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GEOMETRIZATION OF PERFECT FLUID IN 5-D KALUZA-KLEIN THEORY

S.S. Kokarev[1]

Yaroslavl State Pedagogical University

We give a general formulation of the problem of geometrization of matter by the scalar field $\varphi = \sqrt{-G_{55}}$ in the context of the classical 5-dimensional Kaluza-Klein theory. A method of mathematical analysis of such geometrization for the case of perfect fluid is proposed.

1. Introduction

When general relativity (GR) was built and the gravitational interaction within its frame was geometrized, Einstein himself considered this problem to be half solved: the energy-momentum tensor, appearing in the right-hand side of the field equations, in his view, should have a geometric nature too [1].

Such a possibility naturally appears within the frame of multidimensional geometric models of Kaluza-Klein type. The first (5-dimensional) version of such a theory was suggested by T.Kaluza in 1921 [2, 3]. The extra components, represented by the "scalar" sector of the multidimensional metric, have been used for a number of purposes. In the present article the possibility of describing matter sources in terms of the geometric scalar field $\varphi = \sqrt{-G_{55}}$ in the effective 4-D space-time is analyzed. Such an approach is at present worked out by Wesson and others (see References in [7, 8, 9, 10]), but some aspects of this problem were developed earlier by other authors [4, 5].

2. Statement of the problem

The starting point of our investigation are the vacuum 5-D Einstein equations, written in the form

$$^5R_{AB} - \frac{1}{2}G_{AB}\,{}^5R = 0, \tag{1}$$

where $^5R_{AB}$ and ^5R are, respectively, the 5-D Ricci tensor and curvature scalar, corresponding to the metric G_{AB}, which in the 4-D presentation has the form

$$G_{AB} = \left(\begin{array}{c|c} \tilde{g}_{\mu\nu} - (4k/c^4)\varphi^2 A_\mu A_\nu & (2\sqrt{k}/c^2)\varphi^2 A_\mu \\ \hline (2\sqrt{k}/c^2)\varphi^2 A_\nu & -\varphi^2 \end{array}\right) \tag{2}$$

The expression (2) for G_{AB} can be derived by the (1+4) splitting method applied to the 5-D manifold, outlined in the monograph [5]. The "vector" sector $G_{5\mu}$ is identified with the electromagnetic potential vector: $G_{5\mu} = (2\sqrt{k}/c^2)\varphi^2 A_\mu$, thus allowing us to treat the 5-D Kaluza-Klein model as a unified theory of gravitational and electromagnetic interactions.

In identifying the metric of the 4-D space-time section $\tilde{g}_{\mu\nu}$ with the observable 4-D metric $g_{\mu\nu}$, there is a possibility of conformal transformations of the form

$$\tilde{g}_{\mu\nu} = F(\varphi)g_{\mu\nu}, \tag{3}$$

where $F(\varphi)$ is an arbitrary function of the scalar field.

To identify the multidimensional metric components with and 4-D quantities it is necessary to impose the following conditions:

$$\frac{\partial \tilde{g}_{\mu\nu}}{\partial x^5} = 0, \qquad \text{(the cylindricity condition)} \tag{4}$$

and the restrictions on coordinate transformations

$$x'^\mu = x'^\mu(x^0, x^1, x^2, x^3); \tag{5}$$

$$x'^5 = x'^5 + f(x^0, x^1, x^2, x^3). \tag{6}$$

Such transformations, on the one hand, conserve cylindricity (4) and 4-D covariance of $\tilde{g}_{\mu\nu}, A_\mu$ and φ, and, on the other hand, are general 4-D coordinate transformations (5) and gauge transformations of vector potentials A_μ (6).

The (1+4) splitting method allows one to split the 5-D Einstein equations (1) into ten 4-D Einstein equations

$$^4\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\,{}^4\tilde{R} = \varphi^2\kappa T^{(em)}_{\mu\nu} + \frac{1}{\varphi}(\varphi_{;\tilde{\nu};\tilde{\mu}} - \tilde{g}_{\mu\nu}\tilde{\nabla}^2\varphi), \tag{7}$$

four equations corresponding to the Maxwell ones

$$\tilde{F}^{\mu\nu}{}_{;\tilde{\nu}} - \frac{2\varphi_{,\alpha}}{\varphi}\tilde{F}^{\alpha\mu} = 0 \tag{8}$$

[1] e-mail: glad@delta.yaroslavl.su

and the 15th equation that connects the gravitational and electromagnetic field invariants:

$$ {}^4\tilde{R} + \frac{3}{8\pi}\kappa\varphi^2 \tilde{F}_{\alpha\beta}\tilde{F}^{\alpha\beta} = 0. \quad (9)$$

In (7)–(9) tilded quantities correspond to the original metric $\tilde{g}_{\mu\nu}$.

Throughout the present paper, first, there is no electromagnetic field ($F_{\mu\nu} = 0$), and, second, the conformal factor in (3) is taken in the form

$$F(\varphi) = \varphi^{2n} \quad (10)$$

where n is an arbitrary real constant. Then Eqs. (7) and (9), taking into account the conformal transformation (10), can be written in the form

$$ {}^4R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}{}^4R = (1+2n)\phi_{;\mu;\nu} - (2n^2+2n-1)\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}((1+2n)\nabla^2\phi + (n^2+n+1)(\nabla\phi)^2); \quad (11)$$

$$n\nabla^2\phi + n^2(\nabla\phi)^2 - 1/6\,{}^4R = 0, \quad (12)$$

where $\phi = \ln\varphi$. Eqs. (8) are satisfied identically.

So, when all the above conditions are satisfied, the vacuum Einstein equations (1) (after (1+4) splitting) take the form of nonvacuum 4-D Einstein equations (11), (12).

Let us assume now that we have some exact solutions of 4-D nonvacuum Einstein equations with the energy-momentum tensor of perfect fluid in their right-hand side ($\kappa = 1$):

$$R_{\mu\nu} - 1/2g_{\mu\nu}R = (p+\varepsilon)u_\mu u_\nu - pg_{\mu\nu} \equiv T^{(\mathrm{hd})}_{\mu\nu} \quad (13)$$

If the scalar field ϕ and the constant n are such that

$$(p+\varepsilon)u_\mu u_\nu - pg_{\mu\nu} = (1+2n)\phi_{;\mu;\nu} - (2n^2+2n-1)\phi_\mu\phi_\nu - g_{\mu\nu}((1+2n)\nabla^2\phi + (n^2+n+1)(\nabla\phi)^2) \quad (14)$$

and Eq. (12) is satisfied, then we can say that such $T^{(hd)}_{\mu\nu}$ can be geometrized within the frame of the 5-D Kaluza-Klein theory and that the 4-D metric generated by this source, has a 5-D origin. Below we shall derive some restrictions on the geometry of the 4-D space-time and matter motions, providing the validity of Eq. (14).

3. Integrability conditions

Eqs. (14) can be considered as a set of second-order partial differential equations for the components of the gradient $\phi_{,\mu}$, provided u_μ, ε, p and $g_{\mu\nu}$ are known functions. To analyze its integrability, it is convenient to rewrite it in the following more general form:

$$\phi_{;\mu;\nu} = k\phi_\mu\phi_\nu + au_\mu u_\nu + bg_{\mu\nu} \quad (15)$$

where

$$k = -k_2/k_1; \quad a = (p+\varepsilon)/k_1;$$
$$b = \frac{-p - k_3(\nabla\phi)^2}{k_1} = c + \overline{k}(\nabla\phi)^2; \quad (16)$$
$$c = -p/k_1; \quad \overline{k} = -k_3/k_1; \quad k_1 = 1+2n \neq 0; \quad k_2 = -2n^2 - 2n + 1; \quad k_3 = 3n^2 + 3n.$$

Eq. (15) can be derived from (14) with the aid of the expression

$$\nabla^2\phi = -(1+2n)(\nabla\phi)^2, \quad (17)$$

which in turn follows from a linear combination of Eq. (12) and Eq. (11) contracted with $g_{\mu\nu}$.

The integrability conditions for the set (15) consist in the identical fulfilment of the relation

$$2\phi_{;\mu;[\nu;\lambda]} = R^{\sigma}_{\cdot\mu\nu\lambda}\phi_\sigma \quad (18)$$

if in the left-hand side the expressions from the right-hand side of (15) are inserted, with second-order covariant derivatives $\phi_{;\mu;\nu}$ replaced using (15) again. An identical fulfilment of (18) is necessary for the existence of a scalar field ϕ satisfying Eq. (14).

Omitting intermediate calculations, the conditions (18) can be written in the form

$$R^{\sigma}_{\cdot\mu\nu\lambda}\phi_\sigma = u_\mu(u_\nu a_\lambda - u_\lambda a_\nu) + kau_\mu(u_\lambda\phi_\nu - u_\nu\phi_\lambda) + (c_\lambda g_{\mu\nu} - c_\nu g_{\mu\lambda}) + \xi_1(\phi_\lambda g_{\mu\nu} - \phi_\nu g_{\mu\lambda}) + \xi_2(u_\lambda g_{\mu\nu} - u_\nu g_{\mu\lambda}) + a(u_{\mu;\lambda}u_\nu - u_{\mu;\nu}u_\lambda + u_\mu(u_{\nu;\lambda} - u_{\lambda;\nu})), \quad (19)$$

where

$$\xi_1 = 2\overline{k}(k(\nabla\phi)^2 + b) - kb; \quad \xi_2 = 2\overline{k}a(\vec{\nabla}\phi\cdot\vec{u}). \quad (20)$$

4. Tetrad formalism

For the invariant form of the integrability condition (15) it is convenient to pass over to a tetrad representation of all tensors. We choose the tetrad basis as the Lorentz tetrad $\vec{t}, \vec{x}, \vec{y}, \vec{z}$. The first of these vectors can be identified with the matter 4-velocity $\vec{u}$:

$$g = \vec{u}\otimes\vec{u} - \vec{x}\otimes\vec{x} - \vec{y}\otimes\vec{y} - \vec{z}\otimes\vec{z}. \quad (21)$$

The curvature tensor $R_{\mu\nu\lambda\sigma}$ is presented in such a basis in the following form:

$$R_{\mu\nu\lambda\sigma} = a_{ij}X^{(i)}_{\mu\nu}X^{(j)}_{\lambda\sigma} \quad i,j = \overline{1,6}, \quad (22)$$

where $X^{(i)}_{\mu\nu}$ are simple bivectors which form a basis of the 6-dimensional linear bivector space and are built from the basis tetrad vectors:

$$X^{(1)}_{\mu\nu} = 2u_{[\mu}x_{\nu]}; \quad X^{(2)}_{\mu\nu} = 2u_{[\mu}y_{\nu]};$$
$$X^{(3)}_{\mu\nu} = 2u_{[\mu}z_{\nu]}; \quad X^{(4)}_{\mu\nu} = 2x_{[\mu}y_{\nu]}; \quad (23)$$
$$X^{(5)}_{\mu\nu} = 2z_{[\mu}x_{\nu]}; \quad X^{(6)}_{\mu\nu} = 2y_{[\mu}z_{\nu]}.$$

The matrix a_{ij} is symmetric and its elements are the tetrad components of the curvature tensor, for example

$$R_{\mu\nu\lambda\sigma}u^\mu x^\nu u^\lambda x^\sigma = R_{(0)(1)(0)(1)} = a_{11} \quad \text{etc.}\ldots \tag{24}$$

The Einstein equations (13) in the tetrad form are

$$\begin{aligned} &a_{35} = a_{24}; \quad a_{14} = a_{36}; \quad a_{26} = a_{15};\\ &a_{12} = -a_{56}; \quad a_{13} = -a_{46}; \quad a_{23} = -a_{45};\\ &-a_{44} - a_{55} - a_{66} = \varepsilon; \quad a_{66} - a_{22} - a_{33} = p;\\ &a_{55} - a_{11} - a_{33} = p; \quad a_{44} - a_{11} - a_{22} = p. \end{aligned} \tag{25}$$

In the right-hand side of (19), all tensors can be decomposed in a tetrad basis in the following way:

$$\vec{\nabla}\phi = \phi_0\vec{u} + \phi_1\vec{x} + \phi_2\vec{y} + \phi_3\vec{z}; \tag{26}$$

$$\vec{\nabla}\varepsilon = \varepsilon_0\vec{u} + \varepsilon_1\vec{x} + \varepsilon_2\vec{y} + \varepsilon_3\vec{z}; \tag{27}$$

$$\vec{\nabla}a = \frac{1+p_\varepsilon}{k_1}\vec{\varepsilon}; \quad \vec{\nabla}c = -\frac{p_\varepsilon}{k_1}\vec{\nabla}\varepsilon. \tag{28}$$

Here $p_\varepsilon = dp/d\varepsilon$.

To decompose the covariant derivative of the vector u_μ, we use the known expression [5]:

$$u_{;\mu;\nu} = F_\mu u_\nu + \omega_{\mu\nu} + \sigma_{\mu\nu} - \frac{\theta}{3}h_{\mu\nu}, \tag{29}$$

where

$$F_\mu = u_{\mu;\nu}u^\nu$$

is the acceleration vector of a comoving frame of reference, $F_\nu u^\nu = 0$;

$$\omega_{\mu\nu} = u_{[\mu;\nu]} + u_{[\mu}F_{\nu]} \tag{30}$$

is the antisymmetric tensor of rotational velocity of the reference frame, $\omega_{\mu\nu}u^\nu = 0$;

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} - u_{(\mu}F_{\nu)} + \frac{\theta}{3}h_{\mu\nu} \tag{31}$$

is the symmetric shear tensor (the traceless part of the strain tensor) of the reference frame, $\sigma_{\mu\nu}u^\nu = 0$;

$$\theta = u^\mu{}_{;\mu}; \tag{32}$$

is the stretch scalar of the reference frame (the trace of the strain tensor);

$$h_{\mu\nu} = u_\mu u_\nu - g_{\mu\nu} \tag{33}$$

is the metric of a local 3-D space section, orthogonal to u_μ.

The tensors of the reference frame F, ω and σ are given in the chosen tetrad basis by the following expressions:

$$\vec{F} = F_1\vec{x} + F_2\vec{y} + F_3\vec{z}; \tag{34}$$

$$\omega_{\mu\nu} = \omega_4 X^{(4)}_{\mu\nu} + \omega_5 X^{(5)}_{\mu\nu} + \omega_6 X^{(6)}_{\mu\nu}; \tag{35}$$

$$\begin{aligned} \sigma_{\mu\nu} = {} & \sigma_2 Y^{(2)}_{\mu\nu} + \sigma_3 Y^{(3)}_{\mu\nu} - (\sigma_2+\sigma_3)Y^{(4)}_{\mu\nu}\\ & + \sigma_8 Y^{(8)}_{\mu\nu} + \sigma_9 Y^{(9)}_{\mu\nu} + \sigma_{10} Y^{(10)}_{\mu\nu} \end{aligned} \tag{36}$$

In the latter expression $Y^{(i)}_{\mu\nu}$ is a diad basis of the 10-dimensional linear space of symmetric tensors formed by the basis vectors:

$$\begin{aligned} &Y^{(1)}_{\mu\nu} = u_\mu u_\nu; \quad Y^{(2)}_{\mu\nu} = x_\mu x_\nu;\\ &Y^{(3)}_{\mu\nu} = y_\mu y_\nu; \quad Y^{(4)}_{\mu\nu} = z_\mu z_\nu;\\ &Y^{(5)}_{\mu\nu} = 2u_{(\mu}x_{\nu)}; \quad Y^{(6)}_{\mu\nu} = 2u_{(\mu}y_{\nu)};\\ &Y^{(7)}_{\mu\nu} = 2u_{(\mu}z_{\nu)}; \quad Y^{(8)}_{\mu\nu} = 2x_{(\mu}y_{\nu)};\\ &Y^{(9)}_{\mu\nu} = 2x_{(\mu}z_{\nu)}; \quad Y^{(9)}_{\mu\nu} = 2y_{(\mu}z_{\nu)}. \end{aligned} \tag{37}$$

The tensor $h_{\mu\nu}$ has the following form:

$$h = \vec{x}\otimes\vec{x} + \vec{y}\otimes\vec{y} + \vec{z}\otimes\vec{z}. \tag{38}$$

The integrability conditions (19), rewritten in the tetrad form, are invariant under 4-D general coordinates transformations. The freedom in choosing tetrad basis vectors is reduced to local space rotations of the triad $\vec{x}, \vec{y}, \vec{z}$, and can be used to simplify the further obtained equations (39). For maximum generality we assume that the triad $\vec{x}, \vec{y}, \vec{z}$ is arbitrarily oriented.

5. Integrability conditions in tetrad representation

Equating the scalar coefficients in (19) by similar combinations of basic vectors in the right-hand and in left-hand sides, we get the following set of scalar equations, equivalent to the original one (19):

$$\begin{aligned} &a_{11}\phi_1 + a_{12}\phi_2 + a_{13}\phi_3 = e_1 + \lambda\phi_1 - f_1;\\ &a_{12}\phi_1 + a_{22}\phi_2 + a_{23}\phi_3 = e_2 + \lambda\phi_2 - f_2;\\ &a_{13}\phi_1 + a_{23}\phi_2 + a_{33}\phi_3 = e_3 + \lambda\phi_3 - f_3;\\ &a_{14}\phi_1 + a_{24}\phi_2 + a_{34}\phi_3 = 2W_4;\\ &a_{15}\phi_1 + a_{25}\phi_2 + a_{35}\phi_3 = 2W_5;\\ &a_{16}\phi_1 + a_{26}\phi_2 + a_{36}\phi_3 = 2W_6;\\ &a_{11}\phi_0 + a_{14}\phi_2 - a_{15}\phi_3 = \Delta_2 + S;\\ &a_{12}\phi_0 + a_{24}\phi_2 - a_{25}\phi_3 = \Delta_8 + W_4;\\ &a_{13}\phi_0 + a_{34}\phi_2 - a_{35}\phi_3 = \Delta_9 - W_5;\\ &a_{14}\phi_0 + a_{44}\phi_2 - a_{45}\phi_3 = p_\varepsilon e_2 - \xi_1\phi_2;\\ &a_{15}\phi_0 + a_{45}\phi_2 - a_{55}\phi_3 = -p_\varepsilon e_3 + \xi_1\phi_3;\\ &a_{16}\phi_0 + a_{46}\phi_2 - a_{56}\phi_3 = 0;\\ &a_{12}\phi_0 - a_{14}\phi_1 + a_{16}\phi_3 = \Delta_8 - W_4;\\ &a_{22}\phi_0 - a_{24}\phi_1 + a_{26}\phi_3 = \Delta_3 + S;\\ &a_{23}\phi_0 - a_{34}\phi_1 + a_{36}\phi_3 = \Delta_{10} + W_6;\\ &a_{24}\phi_0 - a_{44}\phi_1 + a_{46}\phi_3 = -p_\varepsilon e_1 + \xi_1\phi_1;\\ &a_{25}\phi_0 - a_{45}\phi_1 + a_{56}\phi_3 = 0;\\ &a_{26}\phi_0 - a_{46}\phi_1 + a_{66}\phi_3 = p_\varepsilon e_3 - \xi_1\phi_3;\\ &a_{13}\phi_0 + a_{15}\phi_1 - a_{16}\phi_2 = \Delta_9 + W_5;\\ &a_{23}\phi_0 + a_{25}\phi_1 - a_{26}\phi_2 = \Delta_{10} - W_6;\\ &a_{33}\phi_0 + a_{35}\phi_1 - a_{36}\phi_2 = \Delta_4 + S;\\ &a_{34}\phi_0 + a_{45}\phi_1 - a_{46}\phi_2 = 0;\\ &a_{35}\phi_0 + a_{55}\phi_1 - a_{56}\phi_2 = p_\varepsilon e_1 - \xi_1\phi_1;\\ &a_{36}\phi_0 + a_{56}\phi_1 - a_{66}\phi_2 = -p_\varepsilon e_2 + \xi_1\phi_2. \end{aligned} \tag{39}$$

Here

$$e_i = \varepsilon_i/k_1; \quad f_i = aF_i; \quad W_i = aw_i; \quad \Delta_i = a\sigma_i;$$
$$\lambda = \xi_1 - ka. \quad S = \frac{-p_\varepsilon}{k_1}\varepsilon_0 + \xi_1\phi_0 + \xi_2 - a\theta/3. \quad (40)$$

Eqs. (39) connect a component of a curvature tensor of the 4-D space and a characteristic of motion of matter placed there. Besides, these relations are additions to the Einstein equations (13). If the space-time admits the identical fulfilment of Eqs. (39) and the scalar field satisfies Eq. (12), then this space-time has a 5-D nature and 5-dimensionally geometrized matter.

6. Example: potential equation in the Friedmann flat cosmological model.

Suppose that the expansion $\phi_{,\mu}$ in basis tetrad vectors has the following form:

$$\phi_{,\mu} = \phi_0 u_\mu. \quad (41)$$

This means that the vector field u_μ is orthogonal to the hypersurface $\phi = const$. For simplicity we restrict our attention to the case of motion in the Friedmann flat cosmological model:

$$ds^2 = dt^2 - e^{2\lambda}(dx^2 + dy^2 + dz^2). \quad (42)$$

The nonzero components of the curvature tensor are

$$a_{11} = a_{22} = a_{33} = \ddot{\lambda} + \dot{\lambda}^2,$$
$$-a_{44} = -a_{55} = -a_{66} = \dot{\lambda}^2. \quad (43)$$

The tetrad Einstein equations (25) take the following form:

$$3\dot{\lambda}^2 = \varepsilon; \quad -2\ddot{\lambda} - 3\dot{\lambda}^2 = p. \quad (44)$$

The 4-velocity u_μ has the following components:

$$u_0 = 1, \qquad u_1 = u_2 = u_3 = 0. \quad (45)$$

Among the tensor characteristics of the frame of reference, nonzero is the only stretch coefficient

$$\theta = 3\dot{\lambda}. \quad (46)$$

The integrability conditions (39) are reduced to just

$$a_{11}\phi_0 = S. \quad (47)$$

Taking into account that $\phi_0 = \dot{\phi}$, uncovering the abbreviated designations by (20) and (40) and expressing $\dot{\varepsilon}$ from the Einstein equations (44), the condition (47) can be put in the form

$$\mu_1\dot{\phi}^3 + \dot{\phi}(\dot{\lambda}^2(\mu_2 - 1) - \ddot{\lambda}) - \mu_3\dot{\lambda}\ddot{\lambda} = 0, \quad (48)$$

where

$$\mu_1 = \overline{k}(k + 2\overline{k}); \qquad \mu_2 = 3(2\overline{k}/k_1 + p_\varepsilon k/k_1); \quad (49)$$
$$\mu_3 = 2(3p_\varepsilon - 1)/k_1$$

and the equation of state is taken in the following form:

$$p = p_\varepsilon \varepsilon, \quad p_\varepsilon = \text{const}. \quad (50)$$

Eq. (12), which in metric (42) takes the form

$$\ddot{\phi} + 3\dot{\lambda}\dot{\phi} + (1 + 2n)\dot{\phi}^2 = 0, \quad (51)$$

has the following first integral:

$$\dot{\phi} = \frac{e^{-3\lambda}}{\overline{C} + (1 + 2n)\int e^{-3\lambda}dt}, \quad (52)$$

where $\overline{C}$ is a constant of integration, whence the general solution can be easily obtained:

$$\phi = \frac{1}{1 + 2n}\ln(\overline{C} + (1 + 2n)\int e^{-3\lambda}dt) + C_1 \quad (53)$$

With the substitution of $\dot{\phi}$ from (52) to (48), the expression in the left-hand must identically vanish.

The Einstein equations for the chosen class of equations of state (50) can be easily integrated:

$$\lambda = \begin{cases} \frac{2}{3(1 + p_\varepsilon)}\ln(\frac{3}{2}(1 + p_\varepsilon)t + \overline{C}_0), & p_\varepsilon > -1; \\ C_0 t, & p_\varepsilon = -1. \end{cases} \quad (54)$$

For $p_\varepsilon = -1$ $(p + \varepsilon = 0)$, omitting all intermediate calculations, we obtain that (47) is identically satisfied for $n = 1, -2$. In this case $\phi = C_0 t$. The case $p_\varepsilon = 1$ $(p = \varepsilon)$ will be considered in item (3) of the Conclusion. For $p_\varepsilon \neq \pm 1$ we obtain the following square equation for the conformal transformation index n:

$$\alpha_1 n^2 + \alpha_1 n + \alpha_2 = 0, \quad (55)$$

where

$$\alpha_1 = 27p_\varepsilon^3 + 63p_\varepsilon^2 + 33p_\varepsilon + 5; \quad (56)$$
$$\alpha_2 = 18p_\varepsilon^2 + 24p_\varepsilon - 10.$$

Its solution is

$$n = \frac{3p_\varepsilon(\sigma - 1) - (3\sigma + 1)}{2(3p_\varepsilon + 1)}, \quad \sigma = \pm 1 \quad (57)$$

For dust $(p_\varepsilon = 0)$ we have $n = 1, -2$, for radiation $(p_\varepsilon = 1/3)$ $n = 0, -1$.

The corresponding 5-D vacuum metrics, which in all cases can be obtained by the inverse conformal transformation, are of only two types:

$$dI_1^2 = dt^2 - dx^2 - dy^2 - dz^2 - t^2(dx^5)^2; \quad (58)$$

$$dI_2^2 = dt^2 - t(dx^2 + dy^2 + dz^2) - \frac{1}{t}(dx^5)^2 \quad (59)$$

Its connections with the above 4-D metrics are shown in the table:

	$p_\varepsilon = -1$	$p_\varepsilon = 0$	$p_\varepsilon = 1/3$
(1)	$n = 1$	$n = -2$	$n = -1$
(2)	$n = -2$	$n = 1$	$n = 0$

The cases with $n = 1$ have been considered in [11, 12], the variant with $n = 0$ has been analyzed in Wesson's work [7] .

7. Conclusion

Completing our account, we would like to make some remarks:

1. The approach proposed here can be called "4-dimensional", because the 4-D metric and energy-momentum tensor are given. From the integrability conditions the scalar field ϕ and the index n can be found, or, which is the same, one can find 5-D vacuum space-time which geometrizes the previously given matter. As an example of another approach, a "5-dimensional" one, we can consider Wesson's approach, where a known 5-D vacuum solution is used. The 5-D Einstein equations are split into parts: the 4-D Einstein tensor and a combination of derivatives of the scalar field, which is claimed to represent an effective energy-momentum tensor of induced matter. The type of this tensor is in this approach, in general, arbitrary. In both [8, 9], the obtained energy-momentum tensor is anisotropic. All investigations in Wesson's group are carried out under $n = 0$. The use of the conformal transformation alleviates the statement made in [10]: there is no rigid necessity to introduce the 5th coordinate into the metric to obtain the equation of state for effective matter other than radiation-like.

2. The problem of matter geometrization in Kaluza-Klein theory has many formal analogues with the known problem of isometric embedding of a 4-D Riemannian space into flat space of greater dimension [6].

3. The restriction on the index $n \neq -1/2$ involves the fact that in this case the second-order derivatives of the scalar field in Eqs. (14) disappear. Eqs. (14) became algebraic with respect to the gradient ϕ_μ and the integrability conditions become

$$\phi_{\mu,\nu} - \phi_{\nu,\mu} = 0,$$

where ϕ_μ is algebraically expressed in terms of u_μ, p and ε. In this example the peculiar case $p_\varepsilon = 1$ ($p = \varepsilon$) is realized, namely, for $n = -1/2$ from both 5-D metrics (58,59).

4. Cosmological models of open and closed types have been considered under $n = 1$ in [5](p.230-234) from the viewpoint of a "5-dimensional" approach.

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