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MULTIDIMENSIONAL COSMOLOGY AND THE TIME VARIATION OF G: A DYNAMICAL SYSTEM APPROACH

C. Romero[1]† and V.N. Melnikov[2]‡

† School of Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS, UK;
Dep. de Física, Universidade Federal da Paraíba, C. Postal 5008 - J. Pessoa - Pb 58059-970 - Brazil

‡ Center for Gravitation and Fundamental Metrology, VNIIMS, 3–1 M. Ulyanovoy St., Moscow 117313, Russia

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We investigate the solutions to Einstein field equations in a $(4+n)$-dimensional spacetime generated by a multicomponent perfect fluid, assuming the existence of a compact and Ricci-flat internal space. We study the qualitative behaviour of solutions, using techniques from dynamical system theory. In this way we obtain information on the expansion of the models, the energy density and the time variation of the gravitational "constant", which in this theory has its own dynamics generated by the extra dimensions. Finally, exact solutions corresponding to the invariant rays of the system are obtained in arbitrary dimensions.

1. Introduction

Multidimensional cosmology has since long ago attracted the attention of cosmologists, who were stimulated initially mainly by the Kaluza-Klein theory [1, 2] and more recently by superstring models [3]. Along with studying some particular multidimensional models, a very extensive work was done to investigate general multidimensional cosmological and spherically symmetric models on the basis of exact solutions [4]. In these papers, classical and quantum systems with an arbitrary number of external spaces and different matter sources were analyzed and observational windows to extra dimensions, such as possible variations of the Newton and Coulomb laws, were studied.

The idea that the Universe we live in can be represented as a 4-dimensional hypersurface embedded in a $(4+n)$-spacetime manifold has actually different versions. In particular, we could mention the one put forward by Wesson, who has developed an embedding scheme in which the Friedmann-Robertson-Walker-Lemaître cosmology can be entirely obtained in a rather simple and elegant way from (4+1)-dimensional Ricci-flat spacetimes [5]. A further generalization of this theory to arbitrary dimension with applications to multidimensional cosmology and lower-dimensional gravity was later carried out by Rippl et al. [6]. More general exact solutions for multidimensional and multicomponent models were found and studied in [4, 11].

In addition to the role multidimensional theories might play in providing a theoretical framework in which the most fundamental laws of physics appear to be unified, another motivation comes from a conjecture — originally proposed by Dirac [7] — regarding a time variation of Newton's gravitational constant G. Indeed, this idea, which was to be taken seriously by superstring theory and recent inflationary models, is also present in the context of multidimensional cosmology where G is considered not as a fundamental constant of Nature, but as a cosmological function depending on the geometry of an "internal space" [8, 10, 11].

Among several attempts to construct gravity theories with varying G, are the Brans-Dicke theory [9] and theories with a conformal scalar field [10], where the strength of the gravitational force is determined by a scalar field. Here we find again the same idea underlying the connection between higher dimensions and time variation of G, since it can be shown that n-dimensional Kaluza-Klein models reduce to Brans-Dicke vacuum models with $\omega = 0$.

In this paper we consider, as in [4, 11] , a $(4+n)$-spacetime manifold defined by the topological product $M^{4+n} = R \times M_k^3 \times K^n$, where M_k^3 is a 3-dimensional space of constant curvature (i.e., $M_k^3 = S^3, R^3, L^3$ according to $k = +1,\ 0,\ -1$, respectively), and K^n is an n-dimensional Ricci-flat manifold. We also assume that this spacetime is generated by a $(4+n)$-dimensional multicomponent perfect fluid.

Now, it turns out that the field equations for the special case $k = 0$ may be reduced to a second-order autonomous homogeneous system. This system contains some free parameters, one of them being n (the

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[1] e-mail: car@maths.qmw.ac.uk
[2] e-mail: mel@cvsi.rc.ac.ru

internal space dimension) and the others come from the equations of state of the multicomponent fluid. However, restricting ourselves to "dust-like" matter, we are left with n as the only parameter of the system. Then, we construct a phase diagram of the system to obtain a general picture of the solutions. As a by-product, we also obtain analytical solutions to the equations for arbitrary values of n.

2. Field equations

The gravitational field equations in $(4+n)$-dimensional gravity are postulated to be

$$ {}^{(4+n)}R_{\mu\nu} = \kappa^2 \left({}^{(4+n)}T_{\mu\nu} - g_{\mu\nu} \frac{T}{(n+2)} \right), \tag{1} $$

where all the geometric quantities are defined in $(4+n)$ dimensions and κ^2 is the generalized Einstein constant [11]. We take the metric tensor to be given by the line element

$$ \begin{aligned} ds^2 = dt^2 &- R^2(t) {}^{(3)}g_{ij}(x^k) dx^i dx^j \\ &- b^2(t) {}^{(n)}g_{pq}(y^r) dy^p dy^q, \end{aligned} \tag{2} $$

where $i,\ j,\ k = 1,2,3;\ p,\ q,\ r = 4, ..., n+3;\ {}^{(3)}g_{ij}$, ${}^{(n)}g_{pq}$, $R(t)$ and $b(t)$ are, respectively, the metrics and scale factors for ${}^{(3)}M_k$ and K^n. The $(4+n)$-dimensional energy-momentum tensor for a multicomponent perfect fluid is taken to be

$$ T^{\mu}_{\nu} = \mathrm{diag}\left(\varrho(t), -p_3(t)\delta^i_j, -p_n(t)\delta^m_n\right). \tag{3} $$

From (2) and (3) the Einstein equations become:

$$ 3\frac{\ddot{R}}{R} + n\frac{\ddot{b}}{b} = \frac{\kappa^2}{n+2}\left(-(n+1)\varrho - 3p_3 - np_n\right), \tag{4} $$

$$ \begin{aligned} &\frac{2k}{R^2} + \frac{\ddot{R}}{R} + n\frac{\dot{R}}{R}\frac{\dot{b}}{b} + 2\frac{\dot{R}^2}{R^2} \\ &\qquad = \frac{\kappa^2}{n+2}\left(\varrho + (n-1)p_3 - np_n\right), \end{aligned} \tag{5} $$

$$ \frac{\ddot{b}}{b} + (n-1)\frac{\dot{b}^2}{b^2} + 3\frac{\dot{R}}{R}\frac{\dot{b}}{b} = \frac{\kappa^2}{n+2}(\varrho - 3p_3 + 2p_n). \tag{6} $$

At this point it is worthwhile mentioning the way by which higher-dimensional gravity theories of this type can be naturally related to their 4-dimensional counterparts with varying G [11]. This is simply done by integrating the $(4+n)$-dimensional energy density over the K^n compact space and equating the result to ${}^{(4)}\varrho(t)$, thereby defining the energy density in the 4-dimensional space-time:

$$ {}^{(4)}\varrho(t) = \int_{K^n} dy^n \sqrt{{}^{(n)}g}\, b^n(t)\varrho(t) = \varrho(t) b^n(t), \tag{7} $$

where $\sqrt{{}^{(n)}g}$ is the determinant of ${}^{(n)}g_{pq}$. It is convenient to "normalize" the scale factor $b(t)$ by imposing the condition $\int_{K^n} \sqrt{{}^{(n)}g}\, dy^n = 1$. Thus, in order to get the equations of 4-dimensional gravity we put

$$ 8\pi G(t) \left[{}^{(4)}\varrho(t) \right] = \kappa^2 \varrho(t). \tag{8} $$

This procedure leads us to the definition of an effective gravitational "constant" $G(t)$ given by $8\pi G(t) = \kappa^2 b^{-n}(t)$. In this way the time variation of G is directly related to the time variation of the internal space scale factor $b(t)$ by

$$ \frac{\dot{G}}{G} = -n\frac{\dot{b}}{b}. \tag{9} $$

Clearly for $n = 0$ the Friedmann cosmology in ordinary 4-dimensional spacetime is recovered.

3. The dynamical system and phase portraits

In this section we let $M^3_k = R^3$ and assume that the multicomponent fluid satisfies the equations of state $p_3 = p_n = 0$, i.e., we assume that the matter behaves as a $(n+4)$-dimensional 'dust'. Then, letting $x = 3\dot{R}/R$ and $y = \dot{b}/b$, Eqs. (4)–(6) become

$$ \dot{x} + \frac{x^2}{3} + n\dot{y} + \dot{y}^2 = -\frac{n+1}{n+2}\kappa^2\varrho \tag{10} $$

$$ \dot{x} + x^2 + Nxy = \frac{3\kappa^2\varrho}{n+2} \tag{11} $$

and

$$ \dot{y} + ny^2 + xy = \frac{\kappa^2\varrho}{n+2}. \tag{12} $$

Eliminating ϱ from these equations results in

$$ \begin{aligned} \dot{x} = \frac{1}{2(n+2)} &\left[-2(n+1)x^2 \right. \\ &\left. + 2n(1-n)xy + 3n(n-1)y^2 \right] \end{aligned} \tag{13} $$

and[1]

$$ \dot{y} = \frac{1}{2(n+2)}\left[\frac{2x^2}{3} - 4xy - n(n+5)y^2 \right]. \tag{14} $$

Defined in this way, x can be interpreted as the usual cosmological expansion of the 4-dimensional observable Universe, while y is a measure of the variation of the gravitational constant G or, equivalently, the expansion of the compact space K^n (see Eq. (9)). The above set of equations represents a homogeneous autonomous second-order dynamical system. To carry out an analysis of this system, we first note that, as the system is homogeneous, the origin of the phase space $x = y = 0$ corresponds to an equilibrium point (in fact, an isolated equilibrium point) [12]. Physically, this point

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[1] It is of course possible to absorb the factor $1/[2(n+2)]$ defining a new time $d\tau = 2(n+2)dt$. However, nothing is gained by this in terms of simplicity.

Figure 1: Phase portrait of the solution for $n > 1$.

Figure 2: Phase portrait for $n = 1$

represents nothing else but the flat Minkowski spacetime of General Relativity, with $\varrho = 0$.

In order to construct a phase diagram of a homogeneous dynamical system we first determine the invariant rays of the system [12] by introducing polar coordinates in the phase plane: $x = r\cos\theta,\ y = r\sin\theta$. In these coordinates a general homogeneous dynamical system of order m of the form

$$\dot{x} = X_m(x,y), \quad \dot{y} = Y_m(x,y)$$

transforms into

$$\dot{r} = r^m Z(\theta), \quad \dot{\theta} = r^{m-1} N(\theta),$$

where the functions $Z(\theta)$ and $N(\theta)$ are given by

$$Z(\theta) \ = \ Y_m(\cos\theta,\sin\theta)\sin\theta + X_m(\cos\theta,\sin\theta)\cos\theta \tag{15}$$

$$N(\theta) \ = \ Y_m(\cos\theta,\sin\theta)\cos\theta - X_m(\cos\theta,\sin\theta)\sin\theta. \tag{16}$$

Then, the invariant rays of the system are obtained by solving the equation $N(\theta) = 0$. Clearly, in the phase plane they will be depicted as straight semi-lines starting from the origin, and it is not difficult to see that if they do exist, then they are automatically solutions for the dynamical system [12]. In our case $m = 2$ and a straightforward calculation leads to

$$\begin{aligned} Z(\theta) = {} & \frac{1}{2(n{+}2)}\Big[-n(n{+}5)\sin^3\theta \\ & + (3n^2{-}3n{-}4)\sin^2\theta\cos\theta \\ & + (2n{-}2n^2{+}\frac{2}{3})\sin\theta\cos^2\theta \\ & - 2(n{+}1)\cos^3\theta\Big], \end{aligned} \tag{17}$$

$$\begin{aligned} N(\theta) = {} & \frac{1}{2(n{+}2)}\Big[-3n(n{-}1)\sin^3\theta \\ & + n(n{-}7)\sin^2\theta\cos\theta \\ & + 2(n-1)\cos^2\theta\sin\theta + \frac{2}{3}\cos^3\theta\Big]. \end{aligned} \tag{18}$$

Here let us make some comments. First, we should point out that the dynamical system (13)-(14) is not defined for $n = 0$, since in this case we would not have Eq. (6). If $n = 1$, then the solutions of the equation $N(\theta) = 0$ yield six invariant rays which correspond to the angles $\theta_i = \pm\pi/2$ and $\arctan(\pm 1/3)$, with $i = 1, ..., 6$ (see Fig. 1). For an arbitrary $n > 1$ we can put Eq. (18) in the following factorized form:

$$N(\theta) = \frac{\cos^3\theta}{2n{+}4}\left\{\left(\tfrac{1}{3}{-}a\right)\left[3n(n{-}1)a^2 + 6na + 2\right]\right\} \tag{19}$$

where we have defined $a = \tan\theta$. Then, for $n > 1$ we have again six invariant rays, now corresponding to the angles $\theta_i = \arctan a_i$, with

$$a_0 = \frac{1}{3}, \quad a_\pm = \frac{1}{n-1}\left(-1 \pm \sqrt{\frac{1}{3}\Big(1 + \frac{2}{n}\Big)}\right).$$

The knowledge of the invariant rays, as well as the analytic expressions for the functions $N(\theta)$ and $Z(\theta)$, allow us to draw separately the phase diagrams of Figs. 1 and 2 for the two cases $n = 1$ and $n > 1$ (for details see the Appendix). These diagrams show the behaviour of all solutions to Eqs. (13)–(14) which make up our dynamical system. Each curve corresponds to a specific cosmological model satisfying the field equations (12)–(13), the origin representing the Minkowski space-time M. In order to know the behaviour of the solutions at infinity, we employed a method due to Poincaré, consisting in projecting the phase plane onto a plane circle [16]. In this compactified phase plane the points at infinity correspond to points located at the border of the circle, the directions of the invariant rays being unaffected by the transformation (see the Appendix).

4. The physical picture

Let us begin our analysis considering $n > 1$, and leave the comments on the case $n = 1$ to the end of this section. In Fig. 1 we have a typical diagram for arbitrary $n > 1$. First we note that the invariant rays divide the phase plane into six topologically distinct regions (or sectors) A, B,..., F. Each of these regions contains an infinite number of solutions which represent cosmological models with different physical properties. The arrows on the curves are to be interpreted as the time evolution of the corresponding models.

Since there is no closed curve in the phase plane, we can conclude that all models are singular (the expansion parameter x tends to infinity either in the past, or in the future), some of them starting from a big bang ($x \to +\infty$), while others collapsing to a big crunch ($x \to -\infty$). In this sense the solutions represented by the invariant rays exhibit the same behaviour. It would be rather tedious to describe exhaustively the time evolution of the models corresponding to all the curves of the phase diagram. So, we will pick up some illustrative cases, although the complete information about all solutions is provided by the phase portrait.

To begin with, let us consider the solution represented by the invariant ray depicted in Fig. 1 as the semi-line I^+. This curve clearly describes a universe starting from a big bang ($x = +\infty$) and evolving towards the Minkowski spacetime (depicted in the diagram as the fixed point M located at the origin) . Since $y > 0$ along this trajectory, we see that, as time goes by, the gravitational constant G decreases. This is in agreement with the known hypothesis by Dirac, who postulated, inspired by a different reasoning (the large numbers conjecture), that the Newtonian gravitational constant should decrease as the Universe expands [7].

A similar analysis shows us that the invariant ray II^+ corresponds to an expanding universe starting from a big bang and tending to Minkowski space-time. Since y is negative in this anti-Dirac universe, the gravitational constant G increases with cosmic time.

The invariant rays I^+ and II^+ enclose an infinite class of solutions all lying within the region A. A typical solution of this class describes an expanding and singular universe undergoing a transition from an increasing G (anti-Dirac) era to one with decreasing G (the Dirac phase).

Quite a different situation arises when one examines the solution corresponding to the invariant ray III^+. Here we observe an initially static universe ($x = 0$) entering an expansion regime with the increasing gravitational constant.

At this point it is interesting to note that one might look alternatively at the dynamics of the models corresponding to II^+ and III^+ as describing the usual cosmic expansion taking place in ordinary 4-dimensionality (here expressed by the variable x) followed by a contraction of the internal n-dimensional space (represented here by y). Sector B, which is delimited by II^+ and III^+, contains only solutions which do not approach Minkowski space-time, neither in the future nor in the past. On the other hand, the solutions lying in Sector F, all tend to M and start their trajectories as contracting universes, slowing down and entering an expansion era. In this class of models the gravitational constant is an ever decreasing function of the cosmic time.

We shall not carry out a detailed analysis of the solutions lying in Sectors D and E, as these describe only contracting universes, ipso facto not being physically relevant. (As we shall see later, in Section 6, Sectors E and B both represent classes of solutions with negative energy density.) In Sector C a typical universe comes from Minkowski spacetime in the past and has a contraction era followed by further expansion.

In the case $n = 1$ (see Fig. 2) the physical picture is very similar. However, now, as two of the invariant rays, namely, III^+ and III^- lie exactly on the y axis, they represent vacuum flat solutions with a time-varying G. (In fact, an identical configuration has been already found in the context of the Brans-Dicke theory by Romero and Barros [13]). An alternative way to look at these solutions is to consider them as a topological product of a static Minkowski spacetime by a time-dependent (expanding or contracting) compact internal space.

5. Exact solutions of the field equations

Often the knowledge of the invariant rays present in a homogeneous dynamical system, is helpful in obtaining exact analytic solutions of the system. In that case, the problem of finding the solutions corresponding to the invariant rays reduces to solving an algebraic equation one order higher than the system itself. In our particular case we will have to solve a cubic polynomial equation, whose roots are nothing more than the already known tangents a_i of the arcs defined by the invariant rays.

Let us express the equations of the invariant rays simply by $y = ax$, where clearly a generically denotes a_i. Now, putting this into Eqs. (13)–(14), we get

$$\dot{x} = \frac{x^2}{2(n+2)}\left[-2(n+1) + 2n(1-n)a + 3n(n-1)a^2\right] \tag{20}$$

$$\dot{y} = a\dot{x} = \frac{x^2}{2(n+2)}\left[\tfrac{2}{3} - 4a - n(n+5)a^2\right]. \tag{21}$$

The condition for (20) and (21) to be consistent is the algebraic equation

$$3n(n-1)a^3 + n(7-n)a^2 + 2(1-n)a - \tfrac{2}{3} = 0 \tag{22}$$

which is, in fact, equivalent to Eq. (18). Again, we have to consider two cases: (a) $n > 1$ and (b) $n = 1$.

(a) If $n > 1$, then the roots of (24) are given by

$$a_0 = \frac{1}{3}, \quad a_\pm = \frac{1}{n-1}\left[-1 \pm \sqrt{\frac{1}{3}(1+\frac{2}{n})}\right].$$

Now, going back to Eq. (13) and putting $y = ax$, with $a = a_0,\ a_\pm$, we get, respectively:

$$\dot{x} = \gamma x^2 \tag{23}$$

where $\gamma = \gamma_0,\ \gamma_\pm$ and

$$\gamma_0 = -(n+3)/6, \tag{24}$$

$$\gamma_\pm = -(1 + na_\pm). \tag{25}$$

These last equations can be immediately integrated to give $R(t)$ and $b(t)$. Then, corresponding to the three values of $a = a_0,\ a_\pm$, we have, respectively (after suitable coordinate transformations):

$$R(t) \sim t^{-1/(3\gamma_0)} = R_0 t^{2/(n+3)}, \tag{26}$$

$$b(t) \sim [R(t)]^{3a_0} = b_0 t^{2/(n+3)}, \tag{27}$$

$$R(t) \sim t^{-1/(3\gamma_\pm)} = R_0 t^{-1/[3(1+na_\pm)]}, \tag{28}$$

$$b(t) \sim [R(t)]^{3a_\pm} = b_0 t^{a_\pm/(1+na_\pm)}, \tag{29}$$

where R_0 and b_0 are constants.

(b) If $n = 1$, then the Eq. (24) has two solutions, namely, $a = \pm 1/3$. Naturally, these solutions correspond to the invariant rays defined by $\theta_i = \arctan(\pm 1/3)$ in Section 3. The third solution, corresponding to the other invariant rays, $\theta_i = \pm\pi/2$, can be obtained directly from the dynamical system (Eqs. (13)–(14)) just by putting $n = 1$ and $x = 0$. This procedure leads us back to the static solution mentioned earlier in section 4:

$$R(t) = \text{const}, \tag{30}$$

$$b(t) = b_0 t. \tag{31}$$

The other solutions are:

$$R(t) = R_0 t^{1/3}, \tag{32}$$

$$b(t) = b_0 t^{1/3}, \tag{33}$$

and

$$R(t) = R_0 t^{1/3}, \tag{34}$$

$$b(t) = b_0 t^{-1/3}. \tag{35}$$

We conclude this section by noting that Eqs. (26)–(35) actually represent six distinct pairs of solutions $R(t),\ b(t)$, each being singular at $t = 0$. Indeed, after integrating (25) we obtain (apart from a constant of integration which can be further eliminated by a coordinate transformation)

$$x = -1/(\gamma t), \tag{36}$$

which has in fact to be understood as representing different solutions (for the same γ) according to $t \in (-\infty,\ 0)$ or $t \in (0,\ +\infty)$. In the phase diagrams these twofold degeneracy is reflected by the presence of distinct solutions (including the equilibrium point M) all lying on the same line $y = ax$. Finally, we should mention that, if $n = 0$ in (28), we recover Friedmann's solution for a dust-filled universe.

6. Energy density

So far we have not been concerned with the energy density predicted by the models. A brief glance at the field equations shows us that ϱ must be given by

$$\varrho = \frac{1}{6\kappa^2}\left[2x^2 + 3n(n-1)y^2 + 6nxy\right]. \tag{37}$$

If $n > 1$, the above equation, however, can be put into the factorized form

$$\varrho = \frac{1}{6\kappa^2}(y - a_+x)(y - a_-x), \tag{38}$$

with $a_\pm$ defined in Section 5. This last equation allows us to draw the following conclusions:

(i) For $n > 1$ we verify that the solutions lying on the invariant rays corresponding to $a_\pm$, are vacuum solutions.

(ii) All solutions lying in the sectors B and F, are nonphysical (in the sense that they have negative energy, which is classically forbidden). Incidentally, these are the only solutions which never tend to Minkowski spacetime, neither in the past, nor in the future.

(iii) Solutions lying on the invariant ray corresponding to a_0 have positive energy density for an arbitrary value of $n > 1$. This can be easily verified by computing ϱ for this case, as we have $\varrho = x^2/(36\kappa^2)\left[2n^2 + n + 12\right]$.

All the properties mentioned above are illustrated in Fig. 3 [2].

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[2] One could argue that it is not exactly ϱ , but ${}^{(4)}\varrho$, the physical quantity which would be actually measured. However, from Eq. (7) we see that all that has been said in this section of ϱ is also true for ${}^{(4)}\varrho$.

Figure 3: Energy density diagram for $n > 1$

Figure 4: Energy density diagram for $n = 1$

For $n = 1$ the same procedure leads to the picture displayed by Fig. 4.

7. Conclusions

The idea that the Newtonian constant of gravitation G could indeed vary with time on a cosmic scale, which seems to have ocurred first to Dirac, in 1938, is far from being supported by current experimental data. Recent results [14], based on Solar-system experiments, tend to indicate an upper limit given by $|\dot{G}/G| < 10^{-12}$ to any possible variation of G. Yet even this rather stringent condition has not prevented cosmologists from speculating and investigating what theoretical consequences such hypothesis lead to (for a list of references on past and recent works see [8, 9, 11, 15]). Among other attempts to insert G in gravity theories as a scalar field (e.g., Brans-Dicke-Jordan theories), is the Multidimensional Cosmology approach [11] described in Section 2. The fact that in this scheme the field equations plus some symmetry assumptions may be tractable by mathematical techniques of the dynamical system theory led us to obtain a whole spectrum of cosmic configurations where the matter of the Universe is regarded as a multicomponent perfect fluid in higher dimensions. It turns out that in this scheme some solutions exhibit nonphysical behaviour (at least from a classical standpoint). However, other solutions seem not to be in contradiction with generally accepted and standard models of the Universe, as they manifest properties such as cosmic expansion and the existence of an initial singularity. Also, in some of these expanding solutions the gravitational constant G decreases with time, a property which may justify calling them Dirac universes (we detect the presence of anti-Dirac models as well). Evidently, it was not our aim here to provide a quantitative discussion of the solutions, even the more physically relevant ones, trying to square them with the present observational and experimental data. Rather, our interest in this paper was actually to call the attention of theorists to the extremely rich scenario which arises when one allows for higher dimensions and the varying gravitational constant hypothesis.

Appendix

In order to construct the phase diagrams corresponding to Figs. 1 and 2, all we need is to calculate the values of the functions $N^l(\theta)$ and $Z(\theta)$ at $\theta = \theta_i$, where θ_i is an invariant ray and the superscript l refers to the first non-vanishing derivative evaluated at θ_i [12]. Since the system is quadratic, the phase portraits are symmetric under plane reflections ($x \to -x$, $y \to -y$), although the time orientation of the curves must be reversed in this operation. Such a property means that our analysis should be carried out in the neighbourhood of just three of the six invariant rays. Then, let us summarize the results which come from straightforward calculations.

For both cases $n > 1$ and $n = 1$, we obtain the following:

$l = 1$, $N^1(\theta_1) < 0$, $N^1(\theta_2) < 0$, $N^1(\theta_3) > 0$, $Z(\theta_1) < 0$, $Z(\theta_2) < 0$, and $Z(\theta_3) > 0$, where for the case $n > 1$ the invariant rays are: $\theta_1 = \arctan(1/3)$, $\theta_2 = \arctan a_+$, $\theta_3 = \arctan a_-$, whereas for the case $n = 1$, $\theta_1 = \arctan(+1/3)$, $\theta_2 = \arctan(-1/3)$ and $\theta_3 = -\pi/2$. With these results we can classify for arbitrary values of n the invariant rays θ_1 and θ_2 as being of type (β), while θ_3 is of type (α) [12]. From this classification we are led to the diagrams displayed in Figs. 1 and 2.

To carry out the Poincaré compactification of the phase plane, we perform the transformations of variables $u = y/x$ and $z = 1/x$. Then, starting from the Eqs. (20) and (21), we end up with the dynamical system:

$$\frac{du}{d\tau} = \frac{1}{2(n+2)} \left[\left(\tfrac{1}{3} - u\right) \left(3n(n-1)u^2 + 6nu + 2\right) \right] \quad (39)$$

$$\frac{du}{d\tau} = \frac{z}{2(n+2)} \left[2(n+1) + 2n(n-1)u + 3n(1-n)u^2 \right], \quad (40)$$

where $z\,d\tau = dt$. The equilibrium points of the dynamical system in the plane uz are: $(1/3,\ 0)$, $(u_\pm,\ 0)$, with $u_\pm = a_\pm$. A simple analysis of the topological nature of these points reveals that they correspond to a saddle point and two nodes (unstable and stable), respectively [16].

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References

[1] T Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin, Phys. Math., 33 (1921), 966.

[2] O. Klein, Z. Phys. 37 (1926), 895.

[3] M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory" Cambridge University Press, 1987.

[4] K.A. Bronnikov, V.D. Ivashchuk and V.N. Melnikov, Nuovo Cim. 102 (1988), 209.
V.D. Ivashchuk, V.N. Melnikov and A. I.Zhuk, ibid., 104 B (1989), 575.
V.N. Melnikov, preprint CBPF–N 7–051/93, Rio de Janeiro.
K.A. Bronnikov and V.N. Melnikov, Ann. of Phys. N.Y. 239 (1995), 40.
K.A. Bronnikov, Grav. & Cosmol. 1 (1995), 67.
V.D. Ivashchuk and V.N. Melnikov, Int. J. Mod. Phys. D 3 (1995), 795.
V.D. Ivashchuk and V.N. Melnikov, Class. and Quant. Grav. 12 (1995), 809.

[5] P.S. Wesson,(1992). Astrophys. J. 394 (1992), 19.
P.S. Wesson and J. Ponce de Leon, J. Math. Phys. 33 (1992), 3883.

[6] S. Rippl, C. Romero and R. Tavakol "D-dimensional Gravity from (D+1)-dimensions", Preprint 03/95-DF-UFPb, 1995.

[7] P.A.M. Dirac, Proc. R. Soc. A165 (1938), 199.

[8] D. La and P.J. Steinhardt, Phys. Rev. Lett. 62 (1989), 376.

[9] C. Brans and R.H. Dicke, Phys. Rev. 124 (1961), 539.
J. Narlikar, "Introduction to Cosmology", Cambridge University Press, 1983, Chapter 8.

[10] K.P. Staniukovich and V.N. Melnikov, "Hydrodynamics, Fields and Constants in Gravitation Theory", Energoatomizdat, Moscow, 1983 (in Russian).

[11] V.N. Melnikov, "Multidimensional Classical and Quantum Cosmology", Proceedings of the VII Brazilian School of Cosmology, ed. M. Novello, Editions Frontieres, Singapore, 1994, p. 147.

[12] G. Sansoni and R. Conti, "Nonlinear Differential Equations", Pergamon Press, Oxford, 1964, Chapter 2.

[13] C. Romero and A. Barros, Gen. Rel. Grav. 25 (1993), 491.

[14] C. Will, "Theory and Experiment in Gravitational Physics", Cambridge University Press, Cambridge, 1981, Section 8.4.

[15] T. Damour, G.W. Gibbons and C. Gundlach, Phys. Rev. Lett. 64 (1990), 123.

[16] A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, "Qualitative Theory of Second Order Dynamic Systems", John Wiley & Sons, New York, 1973, Chapter 6.