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MATHEMATICS OF FRACTIONAL-DIMENSION SPACES

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On the basis of the Hausdorff measure the concept of a fractional-dimension space is introduced. A physical realization of these mathematical spaces is some homogeneous medium (also called a fractal). Mathematics of fractal, or fractal mathematics (that is, fractional derivatives and differential equations with them, integral equations with fractional integrals) is considered. A general principle of obtaining differential equations with fractional derivatives is established for the description of physical process. The problem of finding groups of symmetry (by analogy with the Lorentz group) for these equations is posed.

Natura non facit saltus.
(Nature makes no jump.)

. . . It is necessary to realize that a fruitful axiomatics cannot appear without a rich model, a classical theory, rich in important constructions and deep in sense.

K. Maurin[1]

The concept of a space of integer dimension n ($n = 0, 1, 2, ..., 8, ...$) is one of the basic concepts in mathematics and has gained widespread acceptance in physics and many other sciences. Recall, e.g., Minkowski's four-dimensional space-time or Riemannian n-dimensional spaces. But besides integers, there exist fractions. How poor mathematics would have been without them! And are fractional-dimension spaces possible? At first sight, such a formulation of the problem may seem absurd, but, mathematically, this fancy is not unreasonable. Two questions challenge the mathematicians and philosophers: do fractional-dimension spaces exist? For which purpose are they needed?

Unfortunately, it is not always that the use of one or another hypothesis or theory can be evaluated at once in mathematics. Sometimes, decades and even centuries had passed before a mathematical theory gained a vital importance. For example, such curves as the ellipse, hyperbola and parabola, being conic sections, were used in applied problems only fifteen hundred years after their discovery. The binary number system was used as the basis of computer operation 70 years after it had been devised. The finite difference method, belonging to the genial Euler, began to enjoy wide application only with the advent of powerful computers.

Using the logic of constructing integer dimension spaces, we will try to "seek out" fractional dimension spaces. We start from nonstrict visual notions, to be refined later on. Let us take a sheet of roofing iron. This is a two-dimensional object, as its thickness may be neglected. Let the iron sheet begin to rust, and it be spoilt to an extent that it left a fragment of negligible width — a one-dimensional object. We obtained at once a one-dimensional object from a two-dimensional one. But the process was prolonged, and we may assume that the sheet which was not so strongly spoilt, would be described by geometric objects of the dimension $1 + \alpha$, where α is a number within the range between zero and unity. In general, any flat object, which is thinned out or perforated, may be considered intermediate between two- and one-dimensional ones, regardless of the way of its thinning-out. It would be natural to suppose its dimension to be a number in the range between unity and two. Let us now offer an example, inverse in some sense, which proved the first to give impetus to considering, by the mathematicians, geometric objects of fractional dimension. So, the topographers needed to measure England's coast line length, which proved to grow with diminishing the measurement scale. This is easily explicable on the basis of the necessity to take into account more and more shallow creeks and capes while diminishing the scale.

[1]Translated from Russian

The growth of the length was so considerable that one should not speak of a line but an object whose dimension falls within the range between unity and two. Therewith Norway's coast line is more irregular and close to a flat object than England's. Abstracting from specific examples, we may assume that there exist objects having dimension, say, 1.6. These objects can be obtained in two ways: by "thinning out" a two-dimensional object and by increasing the dimension of a one-dimensional object (e. g., by its twisting).

Similarly one can imagine objects of dimension $n = 2+\alpha$ and, generally, $n = np+\alpha$, where p is an integer, and α satisfies the condition $0 \leq \alpha \leq 1$.

The ice of a frozen pond, being hit by a man standing on it, becomes covered with a net of cracks. The cracked ice surface will have a dimension $1+\alpha$, the next layer will be less spoilt, its dimension is $2+\gamma, 0 \leq \alpha, \gamma \leq 1$, the deeper layers conserve their dimension 3. A further mathematical formalization will depend on the way of "thinning out" the $n-$dimensional $(n = p+1)$ set (or increasing the dimension of a p-dimensional one). One can certainly obtain spaces of the dimension $p_1 + \alpha (0 \leq \alpha \leq 1)$ less than p from a space of integer dimension $p+1$. But it will suffice to consider a transition from $p+1$ to $p+\alpha$, since a space of dimension $p_1+\alpha$ can be obtained from a (p_1+1)-dimensional space as well.

These considerations are tentative. They must dispose the reader, provide a possibility of his or her realizing the necessity to build a rigorous theory and catch its meaning. We consider this point to be of utmost importance, since many people who do not understand mathematics, actually do not understand the naturalness of the origin and formulation of mathematical problems. The authors of special literature often pay insufficient attention to such clarifications, which explains little public interest in their books.

To construct a rigorous mathematical theory of fractional dimensions, it is necessary to introduce the notions of Hausdorff's dimension and measure. Hausdorff is an Austrian mathematician of late 19th – early 20th century. To clarify this mathematical construction, consider a simple example. Take the segment $[0;1]$, i.e., the whole set of points from zero to unity, including its ends. Divide the segment by the points 1/3 and 2/3 into three equal parts and delete the middle one without its ends. Divide each of the remaining segments into three equal parts and again delete the middle ones (Fig.1). Continue this process infinitely many times.

What is the measure (length) of the part that remained after deleting? After deleting one third there remained two thirds of the original interval $1-\frac{1}{3}$. After the second deletion the length of the remaining intervals will be $1-1/3-2/3^2$, after the n-th step – $1-(1/3)[1+(2/3)+(2/3)^2+...+(2/3)^n]$. Proceeding to the limit $n \to \infty$, we obtain that the remaining part measure is equal to $1-(1/3)[1/(1-2/3)] = 0$. The set constructed in this way is called the Cantor set after Cantor, German mathematician of the 19th century, who contributed significantly to the development of the set theory. The Cantor set is so thinned out that its measure is equal to zero. This is difficult to understand, since the set is nonempty.

Figure 1:

In 1918 Hausdorff proposed another rule of measuring segments and sets. Following this rule in the calculation, we shall consider the deleted intervals to have lengths, instead of $1/3,\ 1/3^2, \ldots,\ 1/3^n$, equal to $(1/3)^\alpha,\ (1/3^2)^\alpha, \ldots,\ (1/3^n)^\alpha$ where α is a so far indefinite number. For different α the Cantor set may be associated with different quantities:

1. If α is such that $2/3^\alpha < 1$, then for $n \to \infty$ the total length of the remaining intervals vanishes and we obtain nothing new.

2. If $2/3^\alpha > 1$, then the set, belonging to the interval [0;1] will have an infinite length, which is absurd.

3. Finally, the most interesting case, when $2/3^\alpha = 1$, i.e., $\alpha = \ln 2/\ln 3$ and the length of the Cantor set is equal to unity. The number $\alpha = \ln 2 \ln 3 \approx 0.6$ is called the dimension of the Cantor set. The notion of Hausdorff's dimension allowed us to distinguish between an empty set and a set of dimension less than unity. The interval $[0;1]$ may be thinned out by following another rule, then the dimension of the space obtained would be another. The denser is this set, the closer is its dimension to its measure equal to 1. The Hausdorff measure and dimension theory extends the construction presented for the Cantor set. This theory allows not only sets of the dimensions n and $n+1$, but ones of the dimension $\alpha, n < \alpha < n+1$, unmeasurable before, to be distinguished.

Now we will construct from a unit length interval a set which is greater than one-dimensional. At the first step $(n = 1)$, in the middle of the segment, an interval of length 1/3 is cut out, and on its place a base-free equilateral triangle is built. At the next step, an interval of length $(1/3)^2$ is cut out on each of the four segments obtained, and the above construction is repeated (Fig.2).

Figure 2:

This procedure is repeated n times. The constructed set is called Koch's triad figure, or Koch's broken line. What is the length L of this line? Since after each step it increases by a factor of 4/3, at the n-th step we obtain $L_n = (4/3)^n$. It is evident that the conventional measuring procedure this time results (for $n \to \infty$) in an infinite length set. Hausdorff's dimension equal to $\ln 4/\ln 3 \approx 1.263$ helps us to get a way out.

We have presented two classical mathematical examples, wherein the dimension of a one-dimensional set decreased (the Cantor set) but did not vanish, and increased (Koch's curve) but did not reach the next integer dimension. The principles of constructing these sets were similar in many respects. Both of them are geometric models of fractals. By a fractal (fractal set) one conventionally means a self-similar (in the sense of scale change) set whose dimension differs from the topological one (in the above case the topological dimension was equal to unity). We shall understand the concept of fractals more widely. In particular, we shall allow a thinning-out of different parts of a set, following different rules, then the local dimension becomes variable: $\alpha = \alpha(x)$, $0 \le \alpha(x) \le 1$. If we thinned out not a segment, but the whole number axis, then, instead of a one-dimensional space, we would obtain a space whose dimension, possibly variable, does not exceed unity.

Let us perform similar constructions for a bounded set M located in the plane. Choose a finite subset of points M_1 of the set M and cover it with squares η on side (?). The squares may therewith intersect and contain points not belonging to the set M. Count the total area covered by the squares, taking each of them into account only once: $S_\eta = \sum_{j=1}^{n} {\eta_j}^2, \eta_j = \eta, j = \overline{1,n}$. As known from mathematical analysis, a finite set M_1 can be chosen in such a way that its covering will be a covering of the whole set M as well. The number $\mu(M) = \inf_{n\to\infty} S_r$ (infinum), the smallest lower bound for all possible coverings allowing one to get rid of the points not belonging to the set M, is called a measure (an outer measure) of the set M. For example, if our set M is a rectangle a and b on sides, then its measure $\mu(M)$ is equal to ab. Hausdorff's idea consists in considering an expression of the form

$$\rho = \min_{\alpha} \inf_{\eta} \sum_{j} \eta_j^{\alpha} \not\equiv 0, \qquad \eta_j = \eta \tag{1}$$

instead of S_η. The number α_0, at which the minimum is reached, is called Hausdorff's dimension of the set M. It is evident that the dimension defined in such a way will not necessarily be an integer.

Consider two simple examples of fractional-dimension sets. Divide a unit square into nine identical squares and cut out the middle one without its bounds. Do the same on each of the remaining eight squares. Performing this procedure ad infinum, in the limit we obtain a set which is called the Sierpiński carpet. It is not difficult to count that after the $n-$step there will remain 2^{3n} squares of the total area $(8/9)^n$, vanishing as $n \to \infty$, i.e., the Serpiński carpet area, like the Cantor set length, is equal to 0. There is no wonder, since one may consider the Sierpiński carpet to be a Cantor set of a square. Measuring it according to Hausdorff, we can speak of the Sierpiński carpet dimension α, apart from the zero area (measure). If $\alpha = \alpha_0$ so that $2^3/3^{2\alpha_0} = 1$, then this dimension is equal to $\ln 8/\ln 9$, i.e., the number not exceeding unity. One might also perforate a cube by the same "middle" method of deleting the central cube of the 27 equal ones. Then the space composed of these "Sierpiński cubes" will have a dimension less than 3. The three-dimensional space may be thinned out variously in its different parts. Then the dimension of the set obtained will be variable. In particular, there can remain sections which conserve the space dimension 3. Such constructions may be carried out for $(p+1)-$dimensional integer-dimension space, $p = 3, 4, ...,$ as well.

In our considerations the number of elementary cubes (segments, squares) tended to infinity. In practice, an infinite number of steps never occurs. For example, by a Cantor set is meant a large but finite number of small sections lying within the interval [0;1]. One may also count the dimension of a "practical" but not mathematical Sierpiński carpet. The transition from the infinite to the finite is a general principle for applications of mathematics. Integrals are replaced by finite integral sums, infinite series by finite sums, derivatives by finite-difference ratios, etc.

In integer multidimensional spaces there exists an algebra - the set of operations on elements of these spaces. It is necessary to construct an algebra for fractional-dimension sets as well. We have mentioned operations on vectors, the elements of integer multidimensional linear spaces. Recall that the set of elements $x, y, z, ...$ forms a linear space if on this set the operations of vector addition and multiplication by a number, are defined and satisfy the necessary axioms. The definition of operations on a discontinuous set requires some clarification. In search for a fundamental solution, we again turn to visual representations. Imagine a man who makes his way through a marshland by stepping on hummocks. The alternation of solid soil and water may be considered a model of a fractal set. The man jumps from one hummock (point A) to another (point B). It would be natural to consider the vector $\overrightarrow{AB}$, connecting these hummocks, as a possible element of the fractional-dimension space. Further, reaching the next hummock (point C), we obtain the vector $\overrightarrow{BC}$. If we succeed in jumping from A right away to C, then such a jump (the vector $\overrightarrow{AC}$) will be a result of two successive jumps — a sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. But if the jump was not successful, then a sum of the vectors cannot be determined. If a bar laid on the hummocks A and B reaches the hummock D, one may consider the vector $\overrightarrow{AB}$ to be lengthened several times, i.e., multiplied by a number.

Such a simple illustration suggests the general principle of constructing fractional dimensions (fractal spaces). Consider a set M of dimension $n+\alpha, 0 \leq \alpha \leq 1$ in a space of dimension $n+1$. We shall say that the vector $\overrightarrow{AB}$ belongs to the set M if the vector ends, i. e. the points A and B, belong to this set. Therewith all inner points of the vector or some of them may be outside the fractal set. By a linear space L of a fractional dimension α is meant a set of those vectors of the set M on which the operations of addition and multiplication may be defined, i. e. a vector with the ends belonging to the fractional set M may be obtained as a result of performing these operations. The system of axioms will be the same in the constructed space, but one should keep in mind a specific meaning assigned to them. To construct a metric space on the basis of the constructed "affine" one L, it is necessary to associate each vector $\overrightarrow{AB}$ belonging to it with a length, i.e., the Hausdorff measure of the points of the set M belonging to the vector $\overrightarrow{AB}$ of the space P.

There arises a question, natural for mathematicians: what should be the mathematical analysis of fractional dimension spaces? The basis of the classical analysis of integer spaces is the Taylor formula

$$y = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2 + ... + \frac{f^{(n)}(x - x_0)}{n!}(x - x_0)^n + ... \qquad (2)$$

The function $y = f(x)$ is represented in the neighbourhood of the point x_0 in the form of a power series with integer exponents with respect to the argument increment, i.e., the measure of the segment $[x;\ x_0]$. A more general formula is valid in fractional-dimension spaces:

$$y = f(x_0) + C_{\alpha_1}(x - x_0)^{\alpha_1} + C_{\alpha_2}(x - x_0)^{\alpha_2} + \cdots + C_{\alpha_n}(x - x_0)^{\alpha_n} + \cdots, \quad \alpha_1 < \alpha_2 < \alpha_n < \dots . \qquad (3)$$

In the formula proposed the segment length is replaced by its Hausdorff measure $(x - x_0)^{\alpha_1}$, where α_1 is the fractional space dimension. The coefficients C_{α_1} are α_1-order fractional derivatives of the function $f(x)$ with certain absolute coefficients. In a special case it may be $\alpha_2 = 2\alpha_1, \alpha_3 = 3\alpha_1, \dots$ and then at $\alpha_1 = 1$ we obtain the conventional Taylor series. The considerations presented allow one to assume that mathematical analysis of fractional-dimension spaces is "fractional" (fractional derivatives, fractional integrals and fractional order differential equations).

What new possibilities does "fractional" mathematics provide to describe physical processes? Consider an inhomogeneous bar made from two-composite material (foreign additions are embedded into a basic material). Then the line element in the basic material should not be measured conventionally in the form dx, but following Hausdorff in the form $(dx)^\alpha$, where α satisfies the condition $0 < \alpha < 1$. Consider the process of heat transfer in such a bar. The amount of heat ΔQ passing through the bar cross-section during the time δt is given by the formula

$$\Delta Q = k\frac{\partial^\alpha u(x)}{\partial x^\alpha}\Delta S \Delta t,$$

where k is a thermal conductivity coefficient of the basic material, $u(x)$ is a function setting the temperature distribution in the bar, ΔS is the bar cross-section area. Recall that in the classical case the Newton formula

$$\Delta Q = k\frac{\partial u(x)}{\partial x}\Delta S \Delta t,$$

is valid, the one ignoring composite additions in the basic material. Many of the equations of mathematical physics are descriptions of a balance of some substance — heat, matter, energy, etc. If we introduce, reflecting an inhomogeneous structure of the material, fractional derivatives into the balance equations, thereby conserving the structure of the equations contained in the coefficients, then we obtain the equations on fractals. For example, the one-dimensional diffusion equation on fractals has the form

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}, \qquad 0 < \alpha \leq 1.$$

The general form of the three-dimensional diffusion equation is

$$\frac{\partial u}{\partial t} = a_1^2 \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + a_2^2 \frac{\partial^{2\beta} u}{\partial y^{2\beta}} + a_3^2 \frac{\partial^{2\gamma} u}{\partial z^{2\gamma}}, \quad 0 < \alpha, \beta, \gamma \leq 1.$$

This equation provides a possibility of various structure of the material along the three axes. In general, the indices α, β and γ may be variable, if the fractal set, where the equation is considered, has a variable dimension, i. e. was obtained by inhomogeneous embedding of an admixture.

Now we would indicate a possibility of differential equations with a fractional order of differentiation with respect to time and their usefulness in physics. Imagine some physical process, e.g., diffusion, such that the concentration of some substance at a given point is a function of time, $u(t, x)$. If $0 \leq t \leq 1$, the function u is defined on a Cantor set, and on its supplement it does not exist or equals zero. Then at a point t_0 belonging to the Cantor set and being a condensation point, the fractional derivative $\partial^\alpha u(x)/\partial t^\alpha$, $0 < \alpha < 1$, exists and makes sense, since one should take as a measure not dt (as usual), but dt^α. It is this circumstance that leads to the appearance of differential equations with fractional derivatives in time. In the general case, the fractional derivative index α may be variable and even u-dependent. Recall that Special Relativity is connected with the string vibration equation

$$\frac{\partial^{2\alpha} u(x)}{\partial t^{2\alpha}} = a^2 \Delta_x u.$$

This relation manifests itself in the form of the Lorentz group with respect to arguments t and x, leaving this equation invariant. Since it is now clear that the equation

$$\frac{\partial^{2\delta} u(x)}{\partial t^{2\delta}} = a_1^2 \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + a_2^2 \frac{\partial^{2\beta} u}{\partial y^{2\beta}} + a_3^2 \frac{\partial^{2\gamma} u}{\partial z^{2\gamma}}, \quad 0 < \delta, \alpha, \beta, \gamma \leq 1$$

has a real physical meaning in composite media, there arises a problem of studying analogues of the Lorentz group for such equations.

At present the author is unaware of any paper where equations with a variable order of differentiation would be considered. We believe that such equations, emerging in the course of the progress of mathematics are being necessary in applied problems, will be the most important object of studies in the nearest future. “Fractional mathematics”, developing since Leibniz according to the intra-mathematical laws, became absolutely necessary only now. In that the remarkable capability of self-development of mathematics is revealed. After a long period of development according to its own laws, seemingly independent of practice, there necessarily comes a time when either other sciences or practice cannot do without these “unneeded” results. Such a time has come in fractal theory. The further development of this theory, most important for technology, is impossible without the formalism of fractional mathematics.

Let us mention some more details of possible practical applications of fractional-dimension spaces. Studies of multidimensional and infinite-dimensional integer spaces have rapidly promoted physics. Special and general relativity and quantum mechanics have been created. At the intuitive level, the notion of multidimensionality has long been used both in science and men’s everyday life. The same had also happened to fractal dimension spaces long before a rigorous mathematical theory was created. The chemists, geologists and biologists have long been working with discontinuous, inhomogeneous and porous materials — the composite materials, activated carbon, various rocks. But the activated carbon is a physical realization of a fractional-dimension space whose dimension is less than three. A bacterial colony on a human body may be considered to be a fractional-dimension space as well. Information on the Hausdorff measure and dimension of this space and its different parts is important for medical men and biologists. A lot of similar examples may be provided.

The above considerations indicate that mathematical models in materialized spaces of fractional dimensions, which are fractals, are constructed on the basis of fractional mathematics. Since the fractals become of utmost importance in technology, the fractal mathematics as a whole and the fractional-dimension space theory in particular acquire top significance. This process is already taking place in the theory and practice of fractals [1,2].

References

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