Gravitation & Cosmology, Vol. 1 (1995), No. 4, pp. 273–276

ON THE PROBLEM OF CONFORMAL COUPLING IN FIELD THEORY IN CURVED SPACETIME

A.A. Grib† and W.A. Rodrigues, Jr.‡

† Friedmann Lab. for Theoretical Physics, St. Petersburg, Russia.

‡ Depart. de Matemática Aplicada - IMECC, Universidade Estadual de Campinas, CP 6065, 13081-970, Campinas, SP, Brazil

Received 21 September 1995

We study the properties of solutions for minimally and conformally coupled scalar fields in curved space-time. We show that, contrary to some claims in the literature, anomalous R-forces between two "scalarly charged" particles do not exist for conformally coupled scalar fields. Even more, it is the minimally coupled scalar field that has a pathological and unexpected behaviour. The origin of the erroneous claim is investigated due to its great methodological meaning.

As is well known in field theory, in curved space-time we have an ambiguity [1,2] concerning massless scalar particles. The Klein-Gordon equation in curved space-time for these particles can be written in two different forms. The first one results from the principle of minimal coupling:

$$\nabla_i \nabla^i \varphi(x) = 0, \tag{1}$$

where ∇_i is the Levi-Civita connection of the metric g_{ik} which has the signature $(+ \; - \; - \; -)$ and $\varphi(x)$ is a scalar field. The second possible form results from conformal coupling:

$$(\nabla_i \nabla^i + R/6)\varphi(x) = 0. \tag{2}$$

Here R is the scalar curvature of the space-time.

This ambiguity is preserved for massive scalar particles, for which one can write either

$$(\nabla_i \nabla^i + m^2)\varphi(x) = 0, \text{ minimal coupling} \tag{3}$$

or

$$(\nabla_i \nabla^i + R/6 + m^2)\varphi(x) = 0, \text{ conformal coupling.} \tag{4}$$

(a) One of the advantages of Eq. (2) is that it is conformally invariant (invariant under Weyl scale transformations). This means that if one changes the metric g_{ik} into $\widetilde{g}_{ik} = \mathrm{e}^{-2\sigma(x)} g_{ik}$, and the function $\varphi(x)$ into $\tilde{\varphi}(x) = \mathrm{e}^{\sigma(x)}\varphi(x)$, then $\tilde{\varphi}(x)$ is a solution of Eq. (2) in the metric $\widetilde{g}_{ik}$.

Eq. (1) does not share the property of conformal invariance with (2).

It is of interest to notice that the Dirac equation for a massless particle (e.g., a neutrino) in curved space-time is conformally invariant. The same is true for Maxwell's equations. But there is no conformal invariance for a graviton, which is described by a spin-2 massless tensor field $h_{ik}(x)$ [2]. It satisfies the linearized equation

$$\nabla^j \nabla_j h_{ik}(x) + 2R_{jikl} h^{jl}(x) = 0 \tag{5}$$

with the gauge conditions

$$\nabla_k h^k_i(x) = 0, \qquad h(x) = h^i_i(x) = 0, \tag{6}$$

where R_{jikl} is the curvature tensor of the background space-time. Notations are the same as in [2].

(b) Another advantage of Eq. (2), as well as (4) for the massive case, concerns the properties of quasiclassical solutions. As shown by Chernikov and Tagirov [3], only for Eq. (2) or (4) in the quasiclassical limit particles move along geodesics of the corresponding space-time. This occurs if R is large enough, so that it has the order $R \sim m^2$ in units $\hbar = c = 1$ and we cannot neglect this term using the quasiclassical approximation.

For massive vector bosons, when we use the Proca equation in a Riemannian space-time, we have no $R/6$ term. Instead we have

$$\nabla_i f^{ik}(x) + m^2 \varphi^k(x) = 0 \tag{7}$$

where $f^{ik}(x) = \nabla_i \varphi_k - \nabla_k \varphi_i = \partial_i \varphi_k - \partial_k \varphi_i$, φ_i being a vector field. As shown in [4], the longitudinal component of a vector field behaves like some minimally coupled scalar massive field.

From all this it seems that one can have in nature both kinds of fields and we must investigate their different physical manifestations.

In this paper we carefully discuss the scalar case. The minimally coupled scalar field is well-known [5] to play an important role in inflation theory, popular in cosmology. If we choose the conformal coupling, we lose the properties necessary for inflation.

Lack of conformal invariance of Eq. (1) can be interpreted in the sense that this field is not really massless and has some scale, defined by the curvature. On the opposite, Eq. (2) describes really massless particles.

This is confirmed by the structure of the scalar-field Green functions $G(x,x')$ for Eqs. (1) and (2). Generally for an arbitrary coupling of the form ξR the Green function obeys the following equation [2]:

$$\sqrt{-g}(\nabla_i\nabla^i + m^2 + \xi R)G(x,x') = \delta(x-y'), \tag{8}$$

where $\xi = 0$ corresponds to minimal coupling and $\xi = 1/6$ to conformal coupling and g is the metric tensor determinant. The singularities of the Green function can be obtained by the Schwinger-DeWitt technique [6, 7] and have the following representation [1]:

$$\begin{aligned}\underset{x\to x'}{G(x,x')} &= \frac{\sqrt{\Delta}m^2}{4\pi^2}\Big\{-\frac{1}{m^2\sigma} + L\Big(1-\frac{m^2\sigma}{4}\Big) - \frac{1}{2} + \frac{5}{16}m^2\sigma \\ &+ \ldots - \frac{a_1(x,x')}{m^2}\Big[L\Big(1-\frac{m^2\sigma}{2}\Big) + \frac{m^2\sigma}{2} + \ldots\Big] \\ &+ \frac{a_2(x,x')}{m^4}\Big[\frac{1}{2} - \frac{Lm^2\sigma}{2} + \frac{m^2\sigma}{4} + \ldots\Big] + \ldots\Big\},\end{aligned} \tag{9}$$

where $\sigma(x,x') = \eta_{ik}(x^i - x^{i'})(x^k - x^{k'})/2$ and $\eta_{ik} = \operatorname{diag}(1,-1,-1,-1)$ is the Minkowski metric;

$$\Delta(x,x') = -\det\left(\frac{\partial^2\sigma(x,x')}{\partial x'^k}\right)\cdot[g(x)g(x')]^{-1/2}, \tag{10}$$

$$\Delta(x,x) = 1; \quad L = \tfrac{1}{2}\ln\big(\tfrac{1}{2}m^2\sigma\big) + c \tag{11}$$

where $c = 0,577,\ldots$ is the Euler constant, and

$$a_1(x,x) = a_1 = \Big(\frac{1}{6} - \xi\Big)R, \tag{12}$$

$$\begin{aligned}a_2(x,x) = a_2 &= \frac{1}{180}(R^{iklm}R_{iklm} - R^{ik}R_{ik}) \\ &- \frac{1}{6}\Big(\frac{1}{5} - \xi\Big)\nabla^k\nabla_k R + \frac{1}{2}\Big(\frac{1}{6} - \xi\Big)^2 R^2.\end{aligned} \tag{13}$$

Eq. (9) includes terms up to $O(m^2\sigma)$ and $O((m\rho)^{-k})$ where $R_{ijkl}R^{ijkl} \sim \rho^{-k}$.

So one can see from Eq. (9) that as $x \to x'$ and $m \to 0$, the structure of the singularity for $\xi = 1/6$ (when $a_1 = 0$) is the same as in flat space-time. But for $\xi = 0$ there is an additional singularity, as if we had some mass due to the space-time curvature R.

Nevertheless, in the literature (e.g., [8]) we can find claims that Eq. (2) leads to violation of the strong equivalence principle and to the appearance of anomalous R-term forces between two "scalarly charged" particles! In this paper we shall show that the situation is quite the opposite. The argument of [8] is the following: write Eq. (2) in a locally Lorentz coordinate frame (with the origin at a given point P_0) where a point source lives. We have

$$(\Box\,\varphi + R/6)\varphi = \mu_1\delta(\vec{r}). \tag{14}$$

Then a solution is Yukawa's potential

$$\varphi = -\frac{\mu_1}{r}\exp\Big(-\frac{r}{a\sqrt{6}}\Big), \tag{15}$$

where $a = R^{-1/2}$.

It is nevertheless easy to see that, if we find an exact solution to Eq. (2) with a source term, we do not arrive at the Yukawa solution. To see that, take the simplest case of a conformally flat Friedmann quasi-Euclidean space-time with the metric:

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2) \tag{16}$$

where η is the conformal time.

Write Eq. (2) as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}g^{ik}\frac{\partial\varphi}{\partial x^k}\right) + \frac{R}{6}\varphi = 0 \tag{17}$$

or as

$$\frac{1}{a^4}\frac{\partial}{\partial x^i}\left(a^4\frac{1}{a^2}\frac{\partial\varphi}{\partial x^i}\right) + \frac{R}{6}\varphi = 0. \tag{18}$$

Here

$$R = 6a^{-2}\left(\frac{a''}{a}\right), \qquad a'' = \frac{d^2a}{d\eta^2}. \tag{19}$$

From Eq. (18) we have

$$\frac{1}{a^4}\frac{\partial}{\partial\eta}\left(a^2\frac{\partial\varphi}{\partial\eta}\right) - \frac{1}{a^2}\Delta\varphi + \frac{R}{6}\varphi = 0 \tag{20}$$

To solve Eq. (20) we make a transformation of φ

$$\varphi \longmapsto \tilde{\varphi} \quad ; \quad \varphi = \tilde{\varphi}/a. \tag{21}$$

Then we get from Eq. (20):

$$\frac{1}{a^2}\Big[\frac{\partial^2\varphi}{\partial\eta^2} + \frac{2a'}{a}\frac{\partial\varphi}{\partial\eta} - \Delta\varphi + \frac{R}{6}a^2\varphi\Big] = 0 \tag{22}$$

or for $\tilde{\varphi}$

$$\frac{1}{a^2}\Big[\frac{\tilde{\varphi}''}{a} - \tilde{\varphi}\frac{a''}{a^2} + \frac{R}{6}a\tilde{\varphi} - \frac{1}{a}\Delta\tilde{\varphi}\Big] = 0. \tag{23}$$

Using R from (19), due to compensation of terms we obtain

$$\frac{1}{a^3}[\tilde{\varphi}'' - \Delta\tilde{\varphi}] = 0. \tag{24}$$

Now let us put a source on the right-hand side of (2). Instead of Eq. (2) we now have

$$(\nabla_i\nabla^i + R/6)\varphi(x) = \mu\delta(\vec{x})/\sqrt{g^{(3)}}. \tag{25}$$

Here $\delta(\vec{x})$ is the usual δ-function, but we must include $\sqrt{g^{(3)}}$ where $g^{(3)}$ is the 3-metric determinant, so that $\sqrt{g^{(3)}} = a^3$.

Thus by (24), Eq. (25) can be written in the form

$$\frac{1}{a^3}[\tilde{\varphi}'' - \Delta\tilde{\varphi}] = \frac{1}{a^3}\mu\delta(\vec{x}), \tag{26}$$

and a static solution of this equation is the usual Coulomb potential but with a conformal factor:

$$\widetilde{\varphi} = -\frac{\mu}{r} \quad \text{and} \quad \varphi = -\frac{1}{a}\frac{\mu}{r}. \tag{27}$$

On the contrary, for minimal coupling instead of Eq. (24) we get

$$\frac{1}{a^3}\left(\widetilde{\varphi}'' - \widetilde{\varphi}\frac{a''}{a} - \Delta\widetilde{\varphi}\right) = 0, \tag{28}$$

so that, if there is a source, we have

$$\frac{1}{a^3}\left(\widetilde{\varphi}'' - \widetilde{\varphi}''\frac{a''}{a} - \Delta\widetilde{\varphi}\right) = \mu\delta(\vec{x})/\sqrt{g^{(3)}} \tag{29}$$

The situation is the same as if in flat space-time we had some mass term due to $a''/a \neq 0$. This is why, contrary to [8], we here lose the usual massless behaviour.

For a dust-like Universe $R \neq 0$ and $a(\eta) = a_0\eta^2$, so that the term $a''/a > 0$ and for $\eta \approx \text{const}$ it behaves like $m^2a^2 < 0$, i.e., a "tachyonic mass".

Now let us discuss the quasiclassical behaviour for this case. We write

$$\varphi = \frac{\rho}{a}\, e^{iS/\hbar}. \tag{30}$$

Then it is easily seen that for Eqs. (2) and (24) we have the following equations (31) and (32), respectively, valid up to $\hbar^{-2}$:

$$\frac{\partial S}{\partial\eta}\frac{\partial S}{\partial\eta} - \sum_{\alpha=1}^{3}\frac{\partial S}{\partial x_\alpha}\frac{\partial S}{\partial x_\alpha} = 0, \tag{31}$$

$$\frac{1}{a^2}\left\{\frac{\partial S}{\partial\eta}\frac{\partial S}{\partial\eta} - \sum_{\alpha=1}^{3}\frac{\partial S}{\partial x_\alpha}\frac{\partial S}{\partial x_\alpha}\right\} = m^2 \tag{32}$$

which is just $g^{00}\dfrac{\partial\delta}{\partial\eta}\dfrac{\partial S}{\partial\eta} - g^{\alpha\beta}\dfrac{\partial S}{\partial x^\alpha}\dfrac{\partial S}{\partial x^\beta} = m^2$.

For minimal coupling we obtain from Eq. (28)

$$\frac{1}{a^2}\left\{\frac{\partial S}{\partial\eta}\frac{\partial S}{\partial\eta} - \sum_{\alpha=1}^{3}\frac{\partial S}{\partial x_\alpha}\frac{\partial S}{\partial x_\alpha}\right\} = -\frac{a''}{a^3} = -\frac{R}{6}. \tag{33}$$

It is thus easy to see that geodesics for these particles cannot live on light cones. Even worse, they can be spacelike (the meaning of that for gravitons and vector mesons for large η is still to be understood!).

And now, in the end of our presentation, we return to the question: what is wrong when we naively (as was done in [8]) write Eq. (2) in a locally Lorentz frame as

$$(\Box\varphi + R/6)\varphi = \mu_1\delta(\vec{r}), \tag{34}$$

or better

$$(\Box\varphi + R/6)\varphi = \mu_1\delta(\vec{r})/\sqrt{g^{(3)}}. \tag{35}$$

The answer to this question has a great methodological meaning.

The operator $\nabla_i\nabla^i$ in Eq. (2) acting on the scalar field contains only one Christoffel symbol, so it seems that at "a point" (i.e., at P_0, the origin of the locally Lorentz coordinate frame) we can write $\Box\varphi$ as in Minkowski space-time. But, as discussed in [9], in order to find a solution to a given differential equation at a given point of a manifold, one must consider the properties of this solution in the neighbourhood of this point, properly taking into account the boundary conditions. Indeed, if $\langle\xi^\mu\rangle$ are locally Lorentz coordinates and $\langle x^\mu\rangle$ are arbitrary coordinates, and if $\langle X^\mu\rangle$ are the coordinates of P_0 (the origin of the locally Lorentz frame), then, as is well known,

$$\begin{aligned}\xi^\alpha(x) &= a^\alpha + b^\alpha_\mu(x^\mu - X^\mu)\\ &\quad + \tfrac{1}{2}b^\alpha_\lambda\Gamma^\lambda_{\mu\nu}(X)(x^\mu - X^\mu)(x^\nu - X^\nu),\end{aligned} \tag{36}$$

with $a^\alpha = \xi^\alpha(x)\Big|_{x=X}$; $b^\alpha_\lambda = \dfrac{\partial\xi^\alpha}{\partial x^\lambda}\Big|_{x=X}$. Also:

$$\Gamma^\alpha_{\beta\gamma,\mu}(\xi^\mu)\Big|_{\xi^\mu_0} = -\frac{1}{3}(R^\alpha_{\beta\gamma\mu}(\xi^\mu) + R^\alpha_{\gamma\beta\mu}(\xi^\mu))\Big|_{\xi^\mu_0}. \tag{37}$$

Then, for $\xi^\mu \neq \xi^\mu(P_0) = \xi^\mu_0$, the quantities $\Gamma^\alpha_{\beta\mu}(\xi^\mu)$ are nonzero. This means that, in order to get a solution for Eq. (2) with a source term in the $\langle\xi^\mu\rangle$ coordinates, valid for all spacetime and taking into account the boundary conditions, we cannot simply solve Eq. (34). What is necessary is to solve a very complicated equation in these coordinates. If this is done carefully, we must obtain the right result found above.

We find it useful to discuss in conclusion how to solve Eq. (2) in a coordinate frame used by astronomers. To do that, besides the synchronous frame

$$ds^2 = c^2dt^2 - a^2(t)[dr^2 + r^2(\sin^2\theta' d\varphi^2 + d\theta'^2)] \tag{38}$$

connected with the conformal one [(16)] by $cdt = ad\eta$, we introduce another coordinate frame where the space distance is given by $D = a(t)r$, so that

$$dD = adr + rda \tag{39}$$

and

$$\begin{aligned}ds^2 &= \left(1 - D^2\frac{\dot{a}^2}{c^2a^2}\right)c^2dt^2 + 2D\frac{\dot{a}}{ca}dDcdt\\ &\quad - dD^2 - D^2(\sin^2\theta d\varphi^2 + d\theta^2).\end{aligned} \tag{40}$$

An advantage of the reference frame associated with these coordinates is that, for an observer on the Earth, when $\varepsilon = D\dot{a}/(ca)$ is small, the Minkowski metric is a good approximation, and only for large enough D one has a curved space-time. Actually we can use ε as a small parameter. Let us write equation (7) in these coordinates. From

$$\begin{aligned}&g_{00} = 1 - (D\dot{a})^2/(ca)^2, \qquad g_{01} = 2D\dot{a}/(ca);\\ &g_{ii} = -1;\quad g_{\theta\theta} = -D^2,\quad g_{\varphi\varphi} = -D^2\sin^2\theta'\end{aligned} \tag{41}$$

and using the relations [9]

$$\begin{aligned} g^{0r}g_{r0} + g^{00}g_{00} &= 1,\\ g^{r0}g_{00} + g^{rr}g_{r0} &= 0,\\ 2g^{r0}g_{0r} + g^{rr}g_{rr} &= 1,\end{aligned} \tag{42}$$

we obtain

$$g^{0r} = \frac{2D\dot{a}/(ca)}{1 + 7[D\dot{a}/(ca)]^2}. \tag{43}$$

Then, in the first approximation in ε we have

$$g^{0r} = 2D\dot{a}/(ca) = g_{0r}. \tag{44}$$

Putting this into Eq. (17), noting that terms depending on ε^2 after differentiation still contain ε, we can remove them as $\varepsilon \to 0$. The only remaining term is the one containing derivatives in D of g^{0r}. So, taking for $\sqrt{-g}$ the Minkowski value, we obtain the equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right]\varphi + 2\frac{\dot{a}}{ca}\frac{\partial}{\partial t}\varphi + \frac{R}{6}\varphi = 0 \tag{45}$$

Thus the nondiagonal components g_{0r} lead to an extra term in the approximation $\varepsilon \to 0$.

But in order to solve this equation we must take

$$\varphi = \widetilde{\varphi}/a \tag{46}$$

which, as in the previous case of conformal time, immediately leads to a cancellation of the $R/6$ term, so that we finish with

$$\frac{1}{c^2 a}\ddot{\widetilde{\varphi}} - \frac{1}{a}\Delta\widetilde{\varphi} = 0. \tag{47}$$

Can we reject the term $2\frac{\dot{a}}{c^2 a}\frac{\partial}{\partial t}\varphi$ but retain the $\frac{R}{6}$ term? As seen from Einstein's equations, for usual cosmological models we must reject the term $R\varphi/6$ if we put $\dot{a}/a$, the Hubble's constant, equal to zero. At the modern epoch of evolution of the Universe, we can do this and use the Minkowski metric as a very good approximation near the Earth.

But for R large enough in the early epochs this cannot be done. But then the nondiagonal terms $g^{0r} \neq 0$ lead to the impossibility of having a unique time and defining space distances unambiguously. It follows that we cannot write the usual $\delta(\vec{r})$ function for a charge distribution and Eq. (34) does not possess an unambiguous meaning.

To finish the story of "Yukawa forces" for the conformal massless case let us consider the de Sitter Universe. As is well known, the De Sitter metric can be written, depending on coordinates used, in nonstationary or stationary forms. In the "curvature" coordinates the interval can be written as

$$ds^2 = \Big(1 - \frac{r^2}{a_0^2}\Big)dt^2 - \Big(1 - \frac{r^2}{a_0^2}\Big)^{-1}dr^2 - r^2 d\sigma^2,$$
$$d\sigma^2 = \sin^2\theta d\varphi^2 + d\theta^2. \tag{48}$$

In "horospheric" coordinates it can be written as (16) with $a(\eta) = a_0/\eta$.

In the synchronous frame (33), due to $a(\eta)d\eta = d\tau$ one has $a_0 \ln\eta = \tau$, leading to $a(\tau) = \mathrm{e}^{-\tau/a_0}$, and for $k = 0$

$$ds^2 = c^2 dr^2 - e^{-2\tau/a_0}(d\chi^2 + \chi^2 d\sigma^2). \tag{49}$$

Here we use the notations τ, χ instead of t, r used for the stationary case.

The connection between coordinates τ, χ, t, r is given as follows. First from t, r go to

$$\tau = t + \int\frac{f(r)dr}{1 - r^2/a_0^2}, \quad R = t + \int\frac{dr}{(1 - r^2/a_0^2)f(r)} \tag{50}$$

with $f(r) = r/a_0$. Then

$$R - \tau = a_0 \ln r \tag{51}$$

and

$$r = \exp[(R - \tau)/a_0]. \tag{52}$$

The interval (48) is written as

$$\begin{aligned} ds^2 &= d\tau^2 - (r^2/a_0^2)dR^2 - r^2 d\sigma^2 \\ &= d\tau^2 - e^{-2\tau/a_0}(d\chi^2 + \chi^2 d\sigma^2) \end{aligned} \tag{53}$$

if $\chi \equiv \exp(R/a_0)$.

But then it is easy to write our solution (28) of the Klein-Gordon equation for the massless conformal coupling case in the form

$$\begin{aligned} \varphi &= -\frac{1}{a(\tau)}\,\frac{M}{\chi} = M\exp\Big(\frac{1}{a_0}\tau\Big)\exp\Big(-\frac{R}{a_0}\Big) \\ &= -M\exp\bigg(-\frac{R-\tau}{a_0}\bigg) = -\frac{M}{r}. \end{aligned} \tag{54}$$

It is just the usual Coulomb potential and we consider this to be an end of "Yukawa-like" forces for the massless, conformal coupling case!

Acknowledgement

The authors are grateful to CNPq for research grants.

References

[1] N.O. Birrell, P.C.W. Davies, "Quantum Fields in Curved space", Cambridge Univ. Press, 1982.

[2] A.A. Grib, S.G. Mamayev and V.M. Mostepanenko, "Vacuum Quantum Effects in Strong Fields", Friedmann Laboratory Publishing, St. Petersburg, 1994.

[3] N.A. Chernikov and E.A. Tagirov, Ann. Inst. H. Poincaré 9A (1968), 109.

[4] A.A. Grib and A.V. Nesteruk, Gen. Relat. & Grav. 15 (1983), 31.

[5] A.D. Linde, "Elementary Particle Physics and Inflationary Cosmology", Moscow, Nauka, 1990.

[6] J. Schwinger, Phys. Rev. 93 (1951), 615.

[7] B.S. DeWitt, Phys. Rep. C 19 (1975), 296.

[8] A.P. Lightman, W.H. Press, R.H. Price and S.A. Teukolsky, "Problem Book in Relativity and Gravitation", Problem 14.14, p. 85. Princeton Univ. Press., Princeton, N. Jersey, 1975.

[9] W.A. Rodrigues, Jr., "On Locally Inertial Reference Frames in General Relativity", subm. for publ. (1994).

[10] L.D. Landau and E.M. Lifshitz, "Classical Theory of Fields" (fourth revised English edition), Pergamon Press, Oxford, 1975.