Gravitation & Cosmology, Vol. 1 (1995), No. 4, pp. 315–318

ON A POWER SERIES REPRESENTATION OF THE GENERAL SOLUTION OF FEDOROV'S SET OF EQUATIONS

V. Skorobogat'ko[1] and O. Myakinnik[2]

Inst. of Applied Problems of Mechanics and Mathematics, National Acad. Sci. of Ukraine,
3b Nauchnaya Str., MSP, Lvov 290601, Ukraine

Received 16 December 1994
Received in final form 09 July 1995

The efficiency of the alternion linearization as a general line of attacking the quadratic nonlinearity entering into Fedorov's set of equations is demonstrated by the example of finding a power series expansion of its general solution.

F.I. Fedorov's set of 1st order partial differential equations with a quadratic non-linearity

$$\sum_{k=1}^{s} A_k \frac{\partial \overrightarrow{B}}{\partial x_k} = \overrightarrow{C_0} + C_1 \overrightarrow{B} + \overrightarrow{B}^* C_2 \overrightarrow{B} \qquad (1)$$

(where $\overrightarrow{B}^* = (B_1, \ldots, B_n)$; A_k $(k = \overline{1,s})$ and C_1 are square n-by-n numerical matrices, C_2 is a cubic n-by-n-by-n numerical matrix, and $\overrightarrow{C_0}$ is an n-dimensional numerical column vector) provide a unified mathematical basis for describing all types of fundamental interactions. In particular, this formalism was developed as applied to the metric and tetrad formulations of the gravitation theory. New formulations of supersymmetry and supergravity theories were given using the unified field formalism in Refs. [1]–[5].

We shall apply the alternion (spinor) linearization to determine the power series expansion coefficients for the general solution of the set (1). This linearization is a consistent evolution and generalization of Dirac's idea of factorizing the d'Alembert operator.

The general solution of (1) is sought in the form

$$\overrightarrow{B}(x) = \sum_{\|\alpha\|=0}^{\infty} \overrightarrow{\gamma_\alpha} x^\alpha, \qquad (2)$$

where $x = (x_1, \ldots, x_s)$, $\alpha = (\alpha_1, \ldots, \alpha_s)$, $\|\alpha\| = \alpha_1 + \alpha_2 + \cdots + \alpha_s$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_s^{\alpha_s}$. Substituting (2) into (1) and comparing the vector coefficients of similar terms, we obtain an infinite set of quadratic equations. If, starting with some index β, $\|\beta\| \geq N$, one puts $\overrightarrow{\gamma_\beta}$ equal to 0, then to determine the coefficients $\overrightarrow{\gamma_\alpha}$, $\|\alpha\| < N$, one obtains a set of quadratic equations with the number of equations equal to the number of unknowns. If, starting from the subset corresponding to the largest of the indices α, we solve it in a sequential manner, we shall find the sought vector coefficients.

Thus, to present the general solution of the set (1) as a power series and to determine its solutions in the case of a zero left-hand side, $(B_i = \text{const}, \ i = \overline{1,n})$ it is required to solve a set of n quadratic algebraic equations in n unknowns, called a Riccati algebraic system. The latter has the form

$$\begin{aligned} {}^2R_i \equiv a_{ijk} x_j x_k + b_{im} x_m + a_{i0} = 0, \\ i, j, k, m = \overline{1,n}. \end{aligned} \qquad (3)$$

Further we shall deal with this type of system.

In solving the Riccati algebraic system we follow the alternion method proposed by S. Pshenichnikov [6]. This method allows one to construct a system of linear algebraic equations equivalent to a given quadratic equation in terms of the equation itself. The case described by Pshenichnikov corresponds to a Riccati algebraic system possessing a finite number of solutions. In this paper it will be obtained as a special case of a general theory to be presented here.

The alternion theory founded by B. Rosenfeld and developed by G. Zaitsev is a basis for the method to be applied. According to Rosenfeld, an alternion algebra of order n and index q ${}^qA_n$ is a real algebra of rank 2^{n-1} with the generators $1, e_1, \ldots, e_{n-1}$ obeying the multiplication rule $e_i e_j = -e_j e_i$, $e_i^2 = \varepsilon_i$, where for arbitrary q values of the subscripts i, $\varepsilon_i = 1$ and $\varepsilon_i = -1$ otherwise.

The generator 1 denotes the unity element of the algebra A_1. This algebra represents the field of real numbers;

A_2 with the generators 1, e_1, $(e_1)^2 = -1$ is isomorphic to the field of complex numbers;

---

[1] e-mail: skorobog%lms.ipam.lviv.ua@litech.lviv.ua

[2] e-mail: myakino%lms.ipam.lviv.ua@litech.lviv.ua

the algebra ${}^1A_2$ with the generators 1, e_1, $(e_1)^2 = 1$ may be regarded as the algebra of binary numbers;

A_3 with the generators 1, e_1, e_2 and the basis elements $e_1e_2 = e_2e_1$, $(e_1)^2 = (e_2)^2 = (-1)$, $(e_1e_2)^2 = (-1)$ is isomorphic to the field of quaternions;

${}^1A_3$ and ${}^2A_3$ are isomorphic to the antiquaternion (pseudoquaternion) algebra.

Let us give a general definition.

Definition 1. An associative algebra ${}^qA_{2n+1}$ with $2n$ generators e_α related by the law

$$e_{\alpha_1}e_{\alpha_2} + e_{\alpha_2}e_{\alpha_1} = 2\varepsilon_{\alpha_1}\delta_{\alpha_1\alpha_2}e, \qquad \alpha = \overline{1, 2n}, \tag{4}$$

where e is the neutral element of the algebra, e_{α_1} are all equal to e in q cases and to $(-e)$ in the other $(2n-q)$ cases, is called an alternion algebra.

We shall write α_i for the elements whose square is e; and β_i for those whose square is $(-e)$. That is, $\alpha_i^2 = e$, $\beta_i^2 = (-e)$. Alternion algebras having an odd number of generators are not considered since they are subalgebras of the corresponding algebras with an even number of generators. If all $\varepsilon_\alpha = e$, then the algebra is a Clifford algebra. If one considers algebras not over the field of real numbers but over the field of complex numbers or quaternions, then alternion algebras with different q are isomorphic to the Clifford algebra. Therefore, ${}^qA_{2n+1}$ over the field of real numbers are called real forms of the Clifford algebra. By definition, the algebra $B_n = {}^nA_{2n+1}$, with

$$\begin{aligned} \varepsilon_1 &= \varepsilon_2 = \ldots = \varepsilon_n = e, \\ \varepsilon_{n+1} &= \varepsilon_{n+2} = \ldots = \varepsilon_{2n} = (-e) \end{aligned}$$

is considered as a normal real form of the complex Clifford algebra (it is also known as the basis algebra of real spinors). The other real forms can be obtained by replacing some normal form generators e_α by $(e_\alpha)' = (ie_\alpha)$.

The application of alternion algebra for solving a set of algebraic equations is based on some facts presented here without proof.

Theorem 1. An alternion algebra with an even number of generators is a simple algebra (that is, it does not contain any non-vanishing two-sided ideals other than this algebra itself).

Theorem 2. All exact irreducible representations of a simple algebra are equivalent.

Theorem 3. The algebra ${}^qA_{2n+1}$ is isomorphic to the algebra

(1) of real square matrices of order 2^n if

$$p = (-1)^{(n-q+1)(n-q)/2} = 1;$$

(2) of quaternion square matrices of order 2^{n-1} if

$$p = -1.$$

It follows that the basis algebra of real spinors B_n (the one to be dealt with in this paper) is isomorphic to the algebra of real square matrices of order 2^n. There exists a special representation of B_n with -1, 1 and 0 as matrix elements [7]. The existence of these matrices can be proved by induction. Symmetric matrices correspond to α_i and antisymmetric ones to β_i (to be denoted S_i^k and A_i^k, respectively; the matrix S_0^k is auxiliary, $i = \overline{1, 2^{(n-1)}}$; the identity matrix E corresponds to the algebra unity e). Let $n = 1$. Then

$$S_1^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_1^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S_0^1 = S_1^1A_1^1.$$

For $n = 2$

$$\begin{aligned} S_1^2 &= \begin{pmatrix} S_1^1 & 0 \\ 0 & S_1^1 \end{pmatrix}, \qquad S_2^2 = \begin{pmatrix} 0 & S_0^1 \\ S_0^1 & 0 \end{pmatrix}, \\ A_1^2 &= \begin{pmatrix} A_1^1 & 0 \\ 0 & A_1^1 \end{pmatrix}, \qquad A_2^2 = \begin{pmatrix} 0 & -S_0^1 \\ S_0^1 & 0 \end{pmatrix}, \\ S_0^2 &= S_1^2S_2^2A_1^2A_2^2 \text{ and so on.} \end{aligned}$$

At the n-th step of induction

$$\begin{aligned} S_k^n &= \begin{pmatrix} S_k^{n-1} & 0 \\ 0 & S_k^{n-1} \end{pmatrix}, \\ S_{k+1}^n &= \begin{pmatrix} 0 & S_0^{n-1} \\ S_0^{n-1} & 0 \end{pmatrix}, \\ A_k^n &= \begin{pmatrix} A_k^{n-1} & 0 \\ 0 & A_k^{n-1} \end{pmatrix}, \\ A_{k+1}^n &= \begin{pmatrix} 0 & -S_0^{n-1} \\ S_0^{n-1} & 0 \end{pmatrix}, \\ & \qquad k = 1, 2, \ldots, n-1, \\ S_0^n &= S_1^nS_2^n \ldots S_n^nA_1^nA_2^n \ldots A_n^n. \end{aligned}$$

To illustrate the application of alternion algebra to linearization of (3), we first consider the special case of a quadratic form

$$\begin{aligned} {}^2R_i \equiv\ & a_{i1}x_1^2 + a_{i2}x_2^2 + \cdots \\ & - a_{ij}x_j^2 - \ldots + a_{in}x_n^2 + a_{i0}. \end{aligned} \tag{5}$$

Let us associate with (5) the expression, linear in x_i,

$$\begin{aligned} D_i \equiv \big(&\alpha_1\sqrt{a_{i1}}x_1 + \alpha_2\sqrt{a_{i2}}x_2 + \cdots \\ &+\beta_j\sqrt{a_{ij}}x_j + \cdots + \alpha_n\sqrt{a_{in}}x_n + \alpha_0\sqrt{a_{i0}}\big). \end{aligned}$$

Raising D_i to the 2nd power using the alternion multiplication rules (4), we obtain $(D_i)^2 = {}^2R_ie$ (or in matrix notations $(D_i)^2 = {}^2R_iE$). However, if (5) contained terms of the form $a_{ijk}x_jx_k$, the alternions introduced above (they are also called unipotent alternions) would give no way to linearize the expression considered.

Let us introduce another type of alternions with the following multiplication rule [6]:

$$\omega_{p_1}^{t_1}\omega_{p_2}^{t_2}+\omega_{p_2}^{t_2}\omega_{p_1}^{t_1}=\varepsilon_{t_1}\delta^{t_1t_2}\tilde{\delta}_{p_1p_2}e, \tag{6}$$

where $\tilde{\delta}_{p_1p_2}$ is an inverse Kronecker delta ($\tilde{\delta}_{p_1p_2}=1-\delta_{p_1p_2}$); the subscripts take the values 1 and 2. Alternions of this type are called nilpotent. It is possible to unite them with unipotent alternions (4) into a unified algebra and to unify the multiplication rules by

$$e_i\omega_p^t+\omega_p^te_i=0. \tag{7}$$

Nilpotent alternions may be expressed in terms of unipotent ones and hence they also have a special matrix representation. Using nilpotent alternions, let us associate the expression $(a_{ijk}x_jx_k-a_{imn}x_mx_n)$ with

$$\omega_1^1a_{ijk}x_j+\omega_2^1x_k+\omega_1^2a_{imn}x_m+\omega_2^2x_n, \tag{8}$$

where $\omega_1^1=\frac{1}{\sqrt{2}}(\alpha_1+\beta_1)$, $\omega_2^1=\frac{1}{\sqrt{2}}(\alpha_1+\beta_2)$, $\omega_1^2=\frac{1}{\sqrt{2}}(\alpha_2+\beta_3)$, and $\omega_2^2=\frac{1}{\sqrt{2}}(\alpha_3+\beta_3)$. Some explanation for (8) is needed. If $\varepsilon_{t_1}=e$, then the equality of superscripts t of the nilpotent alternions entails the equality of elements α_i in (8). If $\varepsilon_t=-e$, then the elements ω_1^t and ω_2^t have the same element β_j in this formula. To make sure that

$$\begin{aligned}&(\omega_1^1a_{ijk}x_j+\omega_2^1x_k+\omega_1^2a_{imn}x_m+\omega_2^2x_n)^2\\&\qquad=(a_{ijk}x_jx_k-a_{imn}x_mx_n)e.\end{aligned}$$

we use (4), (6), (7).

So, we have demonstrated the fact that it is possible to "take a square root" of the left-hand side of a quadratic equation through the use of alternions. Let us now perform an alternion linearization of (3). It is convenient to put this system in the form

$$\begin{aligned}{}^2R_i\equiv{}&x_1(a_{i11}x_1+a_{i12}x_2+\cdots+a_{i1n}x_n+b_{i1})\\&+x_2(a_{i22}x_2+a_{i23}x_3+\cdots+a_{i2n}x_n+b_{i2})\\&+\cdots\\&+x_n(a_{inn}x_n+b_{in})+a_{i0}=0,\qquad i=\overline{1,n}\end{aligned} \tag{9}$$

for this procedure. Let us associate each equation of the set (9) with the following matrix equation, linear in the variables to be found:

$$\begin{aligned}D_i\overrightarrow{\Phi_i}\equiv{}&(\omega_1^{1i}x_1+\omega_2^{1i}(a_{i11}x_1+\cdots+b_{i1})\\&+\omega_1^{2i}x_2+\omega_2^{2i}(a_{i22}x_2+\cdots+b_{i2})\\&+\cdots\\&+\omega_1^{ni}x_n+\omega_2^{ni}(a_{inn}x_n+b_{in})\\&+e_i\sqrt{a_{i0}})\times\overrightarrow{\Phi_i}=0,\qquad i=\overline{1,n}.\end{aligned} \tag{10}$$

Here e_i are the generators of the algebra (4); ω_j^{ki}, $j=1,2$, are the generators of the algebra (6). The index k corresponds to a summand number in the i-th equation of (9). Consider the alternions involved in (10) as the generators of some unified algebra of nilpotent and unipotent alternions. The number of these generators is determined by a specific form of the system. We need a sufficient number of generators to "split" all the variable containing terms in the i-th equation, $i=\overline{1,n}$, and to "take square roots" of the constant terms. The number of generators does not exceed $3n^2+n$. $\overrightarrow{\Phi_i}$ is an undetermined m-dimensional numerical column vector where m is the order of the matrix representation of the alternions. Suppose that $\overrightarrow{\Phi_i}\not\equiv 0$.

Theorem 4. Every solution to the set (10) is a solution to the original set (9). The inverse is also true.

To prove this assertion we multiply the i-th equation of (10) by the corresponding D_i. Since $\overrightarrow{\Phi_i}\not\equiv 0$, then $\det(D_i)^2$ is necessarily zero. We multiply D_i by D_i term by term. By construction $(D_i)^2={}^2R_iE_{(m)}$. Then

$$\det(D_i)^2=\left({}^2R_i\right)^m. \tag{11}$$

Let $(x_1,\ldots,x_n)$ be a solution of (10) (that is, the determinant $\det D_i$ is zero at these x_k). It follows that ${}^2R_i=0$, i.e. each solution of (10) is a solution of (9). By virtue of (11) the inverse is valid too. •

Let us try to elucidate the manner in which column vectors $\overrightarrow{\Phi_i}$ are related to each other. By construction

$$D_iD_j+D_jD_i=0. \tag{12}$$

Multiply the i-th equation of (10) by D_j and the j-th one by D_i. Using (12) and adding the obtained equations, we write down the following equation with respect to the new column vector $\left(\overrightarrow{\Phi_i}-\overrightarrow{\Phi_j}\right)$

$$D_jD_i\left(\overrightarrow{\Phi_i}-\overrightarrow{\Phi_j}\right)=0.$$

If the set (10) possesses a solution (or, which is the same, the set (9) possesses a solution), then $\det(D_jD_i)=0$ and $\left(\overrightarrow{\Phi_i}-\overrightarrow{\Phi_j}\right)=\overrightarrow{U^{(ij)}}$, $\overrightarrow{U^{(ij)}}\not\equiv 0$, where $\overrightarrow{U^{(ij)}}$ is some column vector. With no loss of generality we can assume that $\overrightarrow{\Phi_j}$, $j=\overline{2,n}$, differ from $\overrightarrow{\Phi}_1\equiv\overrightarrow{\Phi}$ by a vector $\overrightarrow{U^{(j)}}$, $j=\overline{2,n}$. Then (10) takes the form

$$\begin{cases}D_1\overrightarrow{\Phi}=0\\ D_2\left(\overrightarrow{\Phi}+\overrightarrow{U^{(2)}}\right)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ D_n\left(\overrightarrow{\Phi}+\overrightarrow{U^{(n)}}\right)=0\end{cases},$$

or, in new variables,

$$\Lambda\begin{pmatrix}x_1\overrightarrow{\Phi}\\x_2\overrightarrow{\Phi}\\\vdots\\x_n\overrightarrow{\Phi}\end{pmatrix}=\begin{pmatrix}0\\-D_2\overrightarrow{U^{(2)}}\\\vdots\\-D_n\overrightarrow{U^{(n)}}\end{pmatrix}+\begin{pmatrix}Q_1\overrightarrow{\Phi}\\Q_2\overrightarrow{\Phi}\\\vdots\\Q_n\overrightarrow{\Phi}\end{pmatrix}, \tag{13}$$

where

$$Q_i = -\left(\omega_2^{1i} b_{i1} + \omega_2^{2i} b_{i2} + \ldots + \omega_2^{ni} b_{in} + e_i \sqrt{a_{i0}}\right),$$

$i = \overline{1, n}$. The matrix Λ is a square nm-by-nm matrix constructed of the coefficients by the quadratic terms of the original set and the corresponding alternion algebra generators. Therefore, Λ is always invertible. Let $\Lambda^{-1} = \|\lambda_{ki}\|_{k,i=\overline{1,n}}$, where λ_{ki} are square m-by-m matrices, $\lambda_{ki} = \|\mu_{qj}^{(ki)}\|_{q,j=\overline{1,m}}$. Multiplying Eq. (13) by Λ^{-1}, we obtain

$$\begin{pmatrix} x_1 \overrightarrow{\Phi} \\ \vdots \\ x_n \overrightarrow{\Phi} \end{pmatrix} = \begin{pmatrix} -\sum_{i=2}^{n} \lambda_{1i} D_i \overrightarrow{U^{(i)}} \\ \vdots \\ -\sum_{i=2}^{n} \lambda_{ni} D_i \overrightarrow{U^{(i)}} \end{pmatrix} + \begin{pmatrix} C_1 \overrightarrow{\Phi} \\ \vdots \\ C_n \overrightarrow{\Phi} \end{pmatrix} \tag{14}$$

where the column vector $\begin{pmatrix} C_1 \overrightarrow{\Phi} \\ \vdots \\ C_n \overrightarrow{\Phi} \end{pmatrix}$ has been obtained as a result of multiplying $\begin{pmatrix} Q_1 \overrightarrow{\Phi} \\ \vdots \\ Q_n \overrightarrow{\Phi} \end{pmatrix}$ by Λ^{-1}.

Let us recast (14) as

$$\begin{pmatrix} Ex_1 - C_1 & \lambda_{12} & \ldots & \lambda_{1n} \\ Ex_2 - C_2 & \lambda_{22} & \ldots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Ex_n - C_n & \lambda_{2n} & \ldots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} \overrightarrow{\Phi} \\ D_2 \overrightarrow{U^{(2)}} \\ \vdots \\ D_n \overrightarrow{U^{(n)}} \end{pmatrix} = 0. \tag{15}$$

Since (15) possesses a nonzero solution (at least, $\overrightarrow{\Phi} \not\equiv 0$),

$$\begin{vmatrix} Ex_1 - C_1 & \lambda_{12} & \ldots & \lambda_{1n} \\ Ex_2 - C_2 & \lambda_{22} & \ldots & \lambda_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ Ex_n - C_n & \lambda_{2n} & \ldots & \lambda_{nn} \end{vmatrix} = 0. \tag{16}$$

Expanding this determinant, we obtain a single equation of high degree specifying the $(n \times (m-1))$-dimensional manifold of solutions. All the solutions of the original set belong to this manifold. If it turns out that all the correction terms $\overrightarrow{U^{(j)}}$, $j = \overline{2, n}$, are zero, Eq. (16) transforms into a set of n equations of the form

$$(Ex_i - C_i)\, \overrightarrow{\Phi} = 0, \qquad i = \overline{1, n}. \tag{17}$$

That is, we deal with n eigenvalue problems with the sought solutions as eigenvalues and the same eigenvector $\overrightarrow{\Phi}$ for all the eigenvalue problems. If one imposes the additional restriction on $\overrightarrow{\Phi}$,

$$\overrightarrow{\Phi}^* \overrightarrow{\Phi} = 1,$$

then one needs to solve only one of the problems (since $x_j = \overrightarrow{\Phi}^* C_j \overrightarrow{\Phi}$). Thus, the problem of building an ordered sample of n solutions turns out to be solved too. The set (17) corresponds to the case of a finite number of solutions of the Ricatti-type system. This case was described in [6]. Here it was obtained as a special case of (16), which covers all solutions of (3).

We have demonstrated the capabilities of the alternion linearization as a general approach to the nonlinearity appearing in Fedorov's equations system. It is also applicable to other problems connected with Fedorov's set of equations (for example, the Cauchy problem).

References

[1] F.I. Fedorov, Doklady AN SSSR 170 (1968), 4, 802.

[2] A.A. Kirillov and F.I. Fedorov, Acta Phys. Pol. B7 (1976), 3, 161.

[3] A.A. Bogush and F.I. Fedorov, "Universal Matrix Form of First Order Relativistic Wave Equations and Generalized Kronecker Symbols". Preprint Inst. of Phys. of BSSR Acad. of Sci., No. 192, 1980.

[4] L.F. Babichev and V.I. Kuvshinov, Doklady AN SSSR 253 (1980), 5, 1088.

[5] F.I. Fedorov, in: "Absracts of Contributed Papers for the Discussion Groups", 9-th International Conference on General Relativity and Gravitation, July 14–19, 1980 (Friedrich Schiller University, Jena, GDR, 1980, vol. 3).

[6] P.G. Kuznetsov and C.B. Pshenichnikov, Doklady AN SSSR 283 (1985), 5, 1073.

[7] G.A. Zaitsev, Doklady AN SSSR 156 (1964), 2, 294.