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QUANTUM CORRECTIONS TO THE ENTROPY FOR HIGHER SPIN FIELDS IN HYPERBOLIC SPACE

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We calculate one-loop corrections to free energy and entropy for fields of arbitrary spin in the space $S^1 \otimes H^N$. The results obtained for conformally invariant fields are valid, due to a conformal transformation of the metric, for Rindler space of $D = N + 1$ dimensions. We use the zeta regularization technique which yields an ultraviolet-finite result for entropy per unit area. The problem of an infinite area factor in the entropy, which arises equally in Rindler space and in the black hole background, is addressed in the light of a factor space H^N/Γ.

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1. Introduction

Quantum fields against black hole backgrounds have been actively investigated a number of years [1], but only recently an interest in thermodynamic properties, and in particular in entropy, was renewed [2]. Several attempts have been undertaken to calculate quantum corrections to thermodynamic entropy in spaces with horizons. A large number of works have been carried out in the limit of infinite black hole mass when the Schwarzschild spherical horizon surface becomes planar and the metric becomes the Rindler metric.

It is known that the black hole entropy, stored in quanta near the horizon, is ultraviolet divergent. In Rindler space several authors found such a divergence [3, 4, 5, 6]. This is in contrast to the finiteness of the Bekenstein- Hawking entropy [7].

Statistic-mechanical mode counting was recently carried out for the entropy spectrum of scalars in Rindler space with the help of the WKB approximation [4]. Free energy of a massless field in 4-dimensional spacetime, in accordance with previous results, turns out to be proportional to β^{-4} (β is the inverse temperature) and to the horizon area. It has been shown that the entropy per unit area is quadratically divergent near the horizon. Since the level density diverges due to the infinite frequency shift, a cutoff parameter should be introduced [4, 5, 8]. Such quantum corrections to the entropy density, as has been pointed out in Ref. [4], are equivalent to quantum corrections to the gravitational coupling. Therefore, the entropy divergencies obtained by state counting are closely related to the conventional ultraviolet divergencies of canonical quantum gravity.

An independent calculation of the finite temperature stress-energy tensor gave the same results [5]. An interesting method to compute the entropy mostly for $D = 2$ was developed in Ref. [6]. In two dimensions the divergence is logarithmic and the coefficients are cutoff-independent. For $D > 2$ the heat kernel of the Laplace-Beltrami operator in Rindler space was used. As a result, the free energy lower integration limit needs a short-distance regularization. The heat kernel techniques in computing quantum corrections in Rindler space were used in Ref. [8], where the computation was conducted in a geometry with a different topology (with no conical singularity).

In this paper we suggest a powerful method for computing the first quantum correction to the free energy associated with fields of arbitrary spins on the manifold $M = S^1 \otimes H^N$. Such a manifold can be obtained as a result of a conformal transformation of the Euclidean Rindler space. In Sec. 2 we discuss the general technique based on the zeta regularization approach for calculating the one-loop free energy. We show in Sec. 3 that the ultraviolet divergencies of the free energy are removed by the zeta-regularization approach. The free energy per unit area is presented by a series in inverse powers of β (i.e., as a high-temperature expansion) whose coefficients are cutoff-independent. This series is analytically continued to all β. In the conclusion, the results are compared with the corresponding corrections obtained for the conformally invariant and minimally coupled fields. We discuss the problem of regularizing the divergencies in the entropy which result from the infinite horizon area.

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2. Zeta function regularization of free energy

Let us start with a functional integration of the partition function associated with Rindler space. The geometry of the D-dimensional Euclidean Rindler space can be written as follows:

$$ds^2 = \xi^2 d\tau^2 + d\xi^2 + \sum_{i=1}^{N-1} dy_i^2 \,. \qquad (1)$$

Here y_i are the $N-1$ transverse flat coordinates ($D = N+1$), τ is the Euclidean time (periodically identified with a period β) and the lines $\xi = \mathrm{const}$ correspond to uniformly accelerated observers. It is convenient to use the optical metric [9] $\overline{g} = g g_{00}^{-1} = g\xi^{-2}$ in order to define the appropriate functional integration [8, 10]

$$Z[\overline{g}, \beta] = \int D[\Phi]\, e^{-S[\Phi]} \,, \qquad (2)$$

where $S[\Phi]$ is the Euclidean action related to the quantum field Φ in a D-dimensional manifold of the form $M = S^1 \otimes H^N$ and H^N is the simply connected real hyperbolic space.

It should be noted that the measure of the path integral (2) is formally regularized with respect to some inner product of the fields. Generally speaking, the inner product is ill-defined at the horizon (see, for example, Ref. [10]). But actually such a choice of the measure is suitable for obtaining the same thermodynamics as the one associated with the Rindler mode counting [4, 8, 11, 12].

The partition function can be given by

$$\log Z^{\pm}[\overline{g}, \beta] = \pm\tfrac{1}{2} \log\det A^{\pm}(\beta) \,, \qquad (3)$$

where the plus (minus) sign refers to Dirac spinors (resp. integer spin-s fields), $A^{\pm}(\beta) = \partial_\tau^2 + L_N^{\pm}$, where $L_N^{\pm}$ is the self-adjoint Laplace-Beltrami operator acting in the space H^N. The eigenvalues of the operator $A^{\pm}(\beta)$ have the form

$$\omega_n^{(s)} = \left[\frac{2\pi(n+s)}{\beta}\right]^2 + \lambda^2, \quad n \in Z \,. \qquad (4)$$

Here λ^2 are eigenvalues of the operator L_N.

The generalized (Riemann) zeta function related to the operator $A(\beta)$ can be presented using the Mellin transform of the heat kernel $K(t \,|\, A(\beta)) = \mathrm{Tr}\exp(-t(A(\beta))$:

$$\zeta(x \mid A^{\pm}(\beta)) = \frac{1}{\Gamma(s)} \int_0^\infty dt\; t^{x-1} K(t \mid A^{\pm}(\beta)) \,. \qquad (5)$$

Note that the "global" function $\zeta(x \,|\, A(\beta))$ depends only on $x \in M$ since the manifold M is a homogeneous space.

Using the relations

$$\sum_{n=-\infty}^{\infty} e^{-n^2\beta^2/4t} = 2\frac{\sqrt{\pi t}}{\beta} \sum_{n=-\infty}^{\infty} e^{-4pi^2 n^2 t/\beta^2}$$

which are identities for Jacobi's elliptic theta functions $\Theta_3(v,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{i2\pi n v}$ and $\Theta_4(v,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{i2\pi n v}$, one can rewrite $\zeta(x \mid A(\beta))$ in the form (see [13] for more details)

$$\begin{aligned}\zeta(x \mid A^{\pm}(\beta)) &= \sum_{n=-\infty}^{\infty} \zeta(x \mid L_N^{\pm} + [2\pi(n+s)/\beta]^2) \\ &= \frac{\beta\Gamma(x-1/2)}{2\sqrt{\pi}\Gamma(x)} \zeta(x - \tfrac{1}{2} \mid L_N^{\pm}) \\ &+ \frac{\beta}{2\sqrt{\pi}\Gamma(x)} \int_0^\infty dt\; t^{x-3/2} K(t \mid L_N^{\pm}) \big(\Theta^{\pm}(\beta,t) - 1\big) \,, \end{aligned} \qquad (6)$$

where

$$\Theta^{\pm}(\beta, t) = \begin{cases} \Theta_3(0, e^{-\beta^2/4t}), & s = 0,1,2,... \\ \Theta_4(0, e^{-\beta^2/4t}), & s = 1/2. \end{cases} \qquad (7)$$

A complex integral representation for the zeta function can be obtained with the help of the Mellin-Parseval identity

$$\int_0^\infty dt\; f(t)\, g(t) \;=\; \frac{1}{2\pi i} \int_{\Re z = \sigma} \widehat{f}(z)\, \widehat{g}(1-z)\, dz \,, \qquad (8)$$

where $\widehat{f}(z)$ ($\widehat{g}(z)$) is the Mellin transform of $f(t)$ ($g(t)$), $t^{z-1} f(t) \in L(0,\infty)$ and σ is a real number in the strip wherein $\widehat{f}(z)$ and $\widehat{g}(1-z)$ are analytic. Let us suppose that $f(t) = t^{z-3/2} K(t \mid L_N^{\pm})$ and $g(t) = (-1)^{2sm} \exp(-n^2\beta^2/4t)$. Then, using Eq. (6) and the Mellin-Parceval identity (8), one can obtain

$$\begin{aligned}\zeta(x \mid A^{\pm}(\beta)) &= \frac{\beta\Gamma(x-1/2)}{2\sqrt{\pi}\Gamma(x)} \zeta(x - 1/2 \mid L_N^{\pm}) \\ &+ \frac{1}{\sqrt{\pi}\Gamma(x) 2\pi i} \int_{\Re z = c} dz\; \zeta^{\pm}(z) \Gamma\Big(\frac{z}{2}\Big) \Gamma\Big(\frac{z-1}{2} + x\Big) \\ &\times \zeta\Big(\frac{z-1}{2} + x \mid L_N^{\pm}\Big) \Big(\frac{\beta}{2}\Big)^{-(z-1)} \,, \end{aligned} \qquad (9)$$

where $\zeta^-(z) = \zeta_R(z)$ is the Riemann zeta function, $\zeta^+(z) = (1 - 2^{1-z})\zeta_R(z)$ and $c > N+1$.

Let us point out the contribution $Z[\overline{g}, \beta]$ of conformally invariant fields to the partition function $Z[g, \beta]$ in the Rindler space. The renormalized free energy $\mathcal{F}[g, \beta]$ for two conformally related static spaces (with the metrics $\overline{g}_{\mu\nu} = e^{-2\omega} g_{\mu\nu}$) can be rewritten in the form [14]

$$\mathcal{F}^{\pm}[g, \beta] = \pm\beta^{-1} \log Z^{\pm}[g, \beta] = \mathcal{F}^{\pm}[\overline{g}, \beta] + \Delta\mathcal{F}^{\pm}[\omega, g] \,. \qquad (10)$$

We may further apply Eq. (10) to the special case when $\omega = \frac{1}{2}\log\xi^2$ and $\overline{g}_{\mu\nu} = e^{-2\omega} g_{\mu\nu}$ is an ultrastatic metric. It should be noted that the term $\Delta\mathcal{F}^{\pm}[\omega, g]$ for two conformally related theories is proportional to β and hence does not contribute to the entropy. So below we will suppress this term.

From Eq. (9) one can obtain the representation for the free energy

$$\mathcal{F}^{\pm}[\overline{g},\beta] = \mp\tfrac{1}{2}\,\beta^{-1}\zeta'(0 \mid A^{\pm}(\beta))\,. \tag{11}$$

As a result, we have

$$\begin{aligned}\mathcal{F}^{\pm}[\overline{g},\beta] &= \mp\tfrac{1}{2}\,\zeta^{(r)}(\tfrac{1}{2} \mid L_N^{\pm}) \\ &\pm \frac{1}{2\pi\mathrm{i}}\int\limits_{\Re z=c} \mathrm{d}z\;\zeta^{\pm}(z)\Gamma(z-1)\zeta\Big(\frac{z-1}{2}\,\Big|\,L_N^{\pm}\Big)\beta^{-z} \\ &\equiv F_0^{\pm} + F^{\pm}(\beta)\,.\end{aligned} \tag{12}$$

Here $F_0^{\pm}$ is the vacuum energy, $F^{\pm}(\beta)$ is the temperature dependent part of $\mathcal{F}^{\pm}[\overline{g},\beta]$, and we have introduced the notation

$$\zeta^{(r)}(-\tfrac{1}{2} \mid L_N^{\pm}) \;=\; \mathrm{FP}\;\zeta(-\tfrac{1}{2} \mid L_N^{\pm}) + (2-2\log 2)\;\operatorname*{Res}_{z=-1/2}\zeta(z \mid L_N^{\pm})\,. \tag{13}$$

In Eq. (13) the symbols FP and Res denote the finite part and the residue of the function at a specified point, respectively (for more details see Ref. [13]). As was pointed out in [14, 13], the contribution of the fermionic field to the free energy $F^{+}(\beta)$ can be obtained from the bosonic part of $F^{-}(\beta)$. This relation can be easily reproduced from Eq. (12) and the result is

$$F^{+}(\beta) = 2F^{-}(2\beta) - F^{-}(\beta)\,.$$

3. Quantum corrections associated with arbitrary spin fields in $S^1 \otimes H^N$

For a noncompact rank one symmetric space H^N (the rank M is the dimension of the commutative algebra of invariant differential operators, e.g., Laplace operators) the related zeta function can be constructed with the help of the spectral function $\mu(\lambda)$ known as the Plancherel measure [15]. In the case of a Riemann noncompact symmetric space with negative curvature the explicit form of $\mu(\lambda)$ is given by

$$\mu(\lambda) = [C(\lambda)C(-\lambda)]^{-1}\,,$$

where $C(\lambda)$ is the Harish-Chandra function [13, 16]. It can be given in terms of a product over the positive roots of the symmetric space. The spectral function is essential in the construction of the zeta function in a noncompact space. It takes the form

$$\zeta(z \mid L_N^{\pm}) = \int_0^{\infty} \frac{\mathrm{d}\lambda\;\mu(\lambda)}{(\lambda^2 + C_s^2)^z} \tag{14}$$

where C_s are known constants depending on the field masess and on the curvature $R = -N(N-1)a^{-2}$. We choose the radius $a = 1$ and in the final results the dependence on a can be restored.

Here, we add a remark concerning the role of the coefficients C_s. As known, for a conformally invariant massless scalar field $C_0 = 0$ and the corresponding Laplace-Beltrami operator has no gap in its spectrum. In this case we keep the coefficient $C_0 \neq 0$ until of the calculations are finished and use it as a regularization parameter. In this way we get a well defined zeta function suitable for analytic continuation.

The Plancherel measure for spin-s fields in H^N has been recently calculated in [16, 17]. It reads

$$\mu^{\pm}(\lambda,s) = \frac{\pi\lambda}{[2^{N-2}\Gamma(N/2)]^2}\tanh[\pi(\lambda+\mathrm{i}s)]\;\sigma^{\pm}(\lambda,s) \tag{15}$$

with

$$\tanh[\pi(\lambda+\mathrm{i}s)] = \begin{cases}\tanh(\pi\lambda), & s=0,1,...\\ \coth(\pi\lambda), & s=1/2\end{cases}$$

and

$$\sigma^{-}(\lambda,s) = \begin{cases}\Big[\lambda^2 + \big(s+\frac{N-3}{2}\big)^2\Big]\prod\limits_{j=0}^{q-2}(\lambda^2+j^2) \\ \quad\equiv \sum\limits_{k=1}^{q} a_{k,N}^{(s)}\lambda^{2k}, \quad N=2q+1, \\ \Big[\lambda^2 + \big(s+\frac{N-3}{2}\big)^2\Big]\prod\limits_{j=1/2}^{q-5/2}\big(\lambda^2+j^2\big) \\ \quad\equiv \sum\limits_{k=0}^{q-1} b_{k,N}^{(s)}\lambda^{2k}, \quad N=2q,\end{cases} \tag{16}$$

$$\sigma^{+}(\lambda,\tfrac{1}{2}) = \begin{cases}[\lambda\coth(\pi\lambda)]^{-1}\prod\limits_{j=1/2}^{q-1/2}(\lambda^2+j^2) \\ \quad\equiv [\lambda\coth(\pi\lambda)]^{-1}\sum\limits_{k=0}^{q} a_{k,N}^{+}\lambda^{2k}, \\ \qquad N=2q+1, \\ \prod\limits_{j=1}^{q-1}\big(\lambda^2+j^2\big) \\ \quad\equiv \sum\limits_{k=0}^{q-1} b_{k,N}^{+}\lambda^{2k}, \quad N=2q,\end{cases} \tag{17}$$

where $q \in \mathcal{Z}_+$. The coefficients $a_{k,N}^{(s)}$, $b_{k,N}^{(s)}$, $a_{k,N}^{+}$, $b_{k,N}^{+}$ are defined by expanding the products in polynomials in λ^2 in Eqs. (16) and (17). In Eq. (16), for $N=3$, the product is to be omitted and $a_{0,3}^{(s)} = s^2$, $a_{1,3}^{(s)} = 1$. For $N=4$, the product is also omitted and we have $b_{0,4}^{(s)} = (s+1/2)^2$, $b_{1,4}^{(s)} = 1$ (the spectral functions on H^2 for spin 0 and 1 are both given by $\mu^{-}(\lambda,s) = \pi\lambda\tanh(\pi\lambda)$, $b_{0,2}^{+} = 1$).

Finally, the zeta function can be written as [17, 18]

$$\zeta(z \mid L_N^-) = \begin{cases} \frac{1}{2}g(s)A(N) \sum\limits_{k=1}^{q} a_{k,N}^{(s)} C_s^{2k-2z+1} B(k+\frac{1}{2},\ z-k-\frac{1}{2})\ , & N=2q+1; \\ \frac{1}{2}g(s)A(N) \sum\limits_{k=0}^{q-1} b_{k,N}^{(s)} \left[C_s^{2k-2z+2} B(k+1,\ z-k-1) - 4I_k^-(C_s,z)\right]\ , & N=2q; \end{cases} \quad (18)$$

$$\zeta(z \mid L_N^+) = \begin{cases} 2^{[N/2]-1}A(N) \sum\limits_{k=0}^{q} a_{k,N}^{+} C_{1/2}^{2k-2z+1} B(k+\frac{1}{2},\ z-k-\frac{1}{2})\ , & N=2q+1; \\ 2^{N/2-1}A(N) \sum\limits_{k=0}^{q-1} b_{k,N}^{+} \left[C_{1/2}^{2k-2z+2} B(k+1,\ z-k-1) + 4I_k^+(C_{1/2},z)\right]\ , & N=2q \end{cases} \quad (19)$$

where $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ is Euler's beta function,

$$I_k^{\pm}(C,z) \ = \ \int_0^{\infty} \frac{\mathrm{d}\lambda\ \lambda^{2k+1}(\lambda^2+C^2)^{-z}}{\mathrm{e}^{2\pi\lambda}\mp 1}\,, \quad (20)$$

$$g(s) \ = \ \frac{(2s+N-3)(s+N-4)}{(N-3)!s!}, \qquad A(N)=\frac{A}{2^{N-1}\pi^{N/2}\Gamma(N/2)} \quad (21)$$

and A is the area of the manifold M. For $N=3$ we should take $g(0)=1$ and $g(s)=2$ for $s\geq 1$.

For integer spins and odd N, the zeta function (18) is meromorphic in the complex z-plane with simple poles at $z=N/2\ , N/2-1, ...$ and exhibits trivial zeros at $z=0,-1,-2,\dots$. For even N, the integral term in (18) is analytic in z but the first term carries a finite number of first order poles at $z=N/2\ , N/2-1,\dots,1$. Finally, the spinor zeta function (19) has first order poles at the same points (at $z=N/2\ , N/2-1,\dots$ for odd N, and at $z=N/2\ , N/2-1,\dots,1$ for even N).

To obtain a Laurent series representation for the statistical sum, it is convenient to use the Mellin-Barnes representation (12). Using the zeta functions related to the operators $L_N^{\pm}$ given by Eqs. (18) and (19), we obtain

$$F^{\pm}(\beta) = \pm\frac{1}{2\pi\mathrm{i}} \int\limits_{\Re z=c_0} \mathrm{d}z\ \varphi^{\pm}(z,N)\,, \qquad c_0 > N/2\,. \quad (22)$$

Here we have introduced the notations

$$\varphi^-(z,2q+1) \ = \ -\frac{g(s)A(2q+1)}{4\sqrt{\pi}} \sum_{k=1}^{q} a_{k,N}^{(s)} C_s^{2k+2} \Gamma(k+\tfrac{1}{2})\Gamma(z+\tfrac{1}{2})\zeta_R(2z+1)\Gamma(z-k-\tfrac{1}{2}) \left(\tfrac{C_s\beta}{2}\right)^{-(2z+1)}, \quad (23)$$

$$\varphi^-(z,2q) \ = \ -\frac{g(s)A(2q)}{4\sqrt{\pi}} \sum_{k=0}^{q-1} b_{k,N}^{(s)} C_s^{2k+3} \Gamma(k+1)\Gamma(z+\tfrac{1}{2})\zeta_R(2z+1)\Gamma(z-k-1) \left(\tfrac{C_s\beta}{2}\right)^{-(2z+1)}$$
$$+\,4g(s)A(2q) \sum_{k=0}^{q-1} b_{k,N}^{(s)} \zeta_R(2z+1)\Gamma(2z) I_k^-(C_s,z)\beta^{-(2z+1)}\,, \quad (24)$$

$$\varphi^+(z,2q+1) \ = \ \frac{2^{q-2}A(2q+1)}{\sqrt{\pi}} \sum_{k=0}^{q} a_{k,N}^{+} C_{1/2}^{2k+2} \Gamma(k+\tfrac{1}{2})\Gamma(z+\tfrac{1}{2})\zeta^+(2z+1)\Gamma(z-k-\tfrac{1}{2}) \left(\tfrac{C_{1/2}\beta}{2}\right)^{-(2z+1)}, \quad (25)$$

$$\varphi^+(z,2q) \ = \ \frac{2^{q-2}A(2q)}{\sqrt{\pi}} \sum_{k=0}^{q-1} b_{k,N}^{+} C_{1/2}^{2k+3} \Gamma(k+1)\Gamma(z+\tfrac{1}{2})\zeta^+(2z+1)\Gamma(z-k-1) \left(\tfrac{C_{1/2}\beta}{2}\right)^{-(2z+1)}$$
$$+\,2^{q+2}A(2q) \sum_{k=0}^{q-1} b_{k,N}^{+} \zeta^+(2z+1)\Gamma(2z) I_k^+(C_{1/2},z)\beta^{-(2z+1)}\,. \quad (26)$$

For odd (even) dimension $N=2q+1$ ($N=2q$), the function $\varphi^-(z,N)$ is meromorphic in z. It has first order poles at $z=0$, $z=N/2, N/2-1, N/2-2, ...$ ($z=-1/2$, $z=N/2, N/2-1, N/2-2, ...$) and one second order pole at $z=-1/2$ ($z=0$). The function $\varphi^+(z,N)$ has the same properties with the only difference that for even N the poles at $z=0,-1/2$ are simple, while for odd N the pole at $z=-1/2$ is of second order and there is no pole at $z=0$.

Now it is useful to move the integration contour in (22) to the left up to infinity. Thereby it will cross all the poles just mentioned, which results in contributions from the corresponding residues. We obtain the following series representation:

$$F^{-}(\beta)_{2q+1} = -\frac{g(s)A(2q+1)}{4\sqrt{\pi}} \sum_{k=1}^{q} a_{k,N}^{(s)} C_s^{2k+2} \Gamma(k+\tfrac{1}{2}) \Bigg\{ \textstyle\sum_{j=0}^{k} (-1)^j \zeta_R(2k-2j+2) \frac{\Gamma(k-j+1)}{\Gamma(j+1)} \left(\frac{C_s\beta}{2}\right)^{-(2k-2j+2)}$$
$$+ \frac{(-1)^{k+1}\pi^{3/2}}{\Gamma(k+3/2)} \left(\frac{C_s\beta}{2}\right)^{-1} + \frac{(-1)^{k+1}}{\Gamma(k+2)} \left[\gamma + \log\left(\frac{C_s\beta}{4\pi}\right)\right] + \Xi^{-}_{k,2q+1}\left(\frac{C_s\beta}{2\pi}\right) \Bigg\} \tag{27}$$

$$F^{-}(\beta)_{2q} = -\frac{g(s)A(2q)}{4\sqrt{\pi}} \sum_{k=0}^{q-1} b_{k,N}^{(s)} C_s^{2k+3} \Gamma(k+1) \Bigg\{ \sum_{j=0}^{k} (-1)^j \zeta_R(2k-2j+3) \frac{\Gamma(k-j+\frac{3}{2})}{\Gamma(j+1)} \left(\frac{C_s\beta}{2}\right)^{-(2k-2j+3)}$$
$$+ \frac{(-1)^{k}\pi^{1/2}}{\Gamma(k+2)} \log(C_s\beta) \left(\frac{C_s\beta}{2}\right)^{-1} + \frac{(-1)^{k+1}\pi}{2\Gamma(k+5/2)} + \Xi^{-}_{k,2q}\left(\frac{C_s\beta}{2\pi}\right) \Bigg\}$$
$$+ g(s)A(2q) \sum_{k=0}^{q-1} b_{k,N}^{(s)} \Bigg\{ \left[\frac{\mathrm{d}}{\mathrm{d}z} I^{-}(k,z)_{|z=0} + \frac{(-1)^k(1-2^{-2k-1})}{4(k+1)} \left(\gamma - \log\beta^2\right) B_{2k+2} \right] \beta^{-1} - 2I^{-}(k,-\tfrac{1}{2}) + \Psi^{-}\left(\tfrac{\beta}{2\pi}\right) \Bigg\}, \tag{28}$$

$$F^{+}(\beta)_{2q+1} = \frac{2^{q-2}A(2q+1)}{\sqrt{\pi}} \sum_{k=0}^{q} a_{k,N}^{+} C_{1/2}^{2k+2} \Gamma(k+\tfrac{1}{2})$$
$$\times \Bigg\{ \sum_{j=0}^{k} (-1)^j \zeta^{+}(2k-2j+2) \frac{\Gamma(k-j+1)}{\Gamma(j+1)} \left(\frac{C_{1/2}\beta}{2}\right)^{-(2k-2j+2)}$$
$$+ \frac{(-1)^k}{\Gamma(k+2)} \left[\gamma + \log\left(\frac{C_{1/2}\beta}{\pi}\right)\right] + \Xi^{+}_{k2q+1}\left(\frac{C_{1/2}\beta}{2\pi}\right) \Bigg\}, \tag{29}$$

$$F^{+}(\beta)_{2q} = \frac{2^{q-2}A(2q)}{\sqrt{\pi}} \sum_{k=0}^{q-1} b_{k,N}^{+} C_{1/2}^{2k+3} \Gamma(k+1) \Bigg\{ \sum_{j=0}^{k} (-1)^j \zeta^{+}(2k-2j+3) \frac{\Gamma(k-j+\frac{3}{2})}{\Gamma(j+1)} \left(\frac{C_{1/2}\beta}{2}\right)^{-(2k-2j+3)}$$
$$+ \frac{(-1)^{k+1}\pi}{\Gamma(k+2)} \log 2 \left(\frac{C_{1/2}\beta}{2}\right)^{-1} + \frac{(-1)^k\pi}{2\Gamma(k+5/2)} + \Xi^{+}_{k,2q}\left(\frac{C_{1/2}\beta}{2\pi}\right) \Bigg\}$$
$$+ 2^q A(2q) \sum_{k=0}^{q-1} b_{k,N}^{+} \left\{ \frac{(-1)^k}{2(k+1)} \log 2 \; B_{2k+2} \; \beta^{-1} + 2I_k^{+}(C_{1/2},-\tfrac{1}{2}) + \Psi^{+}\left(\tfrac{\beta}{2\pi}\right) \right\}, \tag{30}$$

with

$$\Xi^{\pm}_{k,2q+1}(X) = \frac{1}{\sqrt{\pi}} \sum_{j=k+2}^{\infty} (-1)^j \zeta^{\pm}(2j-2k-1) \frac{\Gamma(j-k-\frac{1}{2})}{\Gamma(j+1)} X^{2j-2k-2}, \tag{31}$$

$$\Xi^{\pm}_{k,2q}(X) = \frac{1}{\sqrt{\pi}} \sum_{j=2+k}^{\infty} (-1)^j \zeta^{\pm}(2j-2k-2) \frac{\Gamma(j-k-1)}{\Gamma(j+1)} X^{2j-2k-3}, \tag{32}$$

$$\Psi^{-}(X) = \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \zeta_R(2j) I_k^{-}(C_s,-j)\,(X)^{2j-1}, \tag{33}$$

$$\Psi^{+}(X) = \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} (1-2^{2j}) \zeta_R(2j) I_k^{+}(C_{1/2},-j)\,(X)^{2j-1}; \tag{34}$$

γ is the Euler-Masceroni constant and B_{2k} are the Bernoulli numbers. Eqs. .(27)–(30) give a series respesentation for the free energy which is very convenient for high-temperature expansion. Let us remark that the regularization has been already removed in these expressions. They are finite. That means that the pole at $s=0$ is absent, as it usually happens in the zeta regularization techniques.

The Laurent series in inverse powers of β we obtained is analogous to the one-loop contributions to the free energy in string theory [19], which has been recently actively investigated.

The infinite series (31) and (32) are convergent for

$$\beta < \beta_C^{\pm}, \quad \begin{cases} \beta_C^{-} = 2\pi/C_s \\ \beta_C^{+} = \pi/C_{1/2} \end{cases}. \tag{35}$$

The series (31) and (32) can be analytically continued in β as follows. Using the integral representation

$$\zeta_R(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{\mathrm{e}^t - 1}$$

for the Riemann zeta function and interchanging the orders of summation and integration, the sums over k can be performed to yield the Bessel functions $J_{k+1}(z)$,

$$\Xi^-_{k,2q+1}(X) = (-1)^{k+1} \int_0^\infty \frac{\mathrm{d}t}{\mathrm{e}^t - 1} \left[\left(\frac{2}{Xt}\right)^{k+1} \mathrm{J}_{k+1}(Xt) - \frac{1}{\Gamma(k+2)} \right] \tag{36}$$

and the Struve functions $H_{k+3/2}(z)$,

$$\Xi^-_{k,2q}(X) = (-1)^k \int_0^\infty \frac{\mathrm{d}t}{\mathrm{e}^t - 1} \left(\frac{2}{Xt}\right)^{k+\frac{1}{2}} \mathrm{H}_{k+\frac{3}{2}}(Xt) \,. \tag{37}$$

Taking into account their asymptotic behaviour for large argument, it is clear that the convergence of the integration in t breaks down for $x \to \pm\mathrm{i}$. For real X, the functions $\Xi^\pm_k(X)$ are smooth. By expanding them back into a series in powers of X, the convergence radius is given by the nearest pole in the complex X-plane which lies in $X = \pm\mathrm{i}$ in accordance with (35). For $\beta = \beta^-_C$ $(X = 1)$ and $\beta = \beta^+_C (X = 1/2)$ the functions (36) and (37) remain finite. From the representations (36) and (37), the behaviour for $X \to \infty$ can be obtained:

$$\Xi^-_k(X) \sim \frac{(-1)^k}{2\Gamma(k+2)} \log X \,, \tag{38}$$

$$\Xi^+_k(X) \sim \frac{(-1)^k}{\sqrt{\pi}\Gamma(k+2)} \log X \,. \tag{39}$$

The functions $\Psi^\pm(X)$ (33) and (34) have similar properties. These series are convergent for $X \le 1/(C\pi)$. Taking into account that their argument is $\beta/2\pi$ (instead of $C\beta/2\pi$ in the case of the functions $\Xi^\pm$), this is equivalent to (35). These functions can be analytically continued and can be represented in the form:

$$\Psi^-(X) = \frac{-1}{\pi X} \int_0^\infty \frac{\mathrm{d}\lambda\ \lambda^{2k+1}}{\mathrm{e}^{2\pi\lambda} + 1} \log \frac{\sinh(\pi X \sqrt{\lambda^2 + C^2})}{\pi X \sqrt{\lambda^2 + C^2}}, \qquad \Psi^+(X) = \frac{-1}{\pi X} \int_0^\infty \frac{\mathrm{d}\lambda\ \lambda^{2k+1}}{\mathrm{e}^{2\pi\lambda} - 1} \log\coth\bigl(\pi X \sqrt{\lambda^2 + C^2}\bigr).$$

They are of the order $O(1/X)$ for $X \to \infty$.

4. Conclusion

Here we would like to make some final remarks concerning the obtained results. We have developed a formalism for calculating the one-loop free energy associated with fields of arbitrary spin in the manifold $M = S^1 \otimes H^N$.

For a minimally coupled scalar field we have $C_0^2 = \rho_N^2 + a^2 m^2$ where $\rho_N = (N-1)/2$ and m is the field mass. For a massless field in 4 dimensions we have $N = 3$, $C_0 = 1$ and $\beta_R = 2\pi$ in agreement with the Rindler temperature $T_R = 1/2\pi$, well known in the theory of conformally invariant fields. In addition, the leading term of the Laurent series has the form $-A\pi^2/90\beta^4$, which is also well known [14, 20]. On the other hand, the constant $C_{1/2}$ is the Dirac spinor field mass. For the vector (spin-1) field, the Hodge-de Rham operator $d\delta + \delta d$ acting on the exact one-forms is associated with the massless operator $[-\Delta^\mu \Delta^\nu + (N-1)a^{-2}]g_{\mu\nu}$. The eigenvalues of that operator are $\lambda^2 + (\rho_N - 1)^2$ and for the Proca field of mass m we find $C_1^2 = (\rho_N - 1)^2 + a^2 m^2$. In general, for a spin-s field the wave operator has the form $L^-_N + m^2 + Qa^{-2}$, where Q is a given constant.

The renormalized free energy $\mathcal{F}^-[\overline{g}, \beta]$ of a conformally invariant scalar field in an ultrastatic space with the metric $\overline{g}$ is given by Eqs. (27) and (28). For a conformal massless field we have $C_0 = 0$ and hence

$$F^-(\beta)_{2q+1} = -\frac{g(0)A(2q+1)}{4\sqrt{\pi}} \sum_{k=1}^{q} a^{(0)}_{k,N} \zeta_R(2k+2)\Gamma(k+\tfrac{1}{2})\Gamma(k+1) \left(\tfrac{\beta}{2}\right)^{-2k-2}, \tag{40}$$

$$\begin{aligned} F^-(\beta)_{2q} &= -\frac{g(0)A(2q)}{4\sqrt{\pi}} \sum_{k=0}^{q-1} b^{(0)}_{k,N} \zeta_R(2k+3)\Gamma(k+\frac{3}{2})\Gamma(k+1) \left(\frac{\beta}{2}\right)^{-2k-3} \\ &+ g(0)A(2q) \sum_{k=0}^{q-1} b^{(0)}_{k,N} \Biggl\{ \left[\frac{\mathrm{d}}{\mathrm{d}z} I^-_k(0,z)_{|z=0} + \frac{(-1)^k(1-2^{-2k-1})}{4(k+1)} \left(\gamma - \log\beta^2\right) B_{2k+2} \right] \beta^{-1} \\ &\qquad -2I^-_k(0, -\tfrac{1}{2}) + \textstyle\sum_{j=1}^\infty \frac{(-1)^j}{\pi j} \zeta_R(2j) I^-_k(0,-j) \left(\frac{\beta}{2\pi}\right)^{2j-1} \Biggr\}. \end{aligned} \tag{41}$$

Using Eq. 39 for $D = 4$, we have $g(0) = 1$, $A(3) = A/2\pi^2$ and $F^-(\beta) = -A\pi^2/90\beta^4$. Thus there is no term proportional to β^{-2}, a standard result [14, 20]. The renormalized stress-energy tensor can be obtained by variation of $\mathcal{F}^-[g,\beta]$ with respect to the metric g. This tensor remains finite at the horizon since the divergence of the thermal contribution obtained by the variation of $\mathcal{F}^-[\overline{g},\beta]$ is compensated by the vacuum polarization obtained by variation of $\Delta\mathcal{F}^-[\omega, g]$ [20]. In general, for a massive conformal field the coefficient reads $C_0 = (\xi - \frac{1}{6})R + m^2$ [14].

The dominant contribution to the one-loop entropy density

$$S(\beta) = \hbar^{-1}\beta^2 \partial F/\partial\beta \tag{42}$$

comes from the region near the horizon. To regulate the divergencies, one can insert in an infrared cutoff by defining a smooth compact manifold M as $M = S^1 \otimes H^N/\Gamma$ and then impose suitable boundary conditions on the fields. Here, H^N/Γ is the quotient of H^N by a discontinuous group Γ of isometries. Under these assumptions the volume of the compact manifold H^N/Γ is $A = V(\mathcal{F}_N)a^N$, $V(\mathcal{F}_N)$ being the volume of the fundamental domain $\mathcal{F}_N$. Making use of the Selberg trace formula associated with H^N/Γ [13], one obtains the zeta function $\zeta(z \in H^N/\Gamma \mid L_N^{\pm}) = V(\mathcal{F}_N)\zeta(z \mid L_N^{\pm}) + \zeta^t(z \mid L_N^{\pm})$, where the topological term (the analytic part of the zeta function) $\zeta^t(z \mid L_N^{\pm})$ can be written as a sum over closed geodesics of H^N/Γ. The leading behaviour of the free energy (27)-(30) is of course independent of the topological contributions. So we obtain the same leading behaviour as above with a finite volume factor.

Moreover, there exist discrete groups Γ of special kind which are related to some noncompact hyperbolic manifolds of finite volume (in this case the group Γ contains parabolic elements as well) [21]. For the symmetric space of rank 1 some concrete examples of such groups Γ are known. Therefore the fundamental domain of Γ is noncompact and the finite invariant volume of the domain can be connected with the area of the horizon.

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