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AN EXACT SOLUTION OF THE STATIONARY VACUUM AXIALLY SYMMETRIC EINSTEIN EQUATIONS

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A class of exact solutions of the stationary vacuum axially symmetric Einstein equations is presented. Asymptotically flat generalizations of the Kerr space–time and stationary generalizations of the Zipoy metric are considered.

The description of the exterior gravitational field of an arbitrary stationary rotating axially symmetric mass is one of the most interesting problems in general relativity. The discovery of the Kerr metric [1], playing a fundamental role in modern relativistic astrophysics, stimulated a further search in the field of exact solutions of the Einstein equations which describe the gravitational field of real astrophysical objects.

Thereafter the investigators in the field of exact solutions were engaged in the problem of the solutions' superposition possessing an arbitrary multipole structure. At the same time, they began to develop new mathematical aspects of the solution generating techniques to apply them to this problem.

For the well–known Kerr solution, which possibly describes the exterior gravitational field of a rotating black hole, the total mass M of the field source, its angular momentum J and quadrupole moment Q are related by [2]

$$Q = -J^2/M, \tag{1}$$

which shows that the solution describes a limited class of rotating objects [3].

In this connection asymptotically flat solutions representing a rotating mass and reducing to the Schwarzchild metric in the static limit are usually considered.

The main result of the present paper is the construction of a non–linear superposition of the Kerr space–time with the static vacuum Weyl field. The obtained asymptotically flat solution describes a stationary generalization of the Zipoy solution [4] and possesses an event horizon containing different number of singular points, which depends upon the distortion parameters.

Consider the Papapetrou line element [5]

$$ds^2 = f^{-1}[\mathrm{e}^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{2}$$

where the functions f, γ and ω, depending on the two Weyl canonical coordinates ρ and z, satisfy the following field equations:

$$\begin{aligned} &f\Delta f = (\nabla f)^2 - f^4\rho^{-2}(\nabla\omega)^2,\\ &\nabla(f^2\rho^{-2}\nabla\omega) = 0,\\ &4\gamma_\rho = f^{-2}\rho[f_\rho^2 - f_z^2 - f^4\rho^{-2}(\omega_\rho^2 - \omega_z^2)],\\ &2\gamma_z = f^{-2}\rho(f_\rho f_z - f^4\rho^{-2}\omega_\rho\omega_z) \end{aligned} \tag{3}$$

(a subscript denotes partial differentiation).

Here Δ and ∇ are the operators of the form

$$\begin{aligned} &\Delta \equiv \partial_{\rho\rho} + \rho^{-1}\partial_\rho + \partial_{zz},\\ &\nabla \equiv \vec{\rho}_0\partial_\rho + \vec{z}_0\partial_z, \end{aligned} \tag{4}$$

$\vec{\rho}_0$ and $\vec{z}_0$ being unit vectors along the ρ and z axes, respectively.

Using the results of [6, 7], one can write down the generating formulas describing a new class of asymptotically flat solutions of the stationary axially symmetric Einstein equations:

$$\begin{aligned} f &= 4e^{2\psi}\frac{A}{B},\\ \gamma &= \frac{1}{2}\ln\frac{A}{16(1-\alpha^2)^2 r_+^4 r_-^4} + \widetilde{\gamma},\\ \omega &= \frac{4k\alpha}{\alpha^2 - 1} - \frac{\alpha(r_+ + r_- - 2k)C}{A}e^{-2\psi},\\ A &= [4r_+^2 r_-^2 - \alpha^2(\rho^2 + z^2 - k^2 + r_+ r_-)^2 ab]^2\\ &\quad - 16\alpha^2 k^2 \rho^2 (r_+^2 a + r_-^2 b)^2, \end{aligned}$$

$$
\begin{aligned}
B &= [8r_+^2 r_-^2 - \alpha^2(\rho^2+z^2-k^2+r_+r_-)(r_++r_-\\
&\quad -2k)^2 ab]^2 + 4\alpha^2(r_++r_--2k)^2[(r_+-r_-\\
&\quad +2k)r_+^2 a + (r_+-r_--2k)r_-^2 b]^2,\\
C &= 8r_+^3 r_-^3[(r_+-r_--2k)r_- b - (r_+-r_-\\
&\quad +2k)r_+ a] - \alpha^2(r_+r_-+\rho^2+z^2-k^2)(r_+\\
&\quad +r_--2k)^2[(r_+-r_--2k)r_-^3 b - (r_+-r_-\\
&\quad +2k)r_+^3 a]ab,
\end{aligned}
\tag{5}
$$

where $r_\pm = \sqrt{\rho^2+(z\pm k)^2}$, k is a parameter of the quadrupole distortion and a special choice of the real constant α ensures the asymptotic flatness of the solution.

In the formulas (5) ψ and $\widetilde{\gamma}$ represent any solution of the static vacuum Weyl class

$$
\begin{aligned}
&\Delta\psi = 0,\\
&\widetilde{\gamma}_\rho = \rho[\psi_\rho^2 - \psi_z^2], \quad \widetilde{\gamma}_z = 2\rho\psi_\rho\psi_z,
\end{aligned}
\tag{6}
$$

and the functions a and b are defined, respectively, by the following first–order differential equations:

$$
\begin{aligned}
r_- a_\rho &= 2a[(z-k)\psi_\rho + \rho\psi_z],\\
r_- a_z &= -2a[\rho\psi_\rho - (z-k)\psi_z],\\
r_+ b_\rho &= -2b[(z+k)\psi_\rho + \rho\psi_z],\\
r_+ b_z &= 2b[\rho\psi_\rho - (z+k)\psi_z].
\end{aligned}
\tag{7}
$$

If one now chooses for ψ and $\widetilde{\gamma}$ the Zipoy solution

$$
\begin{aligned}
\psi &= \frac{\delta}{2}\ln\frac{R_++R_--2m}{R_++R_-+2m},\\
\widetilde{\gamma} &= \frac{\delta^2}{2}\ln\frac{R_+R_-+\rho^2+z^2-m^2}{2R_+R_-},
\end{aligned}
\tag{8}
$$

where $R_\pm = \sqrt{\rho^2+(z\pm m)^2}$, m and δ are real constants, we arrive at the following expressions for the functions a and b:

$$
\begin{aligned}
a &= \left(\frac{\rho^2+(z-k)(z-m)+r_-R_-}{\rho^2+(z-k)(z+m)+r_-R_+}\right)^\delta,\\
b &= \left(\frac{\rho^2+(z+k)(z+m)+r_+R_+}{\rho^2+(z+k)(z-m)+r_+R_-}\right)^\delta.
\end{aligned}
\tag{9}
$$

Let us consider the most interesting special cases of the metric (5)–(9).

1. $\delta = 1$. In this case we obtain the three–parameter solution [8], which has the Schwarzchild solution as a static limit and generalizes the Kerr metric to the case of an arbitrary quadrupole distortion of the mass in the process of rotation.

A calculation of the first four relativistic Geroch–Hansen multipole moments [9, 10, see 11] M_n and J_n, characterizing the mass and angular momentum distribution, respectively, yields

$$
\begin{aligned}
M_0 &= \frac{2\alpha^2 k - \alpha^2 m + m}{1-\alpha^2},\\
M_1 &= M_3 = 0,\\
M_2 &= 2\alpha^2 k^3\frac{\alpha^4+6\alpha^2-3}{(\alpha^2-1)^3} - 4\alpha^2 k^2 m\frac{\alpha^2+2}{(\alpha^2-1)^2}\\
&\quad + \frac{2\alpha^2 k m^2}{\alpha^2-1},\\
J_0 &= J_2 = 0,\\
J_1 &= -2\alpha k\frac{(3\alpha^2-1)k+2m(1-\alpha^2)}{(\alpha^2-1)^2},\\
J_3 &= 2\alpha k^4\frac{5\alpha^6+11\alpha^4-9\alpha^2+1}{(\alpha^2-1)^4}\\
&\quad - 4\alpha k^3 m\frac{(5\alpha^2-1)(\alpha^2+1)}{(\alpha^2-1)^3} + 2\alpha k^2 m^2\frac{5\alpha^2+1}{(\alpha^2-1)^2},
\end{aligned}
\tag{10}
$$

whence one can see that the obtained metric is asymptotically flat ($J_0 = 0$) and the quadrupole moment Q is not connected with the total mass M and angular momentum J by the special relation (1).

2. $m = k$. In this case we obtain the metric, which in the prolate ellipsoidal coordinates (x, y), defined by

$$
\rho = k\sqrt{(x^2-1)(1-y^2)}, \qquad z = kxy,
\tag{11}
$$

has the form

$$
\begin{aligned}
f &= \left(\frac{x-1}{x+1}\right)^\delta \frac{A}{B},\\
\gamma &= \frac{\delta^2}{2}\ln\frac{x^2-1}{x^2-y^2} + \frac{1}{2}\ln\frac{A}{(1-\alpha^2)^2(x^2-1)^{4\delta}},\\
\omega &= \frac{4k\alpha}{\alpha^2-1} - \frac{2k\alpha C}{A}\left(\frac{x+1}{x-1}\right)^\delta,\\
A &= (x^2-1)^4[(x^2-1)^{2\delta-2} - \alpha^2(x^2-y^2)^{2\delta-2}]^2\\
&\quad - \alpha^2(x^2-1)^{2\delta+1}(1-y^2)[(x-y)^{2\delta-2}\\
&\quad + (x+y)^{2\delta-2}]^2,\\
B &= (x^2-1)^2[(x^2-1)^{2\delta-1} - \alpha^2(x-1)^2\\
&\quad \times (x^2-y^2)^{2\delta-2}]^2 + \alpha^2(x-1)^2(x^2-1)^{2\delta}\\
&\quad \times [(y+1)(x-y)^{2\delta-2} + (y-1)(x-y)^{2\delta-2}]^2,\\
C &= (x^2-1)^{3\delta}[(y-1)(x+y)^{2\delta-1}\\
&\quad - (y+1)(x-y)^{2\delta-1}] - \alpha^2(x-1)^2\\
&\quad \times (x^2-1)^{\delta+1}(x^2-y^2)^{2\delta-1}[(y-1)(x+y)^{2\delta-3}\\
&\quad - (y+1)(x-y)^{2\delta-3}].
\end{aligned}
\tag{12}
$$

The obtained solution (12) possibly describes a stationary generalization of the Zipoy metric (8), which, in the prolate ellipsoidal coordinates (x, y), can be rewritten as

$$\psi = \frac{\delta}{2} \ln \frac{x-1}{x+1}, \qquad \widetilde{\gamma} = \frac{\delta^2}{2} \ln \frac{x^2-1}{x^2-y^2}. \tag{13}$$

3. $m = k$, $\delta = 1$. From (5)–(9) we obtain the Kerr metric

$$f = \frac{p^2x^2 + q^2y^2 - 1}{(px+1)^2 + q^2y^2},$$

$$\gamma = \frac{1}{2} \ln \frac{p^2x^2 + q^2y^2 - 1}{p^2(x^2-y^2)},$$

$$\omega = -\frac{2kq(px+1)(1-y^2)}{p(p^2x^2+q^2y^2-1)}, \tag{14}$$

where the real constants p and q are introduced by the expressions

$$p = \frac{1-\alpha^2}{1+\alpha^2}, \quad q = \frac{2\alpha}{1+\alpha^2}, \quad p^2+q^2 = 1. \tag{15}$$

The relativistic Geroch–Hansen multipole moments M_n and J_n have the form

$$M_0 = m\frac{1+\alpha^2}{1-\alpha^2}, \qquad M_1 = 0,$$

$$M_2 = -4\alpha^2 m^3 \frac{1+\alpha^2}{(1-\alpha^2)^3}, \qquad M_3 = 0,$$

$$J_0 = 0, \qquad J_1 = -2\alpha m^2 \frac{1+\alpha^2}{(1-\alpha^2)^2},$$

$$J_2 = 0, \qquad J_3 = 8\alpha^3 m^4 \frac{1+\alpha^2}{(1-\alpha^2)^4}, \tag{16}$$

giving us the relation (1) between the quadrupole moment Q, total mass M_0 and angular momentum J_1 of the Kerr solution

$$Q = -4\alpha^2 m^3 \frac{1+\alpha^2}{(1-\alpha^2)^3} = -J_1^2/M_0. \tag{17}$$

4. $k = -m$. In this case we obtain the solution

$$f = \left(\frac{x-1}{x+1}\right)^{\delta} \frac{A}{B},$$

$$\gamma = \frac{\delta^2}{2} \ln \frac{x^2-1}{x^2-y^2} + \frac{1}{2} \ln \left(\frac{A}{(1-\alpha^2)^2 (x^2-y^2)^{4\delta+4}} \right),$$

$$\omega = \frac{4k\alpha}{\alpha^2-1} - \frac{2k\alpha C}{A} \left(\frac{x+1}{x-1}\right)^{\delta},$$

$$A = [(x^2-y^2)^{2\delta+2} - \alpha^2(x^2-1)^{2\delta+2}]^2 - \alpha^2(x^2-1)^{2\delta+1}(1-y^2)[(x+y)^{2\delta+2} + (x-y)^{2\delta+2}]^2,$$

$$B = [(x^2-y^2)^{2\delta+2} - \alpha^2(x-1)^2(x^2-1)^{2\delta+1}]^2 + \alpha^2(x^2-1)^{2\delta}(x-1)^2[(y+1)(x+y)^{2\delta+2} + (y-1)(x-y)^{2\delta+2}]^2,$$

$$C = (x^2-1)^{\delta}(x^2-y^2)^{2\delta+3}[(y-1)(x-y)^{2\delta+1} - (y+1)(x+y)^{2\delta+1}] - \alpha^2(x^2-1)^{3\delta+1} \times (x-1)^2[(y-1)(x-y)^{2\delta+3} - (y+1)(x+y)^{2\delta+3}], \tag{18}$$

describing a stationary generalization of the Zipoy metric (13).

5. $k = -m$, $\delta = 1$. By this choice we obtain the two–parameter solution [12], which possesses the Schwarzchild static limit but differs from the Kerr metric.

The relativistic Geroch–Hansen multipole moments M_n and J_n for this solution can be expressed in the form

$$M_0 = m\frac{1-3\alpha^2}{1-\alpha^2}, \qquad M_1 = 0,$$

$$M_2 = 4\alpha^2 m^3 \frac{2\alpha^4+3\alpha^2-3}{(1-\alpha^2)^3}, \qquad M_3 = 0,$$

$$J_0 = 0, \qquad J_1 = -2\alpha m^2 \frac{5\alpha^2-3}{(1-\alpha^2)^2},$$

$$J_2 = 0, \qquad J_3 = 8\alpha m^4 \frac{5\alpha^6-4\alpha^2+1}{(1-\alpha^2)^4}. \tag{19}$$

From (19) one can see that the quadrupole moment Q of the examined solution is not related to the total mass M_0 and angular momentum J_1, unlike the case of the Kerr metric.

6. $\delta = 0$. In this case we obtain the solution

$$f = \frac{A}{B}, \quad \gamma = \frac{1}{2} \ln \frac{A}{(1-\alpha^2)^2(x^2-y^2)^4},$$

$$\omega = \frac{4k\alpha}{\alpha^2-1} + 4k\alpha \frac{C}{A},$$

$$A = [(x^2-y^2)^2 - \alpha^2(x^2-1)^2]^2 - 4\alpha^2(x^2-1)(1-y^2)(x^2+y^2)^2,$$

$$B = [(x^2-y^2)^2 - \alpha^2(x-1)^2(x^2-1)]^2 + 4\alpha^2y^2(x-1)^2(x^2+y^2+2x)^2,$$

$$C = (x^2-y^2)^3(x^2+y^2) - \alpha^2(x^2-1)(x-1)^2(x^3+3x^2y^2+3xy^2+y^4), \tag{20}$$

which satisfies the field equations (3) and describes the gravitational field of a stationary rotating deformed mass.

7. In the case $\alpha = 0$ (the absence of rotation) we have the Zipoy solution (8). That the Zipoy solution can be obtained also by nullifying the quadrupole parameter k, reveals the physically evident fact that a nonrotating spherically symmetric source in the process of rotation must necessarily exhibit a distortion.

8. $m = k = 0$ or $\delta = \alpha = 0$. In this case the space–time is flat.

An important property of the solution (5)–(9) is the existence of an event horizon occurring at the hypersurface

$$R_+ + R_- - 2m = 0. \tag{21}$$

This horizon is regular in the case of the Kerr solution ($\delta = 1$, $m = k$) and in the Schwarzchild case ($\delta = 1$, $\alpha = 0$ or $k = 0$). When $|k| > |m|$, the horizon contains only one singular point situated on the symmetry axis (in the spheroidal coordinates $r = M + (R_+ + R_-)/2$, $\cos\vartheta = (R_+ - R_-)/(2k)$ it is the point $r = M + k$, $\vartheta = 0$). If $|k| < |m|$, $k \neq 0$, two ring singularities appeared on the horizon; they correspond to the values $\cos\vartheta = \pm k/m$. In the special case $k = -m$ only the two poles $\cos\vartheta = \pm 1$ are singular.

As for other singularities of the solution at $m \neq k$, they are determined by the equations

$$R_+ + R_- + 2m = 0, \qquad B = 0, \tag{22}$$

whereas real roots of the second equation are placed on the stationary limit surface $A = 0$. From the Robinson theorem [13] it follows that not all of them are hidden under the horizon, but part of them must be outside.

Therefore, the internal perturbations of the Kerr metric can result in singular points outside the event horizon, while the horizon itself remains completely regular.

Although a more detailed examination of the physical properties still remains a task for the future, the above analysis makes us hope that the obtained exact solutions possibly represent the exterior gravitational fields of rotating stars and thus can be of interest for astrophysicists.

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