

MAGNETOHYDRODYNAMIC EQUATIONS IN A GRAVITATIONAL FIELD AND EXCITATION OF MAGNETOHYDRODYNAMIC SHOCK WAVES BY A GRAVITATIONAL WAVE

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On the basis of simple principles we derive and investigate the equations of relativistic plasma magnetohydrodynamics (MHD) in an arbitrary gravitational field. An exact solution describing the motion of magnetoactive plasma against the background of the metric of a plane gravitational wave (PGW) with an arbitrary amplitude is obtained. It is shown that in strong magnetic fields even a sufficiently small amplitude PGW can create a shock MHD wave, propagating at a subluminal velocity. Astrophysical consequences of the anomalous plasma acceleration are considered.

1. Introduction

In [1–4] the effect of PGW on plasmalike media was investigated by the methods of relativistic kinetic theory in the approximation when the back reaction of matter on the PGW is negligible :

$$\varepsilon \ll \omega^2, \quad (1)$$

where ω is the characteristic frequency of a PGW, ε is the matter energy density ($G = \hbar = c = 1$). These papers have revealed a number of phenomena of interest, consisting in the induction of longitudinal electric oscillations in the plasma by PGW. In spite of the strictness of the results obtained in [1–4], the effects discovered in these papers have very little to do with the real problem of GW detection. Moreover, the above calculations show lack of any prospect for GW detectors based on dynamic excitation of electric oscillations by gravitational radiation. There are two reasons for that: the smallness of the ratio $(m^2 G/e^2) = 10^{-43}$ and the small relativistic factor $\langle v^2 \rangle/c^2$ of standard plasmalike systems. The GW energy transformation coefficient to plasmatic oscillations is directly proportional to a product of these factors.

However, the situation may change radically if strong electric or magnetic fields are present in the plasma. In Ref. [5], where the induction of surface currents at a metal-vacuum interface by a PGW was studied, it was shown that the values of currents thus induced can be of experimental interest. In [6], on the basis of relativistic kinetic equations, a set of MHD

equations was obtained, which described the motion of collisionless magnetoactive plasma in the field of a PGW of an arbitrary magnitude in a drift approximation and it was shown that, provided the propagation of the PGW is transversal, there arises a plasma drift in the PGW propagation direction. The set of equations obtained in [6] is rather complex and unwieldy: it is a set of nonlinear partial differential equations. In [7], however, it was shown that, provided the plasma is originally electroneutral and uniform, the solution of the above set of equations is strictly stationary, i.e., it depends only on retarded time. This fact permits us to substantially simplify the problem and to find its exact solution, possessing a number of remarkable peculiarities.

2. The conditions of magnetic field embedding in the plasma

As pointed out above, in [6], on the basis of a self-consistent set of collisionless kinetic equations and the Maxwell equations (i.e., on the basis of general-relativistic Vlasov equations [8]), a set of MHD equations describing the motion of magnetoactive plasma in the field of a PGW, was obtained. This set of equations is obtained in the so-called drift approximation, i.e., in the first approximation in the small parameter ξ

$$\xi = \omega/\omega_B \ll 1, \quad (2)$$

where $\omega_B = eH/(m_e c)$ is the Larmor frequency. However, the equations obtained in [6] are applicable only

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in the case of a strictly transverse PGW propagation, where the original magnetic field is perpendicular to the GW propagation direction. It is not difficult to verify that if the conditions of the strict transversity of PGW propagation are not met, the equations of [6] violate the energy and momentum conservation laws. For our purposes it is necessary to consider a more general case, so in what follows we shall obtain the MHD equations on the basis of other principles.

It is not difficult to see that a consequence of the MHD equations from [6] is the magnetic field embedding in the plasma (MFEP). This reflects the general nature of magnetoactive plasma provided that the condition (2) is met. Therefore, in order to describe the motion of the plasma in a drift approximation, it is simpler to demand at once that the MFEP condition be met.

Mathematically this requirement means a coincidence between the timelike eigenvectors of the plasma energy-momentum tensor (EMT), T_{ik}^p , and that of the electromagnetic field, T_{ik}^f , i.e., according to Synge [9], a coincidence of the dynamic velocities of the plasma and the electromagnetic field:

$$T_k^p u^k = \varepsilon_p u^i, \quad (3)$$

$$T_k^f u^k = \varepsilon_H u^i, \quad (4)$$

where

$$(u, u) \stackrel{\text{def}}{=} g_{ik} u^i u^k = 1. \quad (5)$$

We shall consider in this paper the EMT of the plasma as that of a perfect isotropic fluid

$$T^{ik} = (\varepsilon + p)v^i v^k - pg^{ik}, \quad (6)$$

where

$$(v, v) = 1, \quad (7)$$

and ε and p are the fluid energy density and pressure connected by a certain equation of state:

$$p = p(\varepsilon). \quad (8)$$

Thus v^i is the timelike eigenvector of the plasma EMT ($v^i = u^i$), while ε is the eigenvalue T^{ik} ($\varepsilon_p = \varepsilon$), and there remain the conditions (4):

$$T_k^f v^k = \varepsilon_H v^i. \quad (9)$$

Thus (9) are precisely the conditions of magnetic field embedding in the plasma (MFEP). It is our purpose to clarify all the restrictions imposed by these conditions on the Maxwell tensor F_{ik} . Using the plasma velocity vector v^i , we shall introduce the vectors of the electric field E_i and magnetic field H_i as observed in

the frame of reference (FR) comoving with the plasma [10]:

$$E_i = v^k F_{ki}; \quad H_i = v^k \overset{*}{F}_{ki}, \quad (10)$$

where $\overset{*}{F}_{ki}$ is a tensor dual to the Maxwell antisymmetric tensor F_{ki} :

$$\overset{*}{F}_{ki} = \frac{1}{2} \eta_{kilm} F^{lm}, \quad (11)$$

and η_{kilm} is the covariantly constant discriminant tensor (see, e.g., [9]) satisfying the identity

$$\eta_{iklm} \eta^{iqps} = -\delta_{klm}^{qps} \stackrel{\text{def}}{=} - \begin{vmatrix} \delta_k^q & \delta_l^q & \delta_m^q \\ \delta_k^p & \delta_l^p & \delta_m^p \\ \delta_k^s & \delta_l^s & \delta_m^s \end{vmatrix}. \quad (12)$$

Due to (10), the vectors E and H are spacelike and orthogonal to the velocity vector:

$$(v, E) = 0; \quad (v, H) = 0. \quad (13)$$

The relations (10) can be resolved with respect to the Maxwell tensor [10]:

$$F_{ik} = v_i E_k - v_k E_i - \eta_{iklm} v^l H^m; \quad (14)$$

$$\overset{*}{F}_{ik} = v_i H_k - v_k H_i + \eta_{iklm} v^l E^m.$$

Let us represent the EMT of the electromagnetic field as follows:

$$T_k^f = \frac{1}{4\pi} (F^i_l F^l_k + \frac{1}{4} \delta_k^i F^{lm} F_{lm}), \quad (15)$$

using three vectors ($v; E; H$), one of which, v , is timelike, while two others, E and H , are spacelike:

$$T_k^f = -\frac{1}{8\pi} \left[\delta_k^i (E^2 + H^2) - 2v^i v^k (E^2 + H^2) + 2E^i E_k + 2H^i H_k + 2v^i \eta_{kpqs} E^p v^q H^s + 2v^k \eta^{ipqs} E_p v_q H_s \right], \quad (16)$$

where the following notations are introduced:

$$E^2 \stackrel{\text{def}}{=} -(E, E); \quad H^2 \stackrel{\text{def}}{=} -(H, H). \quad (17)$$

We shall require that the vector v be an eigenvector of the EMT (16). Contracting (16) with v^k and taking into account the identities (12) and (13), we get:

$$\frac{1}{8\pi} [v^i (E^2 + H^2) - 2\eta^{ipqs} E_p v_q H_s] = \varepsilon_H v^i. \quad (18)$$

From (18) we shall find the necessary condition for v^i being an eigenvector of the electromagnetic field EMT:

$$\eta^{ipqs} E_p v_q H_s = \lambda v^i, \quad \forall \lambda \in \mathbb{R}. \quad (19)$$

Contracting this relation with v^i we get $\lambda = 0$. Thus the necessary condition of compatibility (3) and (4) is

$$\eta^{ipqs} E_p v_q H_s = 0. \quad (20)$$

The necessary and sufficient condition for the fulfilment of Eq. (20) is, as we know, the complanarity of the vectors E , H and v , i.e.,

$$\alpha v_i + \beta E_i + \gamma H_i = 0.$$

Contracting this relation with v^i and taking into account (7), (13) we obtain $\alpha = 0$. Thus, Eq. (20) is equivalent to the condition

$$\beta E_i + \gamma H_i = 0.$$

Since we consider magnetoactive plasma, we shall further assume

$$F_{ik}F^{ik} \equiv 2(H^2 - E^2) > 0. \quad (21)$$

Due to (21), the necessary and sufficient condition for the fulfilment of (20) is

$$E_i = \lambda H_i; \quad \forall \lambda \in \mathbb{R} \quad (22)$$

Besides, according to (18),

$$\varepsilon_H = \frac{E^2 + H^2}{8\pi}. \quad (23)$$

However, we have not yet extracted all the algebraic information contained in (3) and (4). With due account of the definitions of the vectors E and H (10), Eqs. (22) can be written as

$$\mathcal{F}_{ik}v^k = 0, \quad (24)$$

where a new antisymmetric tensor \mathcal{F}_{ik} has been introduced:

$$\mathcal{F}_{ik} = F_{ik} - \lambda \overset{*}{F}_{ik}. \quad (25)$$

The relations (24) can be regarded as a set of linear homogeneous algebraic equations with respect to the velocity vector v^i . The necessary and sufficient condition for a nontrivial compatibility of these equations is

$$\det \|\mathcal{F}_{ik}\| = 0. \quad (26)$$

As $\|\mathcal{F}_{ik}\|$ is an even order antisymmetric matrix,

$$\det \|\mathcal{F}_{ik}\| = \frac{1}{16}(\sqrt{-g}\mathcal{F}^{ik} \overset{*}{\mathcal{F}}_{ik})^2. \quad (27)$$

Therefore, the condition (26) reduces to the following:

$$\eta^{ijkl}\mathcal{F}_{ij}\mathcal{F}_{kl} = 0; \quad (28)$$

in this case

$$\text{rank} \|\mathcal{F}_{ik}\| = 2, \quad (29)$$

i.e., the set (24) admits two linearly independent solutions for the eigenvector v^i .

Substituting into (28) the expressions for F_{ik} and $\overset{*}{F}_{ik}$ from (14), we get:

$$\eta^{ijkl}\mathcal{F}_{ij}\mathcal{F}_{kl} = 4\lambda(1 + \lambda^2)H^2 = 0,$$

hence follows the only possible solution under the condition (21):

$$\lambda = 0.$$

Thus we come to the following rigorous conclusion. For the EMT of electromagnetic field (15) to permit as its eigenvector the dynamical velocity vector of the isotropic perfect fluid under the condition (21), it is necessary and sufficient that the electric field intensity vector in the comoving FR should be equal to zero:

$$E_i = 0. \quad (30)$$

In this case the conditions (24)–(29) give:

$$\overset{*}{F}^{ik} F_{ik} = 0; \quad (31)$$

$$\det \|F_{ik}\| = 0 \implies \text{rank} \|F_{ik}\| = 2, \quad (32)$$

while the eigenvector of the fluid EMT must satisfy the set of linear homogeneous algebraic equations

$$F_{ik}v^k = 0. \quad (33)$$

Note that if Eq. (32) is fulfilled, then similar conditions for the dual Maxwell tensor are automatically fulfilled as well:

$$\det \|\overset{*}{F}_{ik}\| = 0 \implies \text{rank} \|\overset{*}{F}_{ik}\| = 2. \quad (34)$$

With (30) write down the Maxwell tensor and the electromagnetic field EMT (15):

$$\begin{aligned} F_{ik} &= -\eta_{iklm}v^l H^m; \\ \overset{*}{F}_{ik} &= v_i H_k - v_k H_i; \end{aligned} \quad (35)$$

$$\overset{f}{T}^i_k = \frac{1}{8\pi}(2H^2 v^i v_k - 2H^i H_k - \delta_k^i H^2). \quad (36)$$

Note that due to Eq. (35) a more severe condition than (31) is fulfilled:

$$\overset{*}{F}_{ik}F^{lk} = 0. \quad (37)$$

The summed EMT of the magnetoactive plasma

$$T_{ik} = \overset{p}{T}_{ik} + \overset{f}{T}_{ik}$$

takes the form

$$T_{ik} = (\mathcal{E} + P)v_i v_k - P g_{ik} - 2P_H n_i n_k, \quad (38)$$

where

$$P_H = \frac{H^2}{8\pi}; \quad \mathcal{E} = \varepsilon + \varepsilon_H; \quad P = p + P_H, \quad (39)$$

P and \mathcal{E} being the summed pressure and energy density of the magnetoactive plasma and

$$n_i = H_i/H \quad (40)$$

is the spacelike unit vector of magnetic field direction:

$$(n, n) = -1, \quad (41)$$

with

$$(n, v) = 0. \quad (42)$$

3. MHD equations for plasma in a gravitational field

The MHD equations are to be obtained on the basis of the vanishing divergence requirement for the summed EMT of the magnetoactive plasma (38) supplemented by the first group of the Maxwell equations:

$$T^k_{i,k} = 0, \quad (43)$$

$$F^*_{i,k} = 0. \quad (44)$$

This set of equations with due account of the equation of state (8), the definition of the plasma EMT (38) and the algebraic relations (7), (13), (14), (30), (35) and (38) completely describes the self-consistent motion of the magnetoactive plasma with an embedded magnetic field in a prescribed gravitational field.

Indeed, Eqs. (43)–(44) represent a set of 8 differential equations with respect to 10 quantities ε, p, H^i, v^i . However, the equation of state (8), the velocity vector normalization (7), the ortogonality condition (13) and (30) raise the total number of equations up to 12. Nevertheless, it turns out that not all of these relations are independent, as we shall see below.

To deduce the MHD equations, let us take into account the well-known relationship (see, e.g., [11]):

$$T^f_{i,k} = -\frac{1}{4\pi} F_{il} F^{kl}_{,k}. \quad (45)$$

Thus, Eq. (43) can be presented in the form

$$F_{ik} \Phi^k = \tau_i \quad (46)$$

where

$$\Phi^k \stackrel{\text{def}}{=} F^k_{,i}, \quad (47)$$

$$\tau_i \stackrel{\text{def}}{=} -4\pi T^p_{i,k}. \quad (48)$$

Eqs. (46) can be regarded as a set of linear inhomogeneous algebraic equations with respect to Φ^k . If $\det \|F_{ik}\| \neq 0$, then the equations are solved in a simple straightforward way:

$$\Phi^k = \frac{\mathcal{A}^k \|F; \tau\|}{\det \|F\|}, \quad (49)$$

where $\mathcal{A}^k \|F; \tau\|$ is a cofactor of the augmented matrix of the set (46). In particular, for the case of vacuum ($\tau_i = 0$) we obtain a trivial solution: $\Phi^k = 0$, and the set of equations (43), (44) reduces to the Maxwell equations in vacuum.

Now we pose the problem of solving the set of equations (46) with respect to Φ^k , i.e., the problem of reducing Eqs. (43)–(44) to the form of the Maxwell equations with minimal requirements upon the electromagnetic field invariants:

$$F^{ik} F^*_{ik} = 0, \quad (50)$$

$$F_{ik} F^{ik} > 0. \quad (51)$$

The existence of a positive invariant (51) means that we can choose a local FR where the electric field is absent [11].

Provided (50) is fulfilled, as before (see (32)),

$$\det \|F\| = 0 \quad (52)$$

$$\Rightarrow \text{rank } \|F\| = 2. \quad (53)$$

Thus for the consistency of the algebraic set of equations (46) under the condition (50) it is necessary and sufficient that

$$\text{rank } \|F; \tau\| = 2. \quad (54)$$

Calculating all the 3rd order minors of the augmented matrix $\|F; \tau\|$, with (50), we obtain a condition equivalent to (54):

$$F^*_{ik} \tau_k = 0 \iff \text{rank } \|F; \tau\| = 2. \quad (55)$$

To solve the set of equations (46), consider the eigenvectors of the matrix $\|F_{ik}\|$:

$$F_{ik} u^k = \lambda u_i. \quad (56)$$

Due to the antisymmetry of F_{ik} , it follows from (56) that either $\lambda = 0$, or u is a null vector. It can be demonstrated that provided Eqs. (50) and (51) are valid, u cannot be a null vector. Thus, the Maxwell tensor admits only nonnull eigenvectors with zero eigenvalues:

$$F_{ik} u^k = 0. \quad (57)$$

By (50), (52) and (53), this eigenvalue is doubly degenerate, and thus, according to a well-known algebraic theorem, two lineary independent eigenvectors correspond to it: $u_{(1)}$ and $u_{(2)}$.

Under the condition (51) we can always choose a local FR where $F_{i4} = 0$. In this FR there always exists an eigenvector of the Maxwell tensor of the form $u^k = \delta^k_4$. Therefore, one of the eigenvectors of the matrix $\|F\|$, e.g., $u_{(1)}$, is timelike and the second one, $u_{(2)}$, is spacelike. Using the standard orthogonalization process, we normalize them as follows:

$$\left(u_{(1)}, u_{(1)}\right) = 1; \quad \left(u_{(2)}, u_{(2)}\right) = -1; \quad \left(u_{(1)}, u_{(2)}\right) = 0. \quad (58)$$

Then the general solution of (57) can be written in the form

$$u^k = \alpha u^k_{(1)} + \beta u^k_{(2)}, \quad (59)$$

where α and β are arbitrary scalars.

Let us now investigate the relations (55), which can be regarded as algebraic equations with respect to τ . Since

$$-F_{ik} F^{ik} = F^*_{ik} F^{*ik} < 0 \quad (60)$$

and Eq. (50) is invariant under the substitution $F \leftrightarrow {}^*F$, as well as the expression for $\det \|F\|$ (27), we conclude that the dual matrix $\|{}^*F_{ik}\|$ also admits two and only two linearly independent spacelike eigenvectors w, u , which correspond to a zero eigenvalue. It is not difficult to verify (e.g., turning to a FR where $F_{\alpha 4} = 0$) that the rank of the unified matrix $\|F, {}^*F\|$ under the condition (50) is equal to 4. Hence the eigenvectors of the matrices $\|F\|$ and $\|{}^*F\|$ are linearly independent and we can choose the following normalization for them:

$$\begin{aligned} (w, w)_{(1)(1)} &= -1; & (w, w)_{(2)(2)} &= -1; \\ (w, w)_{(1)(2)} &= 0; & (w, u)_{(\alpha)(\beta)} &= 0, \\ & & (\alpha, \beta) &= (1, 2). \end{aligned} \quad (61)$$

Thus, the general solution to Eq. (55) is

$$\tau_i = \lambda w_{(1)} + \mu w_{(2)} \quad (62)$$

where λ and μ are arbitrary scalars and due to (61)

$$(\tau, u)_{(\alpha)} = 0. \quad (63)$$

By (53) and (55), the Maxwell tensor and its dual can be represented in terms of the eigenvectors of the matrices $\|F\|$ and $\|{}^*F\|$:

$$F_{ik} = -\sigma \eta_{iklm} u^l u^m; \quad (64)$$

$${}^*F_{ik} = \varrho \eta_{iklm} w^l w^m, \quad (65)$$

where σ and ϱ are certain scalars. Contracting these relations with the discriminant tensor, we obtain the dual relations

$${}^*F^{ik} = \sigma (u^i u^k - u^i u^k); \quad (66)$$

$$F^{ik} = -\varrho (w^i w^k - w^i w^k). \quad (67)$$

In particular, the following relation stems from (64) and (66)

$${}^*F_{ik} F^{kl} = 0. \quad (68)$$

Using the Maxwell tensor representation (64) and the orthonormality relations (58), we get the useful formula

$$F_{ik} F^{il} = \sigma^2 (-u_k u^l + u_k u^l + \delta_k^l). \quad (69)$$

Contracting (69), we get:

$$\frac{1}{2} F_{ik} F^{ik} = \sigma^2 > 0. \quad (70)$$

Contracting Eqs (46) with F^{il} and taking into account (69) and (70), we obtain an equation equivalent to Eq. (46):

$$\sigma^2 \left[\Phi^l - u_{(1)}^l (u, \Phi) + u_{(2)}^l (u, \Phi) \right] = F^{il} \tau_i. \quad (71)$$

A special solution to Eq. (71) is

$$\Phi_{(1)}^i = \frac{1}{\sigma^2} F^{il} \tau_i, \quad (72)$$

Therefore the general solution of Eqs. (46) can be presented in the form

$$F_{,k}^{ik} \equiv \Phi^i = \frac{8\pi F^{ik} T_{k,j}^j}{F^{lm} F_{lm}} + \alpha u_{(1)}^i + \beta u_{(2)}^i. \quad (73)$$

This exhausts the problem of reducing the set of equations (43),(44) to the standard Maxwell form.

If the external currents are absent ($\alpha = \beta = 0$), then Eq. (73) reduces to the form of the second group Maxwell equations:

$$F_{,k}^{ik} = -4\pi J_{dr}^i, \quad (74)$$

where

$$J_{dr}^i = -\frac{2F^{ik} T_{k,l}^l}{F_{jm} F^{jm}} \quad (75)$$

is the drift current, which by (68) satisfies the relation

$${}^*F_{ij} J_{dr}^i = 0, \quad (76)$$

Hence, due to (55) and (63)

$$(J_{dr}, u)_{(\alpha)} = 0, \quad (77)$$

$$(J_{dr}, J_{dr}) < 0, \quad (78)$$

i.e., the drift current is spacelike.

Calculating the covariant divergence ∇_l of Eq. (68) with the aid of the first group Maxwell equations (44), we get the differential implication

$${}^*F^{kl} F_{kl,i} = F^{kl} {}^*F_{kl,i} = 0. \quad (79)$$

Calculating the covariant divergence ∇^i of (68) using the second group Maxwell equations (74) and the relation (76), we get one more differential implication:

$$F^{kl} {}^*F_{i[k,l]} = 0. \quad (80)$$

Setting in Eqs. (57), (58), (61), (64)–(67) $u_{(1)}^i = v^i$, $u_{(2)}^i = n^i \equiv (H^i/H)$, we find a complete coincidence of the above formulas with the relevant expressions from the previous section. Thus

$$\sigma^2 = \frac{1}{2} F_{lm} F^{lm} = -(H, H) \equiv H^2. \quad (81)$$

Thus due to (63) the following differential relations should be valid:

$$v^i T^k_{i,k} = 0, \quad (82)$$

$$H^i T^k_{i,k} = 0. \quad (83)$$

Substituting to (82) and (83) the expression for T^p_{ik} (6) and taking into account (7) and (13), we obtain:

$$v^k_{,k} = -\frac{\varepsilon_{,k} v^k}{\varepsilon + p}; \quad (84)$$

$$v_{i,k} H^i v^k = \frac{p_{,i} H^i}{\varepsilon + p}. \quad (85)$$

Calculating the drift current (75) with Eqs. (6), (33) and (81), we get:

$$J^i_{dr} = -2 \frac{F^{ik} [v_{k,l} v^l (\varepsilon + p) - p_{,k}]}{F_{lm} F^{lm}}. \quad (86)$$

From (35) we obtain the useful relation (for $H \neq 0$)

$$v^i = \frac{\overset{*}{F}{}^{ki} H_k}{H^2}, \quad (87)$$

By (35) and the definition of the vector H^i (10) the orthonormality relations (7) are fulfilled identically.

Note that due to (68) the solution to Eq. (35) is

$$v^i = \overset{*}{F}{}^{ki} S_k, \quad (88)$$

where S_k is an arbitrary spacelike vector, satisfying the only condition

$$\overset{*}{F}{}^{ki} \overset{*}{F}{}^l{}_i S_k S_l = 1. \quad (89)$$

Now let us turn to the first group of the Maxwell equations (44). Using the representation (35) for the Maxwell dual tensor, we obtain for Eq. (44):

$$v^i H^k_{,k} + v^i_{,k} H^k - v^k_{,k} H^i - v^k H^i_{,k} = 0. \quad (90)$$

Consecutively contracting Eqs. (90) with v^i and H^i and using (7) and (13), we obtain:

$$-v_{i,k} H^i v^k = H_{i,k} v^i v^k = H^k_{,k}; \quad (91)$$

$$H_{i,k} v^i H^k = -v_{i,k} H^i H^k = H(Hv^k)_{,k}. \quad (92)$$

Since the rank of the matrix $\|\overset{*}{F}{}_{ik}\|$ equals 2, the relations (90)–(92) are equivalent to the first group of the Maxwell equations (44). By (91), Eq. (85) reduces to a form similar to (84):

$$H^k_{,k} = -\frac{p_{,k} H^k}{\varepsilon + p}. \quad (93)$$

To close the set of hydrodynamic equations, it is necessary to supplement them with the drift current conservation in the form (86):

$$J^i_{dr}{}_{,i} = 0.$$

4. Solution of magnetohydrodynamic equations against the background of a plane gravitational wave

4.1. Initial conditions and symmetry of the problem

The metric of a PGW with the polarization \mathbf{e}_+ is described by the expression [12]

$$ds^2 = 2dudv - L^2[e^{2\beta}(dx^2)^2 + e^{-2\beta}(dx^3)^2] \quad (94)$$

where $\beta(u)$ is an arbitrary function (the PGW amplitude); the function $L(u)$ (the PGW background factor) obeys an ordinary second-order differential equation; $u = (t - x^1)/\sqrt{2}$ is the retarded time and $v = (t + x^1)/\sqrt{2}$ is the advanced time. The absolute future is represented by the region $T^+ : \{u > 0; v > 0\}$, the absolute past by $T^- : \{u < 0; v < 0\}$. The metric (94) admits the group of motions \mathcal{G}_5 , associated with three linearly independent Killing vectors:

$$\xi^i_{(1)} = \delta^i_v; \quad \xi^i_{(2)} = \delta^i_2; \quad \xi^i_{(3)} = \delta^i_3. \quad (95)$$

Let there be no GW at $u \leq 0$, i.e.,

$$\beta(u)|_{u \leq 0} = 0; \quad L(u)|_{u \leq 0} = 1, \quad (96)$$

the plasma is homogeneous and at rest:

$$v^v|_{u \leq 0} = v^u|_{u \leq 0} = \frac{1}{\sqrt{2}}; \quad v^2 = v^3 = 0; \\ \varepsilon|_{u \leq 0} = \varepsilon_0; \quad p|_{u \leq 0} = p_0, \quad (97)$$

and a homogeneous magnetic field vector belongs to the plane $\{x^1, x^2\}$:

$$H_1|_{u \leq 0} = H_0 \cos \Omega; \quad H_2|_{u \leq 0} = H_0 \sin \Omega; \\ H_3|_{u \leq 0} = 0; \quad E_i|_{u \leq 0} = 0, \quad (98)$$

where Ω is the angle between the axis Ox^1 (the PGW propagation direction) and the direction of the magnetic field \mathbf{H} . The conditions (98) agree with the vector potential

$$A_v = A_u = A_2 = 0; \\ A_3 = H_0(x^1 \sin \Omega - x^2 \cos \Omega); \quad (u \leq 0). \quad (99)$$

In [7] it is demonstrated that the solution to the MHD equations in the metric (94) with the initial conditions (96)–(99) with $\varepsilon \neq 0$ is strictly stationary, i.e., all observed quantities are functions of solely the retarded time u . Therefore, we shall immediately require that the solution of our problem inherit the symmetry of the metric (94):

$$\mathcal{L}_{\xi(\alpha)} \mathbf{P} = 0; \quad (\alpha = 1, 2, 3) \quad (100)$$

for all observed quantities \mathbf{P} (\mathcal{L}_ξ is a Lie derivative in the direction ξ):

$$p = p(u); \quad \varepsilon = \varepsilon(u); \quad v^i = v^i(u); \quad (101)$$

$$F_{ik} = F_{ik}(u). \quad (102)$$

With (102) we obtain the following results from the first group of the Maxwell equations:

$$L^2 F^{*u\alpha} = \mathcal{C}_{(\alpha)} (= \text{const}); \quad \alpha = \{v, 2, 3\}. \quad (103)$$

Thus, using the initial conditions (98)–(99), we find:

$$\begin{aligned} L^2 F^{*uv} &= -F_{23} = H_0 \cos \Omega; \\ L^2 F^{*u2} &= F_{v3} = \frac{1}{\sqrt{2}} H_0 \sin \Omega; \\ L^2 F^{*u3} &= -F_{v2} = 0. \end{aligned} \quad (104)$$

The condition (50) ($F_{ik}^* F^{ik} = 0$) with (104) reduces to

$$L^2 F^{*v3} = F_{u2} = \sqrt{2} F_{uv} \cot \Omega, \quad (105)$$

while the MFEP conditions (30) with (104) and (105) yield:

$$v^3 = \frac{\sqrt{2} F_{uv}}{H_0 \sin \Omega} v_v; \quad (106)$$

$$\frac{1}{\sqrt{2}} H_0 v_u \sin \Omega + F_{u3} v_v - H_0 v^2 \cos \Omega = 0. \quad (107)$$

4.2. Conservation laws and the vector potential

A consequence of the second group of the Maxwell equations (74) is, as we know, the current conservation law, which, in view of (100) and the initial conditions (97)–(98), takes the form

$$J_{\text{dr}}^u = 0. \quad (108)$$

Calculating J_{dr}^u using (86), (104) and (106), we reduce (108) to the form

$$v_v (L^2 e^{-2\beta} F_{uv})' = 0, \quad (109)$$

where a prime is a derivative with respect to u . Thus, due to the initial conditions ($F_{uv}(u)|_{u \leq 0} = 0$) and the the velocity vector timelikeness requirement ($v_v \neq 0$) we draw the conclusion that the current conservation law (108) is equivalent to the requirement

$$F_{uv} = 0. \quad (110)$$

But then by (105) and (106)

$$F^{*v3} = F_{u2} = 0; \quad (111)$$

$$v^3 = 0. \quad (112)$$

As we know, the first group of the Maxwell equations (44) is equivalent to the existence condition of a vector potential A_i :

$$F_{ik} = \partial_i A_k - \partial_k A_i. \quad (113)$$

Thus, we can write for the zero component of the Maxwell tensor (104), (110), (111):

$$\partial_\sigma A_\gamma - \partial_\gamma A_\sigma = 0; \quad (\{\gamma, \sigma\} = \{u, v, 2\}). \quad (114)$$

As known, a unique solution to Eqs. (114) on the 3-dimensional hypersurface $\Sigma_3 : \{x^3 = \text{const}\}$ is a gradient vector:

$$A_\sigma = \partial_\sigma \Phi, \quad (\sigma = u, v, 2), \quad (115)$$

where $\Phi = \Phi(u, v, x^2, x^3)$ is an arbitrary scalar function.

The nonzero components of the Maxwell tensor, $F_{\sigma 3}$, can be represented, by (115), as

$$F_{\sigma 3} = \partial_\sigma \tilde{A}_3, \quad (\sigma = \{v, u, 2\}) \quad (116)$$

where

$$\tilde{A}_3 \stackrel{\text{def}}{=} A_3 - \partial_3 \Phi \quad (117)$$

is a gradient-renormalized vector potential. Calculating the nonzero components of the Maxwell tensor $F_{\sigma 3}$ using (116)–(117) and taking into account the relations (102) and (104), we finally find:

$$\tilde{A}_3 = -H_0 x^2 \cos \Omega + \frac{1}{\sqrt{2}} H_0 [v - \psi(u)] \sin \Omega, \quad (118)$$

where $\psi(u)$ is an arbitrary differentiable function satisfying the initial condition

$$\psi|_{u \leq 0} = u. \quad (119)$$

It should be noted that the only ψ -dependent nonvanishing component of the Maxwell tensor is

$$F_{u3} = -\frac{1}{\sqrt{2}} H_0 \psi' \sin \Omega, \quad (120)$$

and the MFEP condition (107) takes the form

$$\frac{1}{\sqrt{2}} (v_v \psi' - v_u) \sin \Omega + v^2 \cos \Omega = 0. \quad (121)$$

Calculating the other components of the drift current using (104), (110) and (111), we find:

$$J^v = J^2 = 0, \quad (122)$$

and for the only nontrivial component J^3 we obtain an expression coinciding with the one found in [6] in the case of rigorously transverse PGW propagation ($\cos \Omega = 0$). However, for other values of Ω the expression for the drift current obtained in [6], as well as that for the drift velocity (cf. (121)), is erroneous. However, we shall not integrate the Maxwell equations with the drift current, since it is a consequence of the conservation laws (43) and the MFEP conditions: it is much simpler in our case to integrate the conservation laws directly.

4.3. Integrals of motion

Due to the conservation laws for the complete plasma EMT (43) and the presence of three linearly independent Killing vectors (95) there are 3 integrals of Eqs. (43):

$$L^2 \xi_{(\alpha)}^k T_k^u = C; \quad (\alpha = 1, 2, 3), \quad (123)$$

where T_k^u is described by Eqs. (36) and (38):

$$T_k^u = \left(\varepsilon + p + \frac{H^2}{4\pi} \right) v_v v_k - \left(p + \frac{H^2}{8\pi} \right) \delta_k^u - \frac{H_v H_k}{4\pi}. \quad (124)$$

Calculating the magnetic field vector H^i and the scalar H^2 according to (10) and (81), we find:

$$\begin{aligned} H_v &= -H_0 L^{-2} (v_v \cos \Omega + \frac{1}{\sqrt{2}} v_2 \sin \Omega); \\ H_u &= H_0 L^{-2} (v_u \cos \Omega - \frac{1}{\sqrt{2}} \psi' v_2 \sin \Omega); \\ H_2 &= -\frac{1}{\sqrt{2}} H_0 e^{2\beta} \sin \Omega (v_u + v_v \psi'); \quad H_3 = 0; \quad (125) \\ H^2 &\stackrel{\text{def}}{=} \frac{1}{2} F_{ik} F^{ik} = H_0^2 \left(\frac{\cos^2 \Omega}{L^4} + \frac{\sin^2 \Omega}{L^2} \psi' e^{2\beta} \right). \end{aligned} \quad (126)$$

It is not difficult to verify that according to (125) the orthogonality relation (13) $((v, H) = 0)$ is satisfied identically. The velocity vector normalization condition (7) with (121) and (125) can be written in the form

$$(v_v \cos \Omega + \frac{1}{\sqrt{2}} v_2 \sin \Omega)^2 = \frac{H^2}{H_0^2} v_v^2 L^4 - \frac{\sin^2 \Omega}{2} L^2 e^{2\beta}. \quad (127)$$

One of the integrals (123), C , proves to be trivial, while the remaining two yield in view of (121), (126) and (127):

$$\begin{aligned} (\varepsilon + p) L^2 v_v^2 + \frac{H_0^2 \sin^2 \Omega}{8\pi} e^{2\beta} &= \frac{1}{2} \left(\varepsilon_0 + p_0 + \frac{H_0^2 \sin^2 \Omega}{4\pi} \right); \quad (128) \\ (\varepsilon + p) L^2 v_v v_2 - \frac{H_0^2 \cos \Omega \sin \Omega}{4\sqrt{2}\pi} e^{2\beta} &= -\frac{H_0^2 \cos \Omega \sin \Omega}{4\sqrt{2}\pi}. \end{aligned} \quad (129)$$

Thus

$$v_v^2 = \frac{\varepsilon_0 + p_0}{2L^2(\varepsilon + p)} \Delta(u); \quad (130)$$

$$v_2/v_v = \sqrt{2}(\Delta^{-1} - 1) \cot \Omega, \quad (131)$$

where

$$\Delta(u) \stackrel{\text{def}}{=} 1 + \alpha^2 (e^{2\beta} - 1) \quad (132)$$

and a dimensionless parameter α has been introduced:

$$\alpha^2 = \frac{H_0^2 \sin^2 \Omega}{4\pi(\varepsilon_0 + p_0)}. \quad (133)$$

Let us now turn to the consequences of the Maxwell equations. Integrating (85), we find one more integral:

$$\sqrt{2} L^2 |v_v| = \exp \left[- \int_{\varepsilon_0}^{\varepsilon} \frac{d\varepsilon}{\varepsilon + p(\varepsilon)} \right]. \quad (134)$$

Thus, if an equation of state is specified, $p = p(\varepsilon)$, then using (130), (131) and (134), the functions $v_v(u)$, $v_2(u)$, $\varepsilon(u)$ and $p(u)$ are determined. However, to be able to determine $v_u(u)$, it is necessary to find the function $\psi(u)$, for which yet one more integral is required. It is precisely the normalization relationship in the form (127) that is the integral in question. Multiplying (128) by $\cos \Omega$ and (129) by $\sin \Omega/\sqrt{2}$ and adding up the values thus obtained, we get¹:

$$L^2 v_v (v_v \cos \Omega + \frac{1}{\sqrt{2}} v_2 \sin \Omega) = \frac{\varepsilon_0 + p_0}{2(\varepsilon + p)} \cos \Omega. \quad (135)$$

Squaring both parts of (135) and taking into account the normalization condition in the form (127) along with (130), we find:

$$H^2 = \frac{H_0^2}{\Delta} \left(\frac{\cos^2 \Omega}{L^4 \Delta} + \frac{\varepsilon + p}{\varepsilon_0 + p_0} e^{2\beta} \sin^2 \Omega \right). \quad (136)$$

Comparing (136) and (126), we finally obtain:

$$\psi' = \frac{L^2(\varepsilon + p)}{\Delta(\varepsilon_0 + p_0)} + \left(\frac{1}{\Delta^2} - 1 \right) e^{-2\beta} \frac{\cot^2 \Omega}{L^2}. \quad (137)$$

Finally, using (121), (131) and (137), we find an expression for v_u :

$$\frac{v_u}{v_v} = \frac{L^2(\varepsilon + p)}{\Delta(\varepsilon_0 + p_0)} + \left(\frac{1}{\Delta} - 1 \right)^2 e^{-2\beta} \frac{\cot^2 \Omega}{L^2}. \quad (138)$$

Thus, with a specified equation of state (8), all unknown functions are found, the solutions to Eqs. (130), (131), (134)–(138) automatically satisfying the initial conditions (97), (98) (see also (119)). Thereby we have found by quadratures an exact solution to the self-consistent problem of motion of a magnetoactive plasma against the background of a PGW.

5. A study of the solution

Squaring both parts of Eq. (134) and using, in the left-hand side of the equation thus obtained, the expression for v_v^2 from (130), we represent (134) after certain obvious transformations in the form

$$\Lambda(u) = e^{-J(\varepsilon; \varepsilon_0)}, \quad (139)$$

¹It is easily demonstrated that this integral is directly obtainable from (93).

where $\Lambda \stackrel{\text{def}}{=} L^2(u)\Delta(u)$ and

$$J(\varepsilon; \varepsilon_0) = \int_{\varepsilon_0}^{\varepsilon} \frac{1 - p'_\varepsilon}{\varepsilon + p(\varepsilon)} d\varepsilon. \tag{140}$$

With a specified equation of state (8), Eq. (139) completely determines the functions $\varepsilon(u)$ and $p(u)$ and thus explicitly determines the solution to the problem posed. Let us investigate these functions, making the most general assumption on the equation of state:

$$p(\varepsilon) < \varepsilon. \tag{141}$$

Then

$$p'_\varepsilon < 1, \tag{142}$$

$$\frac{1 - p'_\varepsilon}{\varepsilon + p(\varepsilon)} > 0$$

and thus

$$\text{sgn}[J(\varepsilon; \varepsilon_0)] = \text{sgn}(\varepsilon - \varepsilon_0) \tag{143}$$

It can be seen from (139) that ε depends on the retarded time only through the function $\Lambda(u)$:

$$\varepsilon = \varepsilon(\Lambda(u)). \tag{144}$$

5.1. Singularity investigation

Let us investigate the dependence $\varepsilon(\Lambda)$. It follows from (130) that the solution is specified in the interval $\Lambda \in [0, +\infty)$. As $\Lambda \rightarrow +0$, according to (139), $J(\varepsilon; \varepsilon_0) \rightarrow +\infty$, which by (143) is possible only with $\varepsilon \rightarrow +\infty$. Thus we can draw a general conclusion that the solution of the MHD equations against the background of a PGW contains a physical singularity on the hypersurfaces $u = u_*$:

$$\Lambda(u_*) = 0. \tag{145}$$

By definition of the functions $\Lambda(u)$ and $\Delta(u)$, there can be two types of such hypersurfaces:

$$\begin{aligned} \text{(A)} \quad & L^2(u) = 0; \\ \text{(B)} \quad & 1 + \alpha^2(e^{2\beta(u)} - 1) = 0. \end{aligned} \tag{146}$$

The first type of singularity is well-known: it is connected with a coordinate singularity of the metric (94) and always arises in plasma (see, for instance, [1]). The second type is new and is not connected with a coordinate singularity of the PGW metric (94) — this is a purely physical singularity [13]. By (146), the conditions for the formation of a second-type singularity are

$$\beta(u) < 0; \tag{147}$$

$$\alpha^2 > 1. \tag{148}$$

It is well-known (see, e.g., [12]) that the values of $\beta(u) < 0$ correspond to a compression phase of the

geodesic tube along the Ox^2 axis, while $\beta > 0$ correspond to an expansion phase. But the condition (148), according to (133), means that in the initial state $\varepsilon_H \sin^2 \Omega > \frac{1}{2}(\varepsilon_0 + p_0)$, i.e., the plasma is highly magnetized. It is an extremely important fact that the B-type singular state is possible even in a weak PGW ($|\beta| \ll 1$) provided the plasma is highly magnetized ($\alpha^2 \gg 1$); in this case, according to (146), the singular state occurs on the hypersurfaces $u = u_*$:

$$\beta(u_*) = -1/(2\alpha^2). \tag{149}$$

Further, according to (139) and (140), at $\Lambda(u) = 1$ one has $\varepsilon = \varepsilon_0$, $p = p_0$ and thus by (130)–(138) the initial conditions are restored on the hypersurfaces $\Lambda(u) = 1$. As $\Lambda(u) \rightarrow +\infty$, by (139) $J(\varepsilon; \varepsilon_0) \rightarrow -\infty$, which is possible by (143) only when $\varepsilon \rightarrow +0$ and $p(\varepsilon) \rightarrow +0$:

$$\Lambda(u) \rightarrow +\infty, \quad \varepsilon \rightarrow 0; \quad p(\varepsilon) \rightarrow 0. \tag{150}$$

Let us investigate the general behaviour of the solution in the vicinity of a B-type singularity, i.e., as $\Delta(u) \rightarrow 0$, imposing a more severe requirement than (141) for the equation of state:

$$p(\varepsilon) \leq \varepsilon/3. \tag{151}$$

Then the following inequalities are valid:

$$\frac{1}{2\varepsilon} \leq \frac{1 - p'_\varepsilon}{\varepsilon + p(\varepsilon)} < \frac{1}{\varepsilon}, \tag{152}$$

the equality in the left-hand side being achieved only in the case of the ultrarelativistic equation of state ($\varepsilon = 3p$). Then, restrictions upon $J(\varepsilon; \varepsilon_0)$ follow from (140), (152):

$$\ln \sqrt{\frac{\varepsilon}{\varepsilon_0}} \leq J(\varepsilon; \varepsilon_0) < \ln \frac{\varepsilon}{\varepsilon_0}, \tag{153}$$

hence by (139) ²:

$$\frac{1}{\Lambda} < \frac{\varepsilon}{\varepsilon_0} \leq \frac{1}{\Lambda^2}. \tag{154}$$

But then we obtain directly from (130):

$$\frac{3}{8}L^2\Delta^3 \leq v_v^2 < \frac{2}{3}\Delta^2. \tag{155}$$

Using (154), (155) and singling out the dominant parts of the expressions (131) and (136) - (138) near a B-type singularity, we obtain the asymptotic estimates

$$\begin{aligned} \Delta \rightarrow 0 : \\ \varepsilon & \sim \varepsilon_0 \Delta^{-\nu}, & \nu \in (1, 2]; \\ v_v & \sim \Delta^\nu; \\ H & \sim H_0 \Delta^{-\mu}, & \mu \in (1, \frac{3}{2}]; \\ v_u/v_v & \sim \Delta^{-\gamma}, & \gamma \in (2, 3]; \\ v_2/v_v & \sim \sqrt{2}\Delta^{-1} \cot \Omega. \end{aligned} \tag{156}$$

²If, instead of (151), we restrict ourselves to the weaker condition (141), then only a lower bound upon the energy density will be preserved.

The components of the 3-vector of physical velocity $V^\alpha = dx^\alpha/dt$ can be presented in the form

$$\begin{aligned} V^1 &= \frac{v_u/v_v - 1}{v_u/v_v + 1}; \\ V^2 &= \frac{\sqrt{2}e^{-2\beta}}{L^2(1 + v_u/v_v)} \frac{v_2}{v_v}. \end{aligned} \quad (157)$$

Using the estimates (156) for the functions near the singularity, we obtain for the physical velocity vector components:

$$\begin{aligned} \Delta \rightarrow 0: \quad V^1 &\rightarrow 1; \\ V^2 &\sim -\frac{2e^{-2\beta}}{L^2} \cot \Omega \Delta^{\gamma-1} \rightarrow 0. \end{aligned} \quad (158)$$

Thus a physical singularity occurs on the hypersurface $u = u_*$: the energy densities of the plasma and the magnetic field tend to infinity, the dynamic speed of the plasma as a whole tends to the speed of light in the PGW propagation direction. Meanwhile the plasma transverse motion vanishes ($V^2 \rightarrow 0$), whereas the ratio of the magnetic field energy density and that of the plasma tend to infinity:

$$\Delta \rightarrow 0; \quad \frac{\varepsilon H}{\varepsilon} \sim \alpha^2 e^{2\beta} \Delta^{-1} \rightarrow \infty. \quad (159)$$

The presence of a B-type singularity naturally poses the question of the applicability of the MHD plasma model in the vicinity of the hypersurface (146), i.e., the question of fulfilling the drift approximation condition (2).

Note that (2) is a local condition, i.e., for the MHD plasma model to be applicable, this condition must be fulfilled in the comoving FR throughout the whole domain under consideration. Let us assume that (2) is fulfilled in the initial state $u \leq 0$:

$$\xi_0 = \omega / \omega_B^0 \ll 1, \quad (160)$$

where $\omega_B^0 = \omega_B|_{(H=H_0)}$. The GW frequency in a frame of reference moving at a speed v^i , is $\omega' = k_i v^i$, where \mathbf{k} is the GW wave vector: $\mathbf{k} = (-\omega, 0, 0, \omega)$. Thus $\omega' = \sqrt{2}v_v \omega$. The scalar H (17) is the modulus of the magnetic field intensity in the comoving FR. So the local value of the drift parameter ξ , measured in the comoving FR, is related to the initial value of (160) by

$$\xi = \sqrt{2} \frac{v_v H_0}{H} \xi_0. \quad (161)$$

Using the asymptotic estimates (156) and (158) of the solution behaviour in the vicinity of a B-type singularity, we obtain from (161):

$$\Delta \rightarrow 0: \quad \xi \sim \xi_0 \Delta^{2\mu} \rightarrow 0.$$

Thus we can conclude: if the drift approximation applicability condition was initially fulfilled, then in

approaching to a B-type singularity the plasma motion is more and more precisely described by the MHD model. Therefore, within the scope of the problem in hand, viz., that of motion of initially homogeneous magnetoactive plasma against the background of the PGW metric there are no mechanism available for preventing the singularity.

We should note an essentially nonlinear nature of the phenomenon detected: if the initial equations are expanded in a Taylor series in the small PGW amplitude, β , this phenomenon is not observed, as confirmed, in particular, by an investigation carried out in Ref. [14]. The reason for this difference of the results lies in the fact that the governing function of the process under study is, as can be seen from the solutions to (130)–(132), the function Δ^{-1} , which in the case of a weak PGW ($|\beta| \ll 1$) takes the form

$$\Delta^{-1} \approx \frac{1}{1 + 2\alpha^2 \beta(u)}. \quad (162)$$

The Taylor expansion of this function in powers of β assumes the smallness of the value $2\alpha^2 \beta$; however, the parameter α in a highly magnetized plasma may prove to be so great that the condition $2\alpha^2 |\beta| > 1$ is fulfilled.

Thus, the discussed phenomenon is a threshold effect arising when the PGW amplitude in the compression phase along the axis Ox^2 reaches the value (149) and can be interpreted as a non-linear threshold GW-generation of a magnetohydrodynamic shock wave propagating at a subluminal speed along the PGW propagation direction. We shall further call this new class of effects gravimagnetic shock waves, or GMSW for short.

Let us note that, firstly, the presence of a plasma qualitatively changes the electromagnetic field nature [7],[14]: in the action of a PGW on a vacuum magnetic field, the solution of the Maxwell equations is essentially nonstationary (an exact solution is given in [6]) and, secondly, neither in a fluid, with any equation of state (exact solutions being given in [15],[16]), nor in a vacuum magnetic field singularities other than an A-type coordinate singularity occur. The only and, at the same time, exotic example is the plasma with a scalar interaction in the case of repulsion of two identically charged particles (an exact solution is given in [17]). It should be also noted that in the case of the ultrastiff equation of state ($p = \varepsilon$) the solution to the hydrodynamic equations is nonstationary, i.e., depends on the variables u and v [16]. The same type of behaviour is detected for a magnetoactive plasma: as seen from (139)–(140), at $p = \varepsilon$ there is no stationary solution.

The shock wave formation mechanism seems to consist in the following. A weak GW is known not to interact with a fluid but to perturb a magnetic field. This in turn causes a drift of the plasma. Particles with smaller values of the coordinate x^1 have a greater

value of the coordinate u , that is why at the compression stage of the geodesic tube in the direction Ox^2 ($\beta < 0$; $\beta' > 0$) such particles have a greater speed than those with larger values of the coordinate x^1 . Therefore, the backward plasmic layers overtake forward ones and thereby contribute to the shock wave formation.

Calculating the drift current with the aid of the Maxwell equations (74), we find (see also [6],[14]):

$$J_3 = \frac{e^{-2\beta}}{4\pi} (L^2 F_v^3)' = -\frac{H_0 \sin \Omega}{2\sqrt{2}\pi} \beta'. \quad (163)$$

For this reason, in a singular B-type state the drift current density remains finite.

5.2. Barotropic equation of state

We shall examine a barotropic equation of state:

$$p = k\varepsilon, \quad 0 \leq k < 1. \quad (164)$$

Then

$$J(\varepsilon; \varepsilon_0) = \frac{(1-k)}{(1+k)} \ln \frac{\varepsilon}{\varepsilon_0} \quad (165)$$

and thus, according to (140), (130)–(133), we obtain an exact solution:

$$\varepsilon = \varepsilon_0 \Lambda^{-1+\kappa}; \quad (166)$$

$$v_v = \frac{1}{\sqrt{2}} L^\kappa \Delta^{1+\kappa/2}; \quad (167)$$

$$\frac{v_u}{v_v} = \Delta^{-2} [\Lambda^{-\kappa} + (\Delta-1)^2 L^{-2} e^{-2\beta} \cot^2 \Omega]; \quad (168)$$

$$H^2 = \frac{H_0^2}{\Lambda^2} = (\cos^2 \Omega + L^2 \Lambda^{-\kappa} e^{2\beta} \sin^2 \Omega), \quad (169)$$

where

$$\kappa = \frac{2k}{1-k} \geq 0. \quad (170)$$

In particular, for a nonrelativistic plasma³ ($p = 0, \Rightarrow \kappa = 0$) (166)–(169) imply:

$$\begin{aligned} \varepsilon &= \varepsilon_0 \Lambda^{-1}; & v_v &= \frac{1}{\sqrt{2}} \Delta; \\ \frac{v_u}{v_v} &= \Delta^{-2} [1 + (\Delta-1)^2 L^{-2} e^{-2\beta} \cot^2 \Omega]; \\ H^2 &= \frac{H_0^2}{\Lambda^2} (\cos^2 \Omega + L^2 \sin^2 \Omega e^{2\beta}); \end{aligned} \quad (171)$$

and for an ultrarelativistic plasma ($p = \varepsilon/3 \Rightarrow \kappa = 1$) we obtain from (162)–(166):

$$\begin{aligned} \varepsilon &= \varepsilon_0 \Lambda^{-1}; & v_v &= \frac{1}{\sqrt{2}} L \Delta^{3/2}; \\ \frac{v_u}{v_v} &= L^{-2} \Delta^{-2} [\Delta^{-1} + (\Delta-1)^2 e^{-2\beta} \cot^2 \Omega]; \\ H^2 &= \frac{H_0^2}{\Lambda^2} (\cos^2 \Omega + \Delta^{-1} e^{2\beta} \sin^2 \Omega). \end{aligned} \quad (172)$$

³The exact solution for a nonrelativistic plasma was obtained in [18]

The exact solutions (166)–(169) confirm the asymptotic estimates for the general behaviour of plasma near the singularity (156).

5.3. Drift of a nonrelativistic plasma in a weak PGW

Let us examine the practically important case of a weak PGW :

$$|\beta(u)| \ll 1 \quad (173)$$

and a nonrelativistic plasma ($k = 0$). It is known (see, for instance, [12]) that $L^2 \sim O(\beta^2)$. In view of this fact, we obtain from (171) in the first approximation with respect to β (but not with respect to $\alpha^2 \beta!$):

$$\begin{aligned} \varepsilon &= \varepsilon_0 \Delta^{-1}; & v_v &= \frac{1}{\sqrt{2}} \Delta; \\ \frac{v_u}{v_v} &= \Delta^{-2} [1 + (2\alpha^2 \beta \cot \Omega)^2]; \\ H &= \frac{H_0}{\Delta}; & \frac{v_2}{v_v} &= \sqrt{2} (\Delta^{-1} - 1) \cot \Omega, \end{aligned} \quad (174)$$

where it is necessary to use the expression (162) for Δ^{-1} . Then we find from (157) the nonzero physical speed components in an explicit form:

$$\begin{aligned} V^1 &= -2\alpha^2 \beta \frac{1 - \alpha_0^2 \beta \cos 2\Omega}{1 + 2\alpha^2 \beta (1 + \alpha_0^2 \beta)}; \\ V^2 &= \frac{1}{\sqrt{2}} \alpha_0^2 \beta \sin 2\Omega \frac{1 + 2\alpha^2 \beta}{1 + \alpha^2 \beta + 2\alpha_0^4 \beta^2 \cos^2 \Omega} \end{aligned} \quad (175)$$

with $\alpha_0^2 = H_0^2/4\pi(\varepsilon_0 + p_0)$.

It follows from (175) that in the case of a sufficiently weak PGW ($2\alpha^2|\beta| < 1$), $V^1 > 0$ for $\beta < 0$ and $V^1 < 0$ for $\beta > 0$, i.e., in the compression phase of the geodesic tube in the direction of the Ox^2 axis, the plasma drifts as a single whole in the the PGW propagation direction, whereas in the expansion phase it drifts in the opposite direction.

Let $\beta(u)$ be a quasiperiodical function with the period T , so that

$$\langle \beta(u) \rangle = 0, \quad (176)$$

where $\langle \dots \rangle$ denotes averaging over the interval $\Delta u = u - u_0 \gg T$;

$$\begin{aligned} |e^{2\beta(u+T)} - e^{2\beta(u)}| &\ll 1/\alpha^2; \\ |e^{2\beta(u+T/2)} + e^{2\beta(u)} - 2| &\ll 1/\alpha^2, \end{aligned} \quad (177)$$

i.e., the PGW amplitude is little changed in the course of the “period” T . Setting in (175) $\Omega = \pi/2$ and averaging V^1 under the conditions (176)–(177) over a sufficiently large interval of retarded time, we find:

$$\langle V^1 \rangle \sim 2\alpha_0^4 \langle \beta^2(u) \rangle, \quad (178)$$

thus the average plasma drift proceeds in the PGW propagation direction and even in a weak GW it is

nonzero. At $\Omega = \pi/2$ by (174) $v_u/v_v \equiv v^v/v^u = \Delta^{-2}$, therefore

$$\frac{dv}{du} = \frac{1}{\Delta^2} = \frac{1}{(1 + 2\alpha_0^2\beta(u))^2}$$

$$\Rightarrow v - v_0 = \int_0^u \frac{du'}{(1 + 2\alpha_0^2\beta(u'))^2}. \quad (179)$$

Passing on in (179) to the limit $u \rightarrow u_*$ under the condition that the function $\beta(u)$ be continuous, we get:

$$u \rightarrow u_*;$$

$$v(u) - v_0 \simeq \frac{1}{(2\alpha^2\beta'(u_*))^2} \frac{1}{(u_* - u)} \rightarrow \infty \quad (180)$$

Eq. (179) described in an implicit form the trajectory $x \equiv x^1 = f(t)$ of an infinitely small volume of plasma. Let this volume be placed at a point x_0 prior to PGW arrival, then $t_0 = x_0$ is the moment of PGW arrival at this point. The singularity occurs at $u = u_* = (t - x)_*/\sqrt{2}$. Then it follows from (180) that in this state $x_* \rightarrow +\infty$ and $t_* \rightarrow +\infty$. Thus, by an external observer's time, a plasmic particle reaches the singular hypersurface $u = u_*$ (which the plasma velocity tends to that of light and the energy density tends to infinity) for an infinitely long time, and, in doing so, it finds itself at an infinitely remote point. If the values $|\beta(u)| > 1/2\alpha_0^2$ are possible in the PGW metric (94), it makes no sense to continue the solution of magnetohydrodynamics beyond the hypersurface $u = u_*$, since at the shock wave front all the invariants suffer a second-type discontinuity.

6. Source of energy

Since on a singular hypersurface $u = u_*$ the energy densities of the plasma and the magnetic field tend to infinity, while the velocity of plasma as a whole tends to the speed of light, the total energy of a MHD shock wave tends to infinity. Thus, as a result of the drift, the complete plasma energy grows (indeed, it grows to infinity), and it is necessary to reveal the nature of a source of this energy. Note that the MHD equations were solved against the background of the PGW metric. In this sense the PGW is an inexhaustible source of energy for the magnetoactive plasma; it is precisely for this reason that a singular state is formed. A singular state originating in the plasma under the influence of PGW, disturbs the basic assumption (1) on the weakness of the GW-plasma interaction. In a more comprehensive self-consistent problem with regard to gravitation the allowance made for the impact of the shock wave on the PGW should lead to a PGW amplitude damping to the values

$$\max(|\beta|) < 1/2\alpha^2. \quad (181)$$

Thus the discovered effect of MHD wave generation by a gravitational wave (GMSW) can be an effective mechanism of GW energy absorption. The author is unaware of any other GW energy absorption mechanism of similar efficiency.

A strict solution to the problem of PGW energy transformation to the energy of a shock wave is possible only on the basis of a study of the self-consistent set of the Einstein and MHD equations. However, a qualitative analysis of the situation may be performed using a simpler model of the energy balance. Let us consider the case of strictly transverse PGW propagation ($\Omega = \pi/2$) in a nonrelativistic plasma. Then the energy flux of magnetoactive plasma should have the direction of PGW propagation, i.e., along the Ox^1 axis. Let $\beta_0(u)$ be the PGW vacuum amplitude and $\beta(u)$ be its amplitude with allowance made for absorption in the plasma. From summed energy flux conservation it follows:

$$T^{41}(\beta) + t^{41}(\beta) = t^{41}(\beta_0), \quad (182)$$

where t^{14} is the energy flux of a weak GW in the direction Ox^1 (see [11]). Using the solution to the MHD equations for nonrelativistic plasma (174) in the case of strictly transverse PGW propagation, as well as the expressions for the summed plasma EMT (38) and weak PGW energy flux [11], we reduce (182) to the form (returning to the standard system of units)

$$\frac{\pi G \mathcal{E}_0}{c^2} [\Delta^{-4}(\beta) - 1] [\alpha_0^2 e^{2\beta} + \Delta(\beta)] + \dot{\beta}^2 = \dot{\beta}_0^2,$$

where $\dot{\beta}$ means the derivative with respect to the time t . Setting in what follows $\dot{\beta}^2 = \omega^2 \beta^2$ (ω being the GW frequency), $\alpha_0^2 \gg 1$ and $|\beta| \ll 1$ and tending Δ to zero, we reduce the latter equation to the form

$$\frac{\chi^2}{(1 + 2\alpha_0^2\beta)^4} + \beta^2 = \beta_0^2, \quad (183)$$

where χ is a dimensionless parameter:

$$\chi^2 = \frac{GH_0^2}{c^2\omega^2} \sim \frac{\omega_g^2}{\omega^2}, \quad (184)$$

$\omega_g^2 = 8\pi G \mathcal{E}_0/c^2$. The approximation (1) is equivalent to the condition

$$\chi^2 \ll 1. \quad (185)$$

In the conditions of strong PGW energy absorption we have $\beta^2 \ll \beta_0^2$, therefore in this case (184) implies:

$$1 + 2\alpha_0^2\beta \approx \left(\frac{\chi}{|\beta_0|} \right)^{1/2}$$

$$\Rightarrow \beta = -\frac{1}{2\alpha_0^2} \left[1 - \left(\frac{\chi}{|\beta_0|} \right)^{1/2} \right]. \quad (186)$$

It follows from (186) that the conditions for a sufficiently strong absorption of a weak PGW are

$$\alpha^2 \gg 1; \quad |\beta_0| > \frac{1}{2\alpha_0^2}; \quad |\beta_0| > \chi. \quad (187)$$

The PGW amplitude damping factor γ is

$$\gamma = \frac{|\beta|}{|\beta_0|} = \frac{1}{2\alpha_0^2|\beta_0|} \left[1 - \left(\frac{\chi}{|\beta_0|} \right)^{1/2} \right]. \quad (188)$$

7. Astrophysical consequences of GMSW

There naturally arises a question, whether or not the GMSW exist in nature, i.e., can the condition for their creation (149) be realized? In laboratory experiments the above necessary condition ($2\alpha^2|\beta| \geq 1$) is unattainable due to the smallness of both the gravitational waves in terrestrial conditions and stable laboratory magnetic fields. For galactic magnetic fields ($H \sim 10^{-5} \div 10^{-6}$ Gauss) and interstellar medium ($\rho \sim 10^{-24}$ g/cm³) $\rightarrow \alpha^2 \sim 10^{-8}$, so GMSW in the interstellar medium do not arise.

The only possible source of a GMSW can apparently be neutron star magnetospheres on the stage of quadrupole neutron oscillations, as well as on stage of a Supernova. Table 1 represents the results of calculating the parameter $2\alpha^2\langle|\beta|\rangle$ ($\langle|\beta|\rangle$ is the average amplitude of irradiated GW) for neutron stars magnetospheres, performing quadrupole oscillations of the fundamental quadrupole mode ($n = 0$). The data placed in the first four columns are taken (or recalculated) from the book [12]. The average GW amplitude is calculated using standard formulas, from the average gravitational radiation power, N , (see for instance [11]) under the assumption that the average relative amplitude of quadrupole oscillations of a neutron star (the parameter $\langle(\delta R/R)^2\rangle^{1/2}$ in the book [12]) is equal to 10^{-6} . It is also assumed that the neutron star magnetosphere consists of completely ionized hydrogen. In addition, the parameter n_0 (electron number density on the neutron star surface) was found from the condition $\tau = 1$, where $\tau = \int n\sigma dl$ is the optical thickness of the magnetosphere. It is assumed also, that the scale of dense magnetosphere is of the order of the neutron star radius R .

Note that the electron concentration values near a neutron star surface, obtained from the condition that the optical thickness of the magnetosphere equals unity, is approximately by 4 orders of magnitude larger than the estimates of n_0 from [19], obtained on the basis of a dimensional analysis of the Maxwell equations. If we accept Pacini's estimates for n_0 [19], then the GMSW factor will grow by 4 orders as compared with that of Table 1.

For a star with a mass of $0,682 M_\odot$ we get from Table 1: $\chi = 1,33 \cdot 10^{-11}$; $\chi/\langle|\beta_0|\rangle = 0,317$. According to (188) we find the GW damping factor: $\gamma = 0,206$.

Thus, the GW amplitude in this case is damped by a factor of 5 and the gravitational radiation power by a factor of 25. For a star with a mass of $1,954 M_\odot$, $\chi = 1,42 \cdot 10^{-11}$, $\chi/\langle|\beta_0|\rangle = 1,01 \cdot 10^{-3} \Rightarrow \gamma = 0,032$, i.e., in this case the GW amplitude can be diminished by a factor of 30 times and the GW power by a factor of 900! So in neutron star magnetospheres there exist necessary conditions for GMSW excitation.

If the magnetic field of a neutron star is described as that of a dipole, then the geographic angle Θ (counted relative to the magnetic equator) will be connected with the above angle Ω by the relation $\Omega = \pi/2 - \Theta$. Therefore the GMSW excitation condition depends on the angle Θ :

$$\sin^2 \Theta < 1 - \frac{1}{2\alpha_0^2|\beta|}.$$

Thus, in the magnetosphere of a neutron star (or a Supernova) a GMSW can be excited in the region of the magnetic equator, similarly to pulsars with a knife-like radiation pattern. In this region, as was demonstrated by the above examples, the gravitational radiation can be absorbed practically completely by the excitation of shock waves. A neutron star of such type should radiate gravitational waves only from its magnetic poles, similarly to pulsars with a pencil-like radiation pattern. In this case the probability of observing a GW sources can be sharply dropped. However, suddenly the GMSW open another way of observing gravitational waves. If such a shock wave is formed and takes off the magnetosphere, it will carry with itself (at a subluminal speed) superstrong magnetic fields into the interstellar space. The interaction of cosmic plasma with such magnetic fields can lead to the anomalous electromagnetic phenomena in the radio and optical spectral ranges. It should be stressed that there is no other mechanism able to accelerate a shock wave to subluminal velocities.

Thus, the GMSW phenomenon may displace the "centroid" of GW experiments from direct GW detection to optical observations of the Supernovae and their remainders. However, to be quite certain that the GMSW do exist in neutron star magnetospheres and to know the observational manifestations of this phenomenon, it is necessary to solve a number of problems:

1. To solve the self-consistent problem of GW propagation in a homogeneous plasma with an embedded magnetic field and calculate the damping decrement;
2. To clarify the effects on the this phenomenon from plasma inhomogeneity and external gravitational fields;
3. To clarify the effect on this phenomenon from the curvature of magnetic lines of force;
4. To investigate optical manifestations of GMSW.

Table 1*

The GMSW parameter: $2\alpha^2\langle|\beta|\rangle$ in a neutron star magnetosphere

M/M_\odot	R	ω	$N/\langle(\delta R/R)^2\rangle$	$\langle \beta \rangle$	n_0	$2\alpha^2\langle \beta \rangle$
0,405	5,00	5249	1,2(50)	2,93(-10)	5,97(18)	0,260
0,682	8,42	2,02(4)	2,9(53)	4,20(-9)	3,54(18)	3,32
0,677	12,60	8987	7,0(52)	1,64(-9)	2,36(18)	3,670
1,954	9,99	1,66(4)	1,6(55)	1,41(-8)	2,99(18)	30,0

- * quantities in parantheses mean the degrees of 10;
 M/M_\odot is the neutron star mass related to the Solar mass;
 R is the star radius in km.;
 ω is the gravitational radiation frequency in sec^{-1} ;
 $N/\langle(\delta R/R)^2\rangle$ is the star's gravitational radiation power in erg/sec ;
 $\langle|\beta|\rangle$ is the GW average amplitude;
 n_0 is the concentration of electrons on the stellar surface in cm^{-3} .

We intend to study these problems in future papers.

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