melnikov, Proc. 7th Soviet Grav. Conf, Erevan, EGU, 1988, p. 405.

[13] S.B. Fadeev, V.D. Ivashchuk and V.N. Melnikov, Phys. Lett. A161 (1991), 98.

[14] V.D. Ivashchuk and V.N. Melnikov, Class. Quantum Grav. 11 (1994), 1793.

[15] V.A. Berezin, G. Domenech, M.L. Levinas, C.O. Lousto and N.D. Umerez, Gen. Relat. Grav. 21 (1989), 1177.

[16] K.A. Bronnikov, U. Bleyer, V.N. Melnikov and S.B. Fadeev, Astron. Nachr. 315 (1994), 399.

[17] K.A.Bronnikov and V.N.Melnikov, Ann. Phys. (N.Y.) 239 (1995), 40.

[18] K.A.Bronnikov, Grav. & Cosmol. 1 (1995), 67.