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p-ADIC MATHEMATICAL PHYSICS AND SPACE-TIME

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The possibility of construction a physical theory with a non-Archimedean space-time is discussed. Physical motivations are based on the fact that in quantum gravity it is impossible to measure distances smaller than the Planck length. Representations of commutation relations over finite- and infinite-dimensional p-adic symplectic spaces and related topics are considered.

1. Philosophy

1.1. Introduction

We discuss here a mathematical model for the physical space-time. In our approach the last means a triple system (triad) (M, G, N), where M is a model of the space-time, G is a system of geometric axioms and N is a number system naturally corresponding to G. A well-known example of such a triad is (R^4, E, R), where R^4 is the four-dimensional Euclidean space, E is the set of axioms of Euclidean geometry and R is the field of real numbers.

All the components in our triad are closely connected with each other and the second component, the system of geometric axioms, plays a fundamental role.

1.2. Measurability

The basic point of our discussion of geometric axioms is measurability. We begin with an analysis of some fundamental physical principles.

The situation with measurability in classical physics is very simple: all distances can be measured with no restriction.

In quantum physics the situation is more complicated because of the Heisenberg uncertainty principle. But nevertheless all distances can be measured. The greater is the momentum of a test particle, the less is the uncertainty in measuring a particle position. That means that all distances can be measured by applying sufficient energy.

Let us consider now the situation in quantum gravity. If Δx is an uncertainty in a distance measurement, the following inequality is valid:

$$\Delta x \geq l_{pl} = \sqrt{\frac{hG}{c^3}} \tag{1}$$

Here l_{pl} is the so-called Planck length (approximately 10^{-33} cm), h is the Planck constant, G is the gravitational constant and c is the velocity of light. As follows from Eq. (1), a measurement of distances smaller that the Planck length is impossible. The last statement will be a basic point in our analysis of geometrical axioms.

1.3. Non-Archimedean space-time

1.3.1. Geometric axioms

It is easy to see that the inequality (1) comes into conflict with the axioms of Euclidean geometry, namely, with the so-called Archimedean axiom. Indeed, according to the Archimedean axiom any given large segment on a straight line can be surpassed by successive addition of small segments along the same line. It means we can measure distances as small as we want. This leads to non-Archimedean geometry of space-time at small distances. The possibility of constructing a non-Archimedean geometry was pointed out by Veroneze and Hilbert.

Thus we find out a candidate for the second component of the triad (M, G, N). Namely, G is a system of axioms of a non-Archimedean geometry, that is the geometry in which the Archimedean axiom is not valid.

1.3.2. Number system

There is a strong correlation between geometric axioms and a number system providing a possibility of expressing geometric axioms in an analytic way using coordinates.

In order to construct a number system corresponding to a non-Archimedean geometry, we need some additional requirements. Let us first point out that any result of a physical experiment is expressed in terms of rational numbers. We never deal with irrational numbers, i.e., infinite nonperiodic decimals, in a real physical experiment. Thus our number system must contain the field Q of rationals. We also want to operate with the notion of distance, so we need a normed number field as a number system. And the last requirement, from the considerations of convenience: our number

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system must be complete with respect to topology generated by the norm.

Thus, the sought number system should be a complete normed field containing the field of rationals as a subfield.

Let us consider norms in the field of rationals. A norm is a real-valued function $\mid x \mid$ satisfying the following properties:

$$\begin{array}{ll} 1) & \mid x \mid \geq 0, \quad \mid x \mid = 0 \quad \Leftrightarrow \quad x = 0, \\ 2) & \mid xy \mid = \mid x \mid \mid y \mid, \\ 3) & \mid x + y \mid \leq \mid x \mid + \mid y \mid \end{array}$$

for any rationals x and y.

The standard absolute value ($\mid \pm x \mid = x, \quad x \in Q, \quad x \geq 0$) gives an example of a norm.

Another example is given by the following construction. Let p be a prime. The norm $\mid x \mid_p, \quad x \in Q$ is defined by the formula

$$\mid 0 \mid_p = 0, \quad \mid x \mid_p = p^{-\gamma},$$

where an integer $\gamma = \gamma(x)$ is determined from the representation

$$x = p^{\gamma} \frac{m}{n},$$

integers m and n are not divisible by p. The norm $\mid x \mid_p$ is called the p-adic norm.

The following Ostrowski theorem is valid.

Theorem 1. The norms $\mid x \mid$ and $\mid x \mid_p, \quad p = 2, 3, ...$ exhaust all nonequivalent nontrivial norms on the field of rationals.

The p-adic norm satisfies the so-called ultrametric inequality

$$\mid x + y \mid_p \leq \max \mid x \mid_p, \mid y \mid_p$$

which is stronger than the standard triangle inequality. This leads to the following property of the p-adic norm: for any integer n and rational x we have $\mid nx \mid_p \leq \mid x \mid_p$. It means that the p-adic norm is non-Archimedean.

The completion of rationals with respect to the p-adic norm gives the p-adic number field Q_p. Thus we construct the third component of our triad (M, G, N), $N = Q_p$. Note that, as follows from Theorem 1, if we want to consider a non-Archimedean number system "based on" rationals, we have to consider the p-adic number field and only it.

1.3.3. Model of space-time

It is clear from the previous discussion, what can be considered as a candidate for the first component of our triad (M, G, N). It can be the four-dimensional vector space Q_p^4 over the p-adic number field. In this case both space and time have a non-Archimedean nature. It is also interesting to consider $Q_p^3 \times R$, which corresponds to models with non-Archimedean space and Archimedean time. In what follows we consider the first, "purely non-Archimedean" case.

2. Apparatus

2.1. Introduction

In this section we finish with philosophy and try to do mathematics. From the mathematical point of view just the Ostrowski theorem provides a sufficient reason for studying the opportunity to construct mathematical physics based on p-adic numbers.

The aim of this section is not to give an overview of methods of p-adic mathematical physics, but just to show some specific constructions and to prove the possibility of constructing a substantial and nontrivial theory.

2.2. p-adic phase space and classical dynamics

As a phase space, we consider a symplectic (finite- or infinite-dimensional) space over p-adic numbers, that is, the pair (V, B), where V is a vector space over Q_p and B is a nondegenerate skew-symmetric p-adic-valued bilinear form over V. Let $Sp(V, B) \equiv Sp(V)$ denote the group of linear automorphisms of V which preserve B, the so-called symplectic group.

A classical dynamic model is given by the pair $((V, B), E_t, t \in \Omega)$, where E_t is a one-parametric subgroup of $Sp(V)$ and Ω is a subgroup of Q_p. Here the parameter t plays the role of time.

Example. Let (V, B) be the two-dimensional symplectic space over Q_p, $p \neq 2$, $\Omega = \{t \in Q_p, \mid t \mid_p \leq 1/p\}$ and E_t be given by the formula

$$E_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

This dynamic system describes a one-dimensional p-adic harmonic oscillator. The functions $\sin t$ and $\cos t$ are defined by their power series expansions by analogy with the real-number case. These series converge in Ω.

2.3. p-Adic Heisenberg group and commutation relations

Let χ denote an additive character of Q_p of rank 0, that is, a function from Q_p into the set T of complex numbers of modulus 1 which satisfies the properties:

$$\begin{array}{ll} \chi(x + y) = \chi(x)\chi(y), & x, y \in Q_p, \\ \chi(x) = 1 \quad \Leftrightarrow \quad x \in Z_p, & \end{array}$$

where $Z_p = \{x \in Q_p, \mid x \mid_p \leq 1\}$ is the set of p-adic integers. Note that χ is an analog of $\exp(2\pi i)$ defined for real numbers.

Definition 1. The Heisenberg group $\mathcal{V}$ of the symplectic space (V, B) over Q_p is defined to be the set of pairs

$$\mathcal{V} = \{(\alpha, x), \alpha \in T, x \in V\}$$

with the multiplication

$$(\alpha, x)(\beta, y) = \Big(\alpha\beta\chi\big(1/2B(x, y)\big), x + y\Big).$$

Note that the center $\mathcal{C}$ of $\mathcal{V}$ consists of the elements

$$\mathcal{C} = \{(\alpha, 0), \alpha \in T\} \simeq T,$$

thus $\mathcal{V}$ is a central extension of the additive group V by T.

Definition 2. A Weyl system on (V, B) is defined to be a pair (H, W), where H is a complex Hilbert space, and W is a mapping from V to the family of unitary operators on H satisfying the condition (the Weyl relation)

$$W(x)W(y) = \chi(1/2B(x,y))W(x+y).$$

It is clear that a Weyl system is a unitary representation of the Heisenberg group $\mathcal{V}$, identical on the center $\mathcal{C}$. Weyl systems are also called representations of the commutation relations in the Weyl form. In other words, an irreducible Weyl system defines quantization.

2.4. Lattices

Let (V, B) be a symplectic vector space over Q_p of arbitrary dimension.

Definition 3. A subset L of V is called a lattice if: (1) L is a Z_p-submodule of V; (2) L generates V ($\forall x \in V \exists s \in Q_p \backslash \{0\} : sx \in L$); (3) L does not contain nonzero subspaces of V.

Definition 4. Let L be a lattice in (V, B). The dual lattice L^* is defined by the formula:

$$L^* = \{x \in V : B(x,y) \in Z_p \forall y \in L\}.$$

If we have $L = L^*$, than L is a self-dual lattice.

Henceforth, a “lattice” means a “self-dual lattice”.

Denote by $\Lambda = \Lambda(V, B)$ the set of all self-dual lattices in (V, B) and define the functional $d : \Lambda \times \Lambda \to Z_+ \cup \{\infty\}$ by the formula

$$d(L_1, L_2) = \frac{1}{2} \log_p[(L_1 + L_2) : (L_1 \cap L_2)], \qquad (2)$$

where the expression in the square brackets denotes the order of the quotient group $(L_1 + L_2)/(L_1 \cap L_2)$.

Theorem 2. Let $L_1, L_2, L_3 \in \Lambda,\ g \in Sp(V)$. Then

1) $d(L_1, L_2) \geq 0, \quad d(L_1, L_2) = 0 \Leftrightarrow L_1 = L_2$,
2) $d(L_1, L_2) = d(L_2, L_1)$,
3) $d(L_1, L_3) \leq d(L_1, L_2) + d(L_2, L_3)$,
4) $d(gL_1, gL_2) = d(L_1, L_2)$.

If dim $V < \infty$, then $d(L_1, L_2) < \infty \quad \forall L_1, L_2$ and d defines an integer-valued metric on Λ, invariant under the action of $Sp(V)$ on Λ.

2.5. Weyl systems

Definition 5. A quasicharacter of a lattice L is defined to be the function $\kappa_L : L \to T$, which satisfies $\forall x, y \in L$ the relation:

$$\kappa_L(x)\kappa_L(y) = \chi(1/2B(x,y))\kappa_L(x+y).$$

Definition 6. An L-quasicharacter of the symplectic space (V, B) is defined to be the mapping $\eta_L : V \to T$ which satisfies $\forall z \in V, x \in L$ the relation

$$\eta_L(z+x) = \chi(1/2B(z,x))\eta_L(z).$$

Definition 7. Let L be a self-dual lattice and κ_L be a quasicharacter of L. An L-vacuum vector ϕ_L of the Weyl system (H, W) is defined to be a unit vector satisfying the following condition $\forall x \in L$:

$$\kappa_L(x)W(x)\phi_L = \phi_L.$$

Proposition. The mapping $f_L : V \to H$ defined by the formula

$$V \ni x \to f_L(x) = \eta_L(x)W(x)\phi_L \in H$$

is constant on each coset in V/L.

The mapping f_L thus defines the mapping $\tilde{f}_L : V/L \to H$.

Definition 8. The range of $\tilde{f}_L$ is called the system of L-coherent states of the Weyl system (H, W).

2.5.1. Finite-dimensional case

In this subsection we consider Weyl systems over a finite-dimensional symplectic space (V, B). The following two theorems show that Definitions 7 and 8 are natural and indeed give the notions of a vacuum vector and coherent states.

Theorem 3. For any continuous Weyl system and self-dual lattice there exists a vacuum vector.

Theorem 4. Let L be a self-dual lattice in (V, B) and (H, W) be a continuous Weyl system over (V, B). The following statements are equivalent:

(1) The Weyl system (H, W) is irreducible.
(2) The vacuum subspace of (H, W) is one-dimensional.
(3) The system of L-coherent states forms an orthonormal basis in H.

Corollary. Any two continuous irreducible Weyl systems over finite dimensional space are unitary-equivalent.

Example. Let L be a self-dual lattice in (V, B). The complex Hilbert space H_L consists of functions $f : V \to C$ possessing the property

$$f(x+u) = \chi(1/2B(x,u))f(x)$$

$\forall x \in V, u \in L$. The scalar product is defined by the formula

$$(f, g) = \sum_{\alpha \in V/L} f(\alpha)\bar{g}(\alpha).$$

The operators W_L are defined by the relation

$$(W_L(x)f)(y) = \chi(1/2B(x,y))f(y-x), \quad f \in H_L.$$

It is easy to see that (H_L, W_L) is an irreducible Weyl system with the following vacuum vector:

$$\phi_L(x) = \begin{cases} 1, & x \in L, \\ 0, & x \notin L. \end{cases}$$

As follows from Theorem 4, the Weyl systems (H_{L_1}, W_{L_1}) and (H_{L_2}, W_{L_2}) are unitary-equivalent. The intertwining operator $F_{L_1L_2} : H_{L_1} \to H_{L_2}$ is given by the formula

$$(F_{L_1L_2}f)(u) = \rho(L_1, L_2) \sum_{\alpha \in L_2/(L_1 \cap L_2)} \chi(1/2B(\alpha, u))f(u+\alpha). \quad (3)$$

Here $\rho^{-2}(L_1, L_2)$ denotes the order of the quotient group $L_1/(L_1 \cap L_2)$.

2.5.2. Maslov index

Let $L_1, L_2, L_3 \in \Lambda$. Then the corresponding Weyl systems are unitary-equivalent and the operator $F = F_{L_1L_3}F_{L_3L_2}F_{L_2L_1}$ commutes with all operators $W_{L_1}(x)$, $x \in V$ and is thus proportional to the identity operator on H_{L_1}:

$$f = \mu(L_1, L_2, L_3)Id.$$

The complex number $\mu(L_1, L_2, L_3)$ will be called the Maslov index of the triad of self-dual lattices.

Theorem 5. Let $L_1, L_2, L_3, L_4 \in \Lambda$. The following statements are valid.

(1) $\mu(L_1, L_2, L_3) = \mu(gL_1, gL_2, dL_3) \quad \forall g \in Sp(V)$.
(2) $\mu(L_1, L_2, L_3) = 1$ if at least two lattices in the triad coincide.
(3) $\mu(L_1, L_2, L_3)$ remains the same under an even permutation of lattices and transfers to conjugate under an odd one.
(4) the following cocycle relation holds:

$$\mu(L_1, L_2, L_3)\mu(L_1, L_3, L_4) = \mu(L_2, L_3, L_4)\mu(L_2, L_4, L_1).$$

Two triad $(L_1, L_2, L_3), (L_1', L_2', L_3')$ are congruent if

$$d(L_1, L_2) = d(L_1', L_2'), \quad d(L_2, L_3) = d(L_2', L_3'), \quad d(L_3, L_1) = d(L_3', L_1').$$

The Maslov index has the following interpretation in terms of action of $Sp(V)$ on Λ.

Theorem 6. For two congruent triads of lattices (L_1, L_2, L_3) and (L_1', L_2', L_3') there is a symplectic transformation $g \in Sp(V)$ that maps one triad to another ($L_{1,2,3}' = gL_{1,2,3}$) if and only if $\mu(L_1, L_2, L_3) = \mu(L_1', L_2', L_3')$.

2.5.3. Infinite-dimensional case

Everywhere in this subsection (V, B) denotes a symplectic vector space over Q_p of arbitrary dimension.

Definition 9. A complex-valued functional π on the symplectic space (V, B) is called a positive functional ($\pi \gg 0$) if $\pi(0) = 1$ and

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \pi(x_j - x_i)\chi(1/2B(x_i, x_j)) \geq 0$$

for any finite subsets $\lambda_1, ..., \lambda_n \in C$ and $x_1, ..., x_n \in V$.

Positive functionals give a method to construct Weyl systems for the infinite-dimensional case.

Theorem 7. For any positive functional π on (V, B) there is a Weyl system (H, W) and a cyclic vector $\phi \in H$ such that

$$\pi(x) = (\phi, W(x)\phi).$$

Such a system is unique up to equivalence.

Example. The functional π_L defined by the formula

$$\pi_L = \begin{cases} 1, & x \in L, \\ 0, & x \notin L \end{cases}$$

is positive and, by Theorem 7, defines an L-Weyl system. The L-Weyl system is irreducible, continuous, has a one-dimensional vacuum subspace and an orthonormal system of coherent states. It gives us an analog of the Fock representation of commutation relations.

For finite-dimensional (V, B) all irreducible Weyl systems are unitary-equivalent. But it is not the fact for the case dim $V = \infty$.

Theorem 8. The L_1- and L_2-Weyl systems are unitary-equivalent if and only if $d(L_1, L_2) < \infty$.

3. Bibliographic sketch

Although this paper is designed as a review, I deliberately omit all bibliographical references. A more or less complete bibliography can be found in the book: V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, "p-Adic Analysis and Mathematical Physics", World Scientific, 1994. I would like to note the role of papers of prof. Vladimirov and prof. Volovich from Steklov Mathematical Institute, which initiated the activity in this field.

The technical part of the paper embraces only a small part of applications of p-adic numbers in mathematical physics and more or less reflects the interest and activity of the author.

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