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GENERALLY COVARIANT CONSERVATIVE ANGULAR MOMENTUM AND ITS RADIATION IN GENERAL RELATIVITY

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The Noether conservative current, corresponding to the invariance of the action of a gravity-matter system under local Lorentz transformations, can be interpreted as the angular-momentum tensor of the system. The existence of superpotentials, expressed solely in terms of a tetrad, make the total current identically conserved and the total angular momentum gauge-covariant. Two examples are given. Quadrupole radiation, which is exactly that derived from the Landau-Lifshitz energy-momentum pseudotensor, is also obtained.

1. Introduction

The construction of generally covariant conservation laws has been one of the fundamental problems in general relativity [1]. One of the present authors (Duan) has succesfully obtained a generally covariant energy-momentum tensor of gravity-matter systems by means of Noether's theorem and the general displacment transformation [2]. The expression has some advantages, overcoming the difficulties possessed by other kinds of expressions [3]. The angular momentum (AM) conservation law in general relativity has been discussed by some authors. In [4], the AM of a gravity-matter system is given in terms of a pseudotensor instead of a tensor. The same difficulties exist in [5-6]. A. Komar [7] first suggested that some integrals defined in terms of Killing fields be interpreted as the gravitational field energy or AM. These integrals can be used to analyze only symmetric space-times. For those which are not symmetric but asymptotically symmetric at null infinity, there are some integrals called linkages which can also be interpreted physically [8]. But the resulting definitions involve some ambiguities and lack adequate useful inequalities, as pointed out in [1]. Recently J. Chevalier [9] presented an expression of the AM conservation law, such that the total AM depends on the tetrad, i. e., is gauge-dependent, hence this theory cannot be called satisfactory.

In this paper, we will show that in a general spacetime there exist Noether conservative currents corresponding to local Lorentz invariance, whose charges can be interpreted as the components of the system angular momentum . This AM conservation law has some advantages over those aforesaid. First, for an isolated system, it does not require the spacetime to be symmetric in any way. Second, it takes the form of the usual Noether current and possesses a superpotential which is directly expressed in terms of a tetrad. This definition is local and the quadrupole radiation is exactly that derived from the Landau-Lifshitz pseudotensor. Third, the charges carry tetrad indices and do not depend on the choice of the Riemann coordinates. The whole theory is generally covariant. Fourth, the total AM is gauge- covariant, i.e., it is independent of the tetrad choice.

In Section 2 we present the derivation of the conservation law. In Section 3 we apply our conservation law to Kerr spacetime and Fock's approximate solution. In Section 4 we derive the radiation formula. Some discussions are presented in Section 5.

2. Generally covariant angular momentum conservation law

It is known that in deriving the generally covariant conservation law of energy-momentum in general relativity [2], the general displacement transformation, which is a generalization of the displacement transfomation in Minkowski spacetime, was used. In a local Lorentz reference frame, the general displacement transformation takes the same form as that in Minkowski spacetime. This implies that generally covariant conservation laws correspond to the invariance of the action under local transformations. We may conjecture that, since the angular momentum conservation law in special relativity corresponds to the invariance of the action under the Lorentz transformation, the generally covariant AM conservation law in general relativity can be obtained using the local Lorentz invariance. In this paper we will show that this conjecture is reasonable.

In general relativity the total action of the gravity-matter system is expressed as [10]

$$I = \int_M \mathcal{L} d^4x = \int_M (\mathcal{L}_g + \mathcal{L}_m) d^4x, \qquad (1)$$

$$\mathcal{L}_g = \frac{c^4}{16\pi G}\sqrt{-g}g^{\alpha\beta}(\Gamma^\nu_{\mu\alpha}\Gamma^\mu_{\nu\beta} - \Gamma^\sigma_{\mu\sigma}\Gamma^\mu_{\alpha\beta}) \qquad (2)$$

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where $\Gamma^{\nu}_{\mu\alpha}$ are the Christoffel symbols, $\mathcal{L}_m$ is the matter Lagrangian and G is the Newton gravitatinal constant. Further on we use the tetrad description and our notations are as follows: e^a_μ are the tetrad components and e^μ_a are their inverse, $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$, $\eta_{ab} = (1,-1,-1,-1)$, $\omega_{\mu ab}$ are the spin connections defined by

$$\mathcal{D}_\mu e^a_\nu \equiv \partial_\mu e^a_\nu - \omega_{\mu ab}e^b_\nu - \Gamma^\lambda_{\mu\nu}e^a_\lambda = 0; \tag{3}$$

$\omega_{abc} = e^\mu_a \omega_{\mu bc}$, $\omega_a = \eta^{bc}\omega_{bac}$, It can be proved that

$$\mathcal{L}_g = \frac{c^4}{16\pi G} e(D_\mu e^\nu_a D_\nu e^{a\mu} - e^{a\nu}e^b_\lambda D_\nu e^\lambda_a D_\sigma e^\sigma_b); \tag{4}$$

$$\mathcal{L}_g = \mathcal{L}_\omega - \frac{c^4}{16\pi G}\Delta; \tag{5}$$

$$\mathcal{L}_\omega = \frac{c^4}{16\pi G}(\omega_a\omega^a - \omega_{abc}\omega^{cba}); \tag{6}$$

$$\Delta = \partial_\mu(ee^{a\mu}\partial_\nu e^\nu_a - ee^\nu_a\partial_\nu e^{a\mu}), \tag{7}$$

where

$$D_\mu e^\nu_a \equiv \partial_\mu e^\nu_a - \omega_{\mu ab}e^{\nu b}, \qquad e = \sqrt{-g} \tag{8}$$

and Δ is a divergence term.

The local Lorentz transformation of tetrads takes the following form:

$$e^a_\mu(x) \rightarrow e'^a_\mu = \Lambda^a{}_b(x)e^\mu(x),$$
$$\eta_{ab}\Lambda^a{}_c(x)\Lambda^b{}_d(x) = \eta_{cd}. \tag{9}$$

It is required that $\mathcal{L}_m$ be invariant under (9) and $\mathcal{L}_g$ is obviously invariant. Thus $\mathcal{L}$ is invariant under the transformation (9), i.e.,

$$[\mathcal{L}]_{e^\nu_a}\delta e^\nu_a + [\mathcal{L}]_{\phi^A}\delta\phi^A + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu e^\nu_a}\delta e^\nu_a + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^A}\delta\phi^A\right) = 0 \tag{10}$$

where $[\mathcal{L}]_{e^\nu_a}$ and $[\mathcal{L}]_{\phi^A}$ are the Euler expressions defined as

$$[\mathcal{L}]_{e^\nu_a} = \frac{\partial\mathcal{L}}{\partial e^\nu_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu e^\nu_a}, \tag{11}$$

$$[\mathcal{L}]_{\phi^A} = \frac{\partial\mathcal{L}}{\partial\phi^A} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^A}. \tag{12}$$

Using the Einstein equation $[\mathcal{L}]_{e^\mu_a} = 0$, i.e.,

$$[\mathcal{L}_g]_{e^\mu_a} + [\mathcal{L}_m]_{e^\mu_a} = 0,$$

and the matter equation of motion

$$[\mathcal{L}]_{\phi^A} = 0, \tag{13}$$

by (10) we obtain the following:

$$\partial_\mu\Big(\frac{\partial\mathcal{L}_g}{\partial\partial_\mu e^\nu_a}\delta e^\nu_a\Big) + \partial_\mu\Big(\frac{\partial\mathcal{L}_m}{\partial\partial_\mu e^\nu_a}\delta e^\nu_a + \frac{\partial\mathcal{L}_m}{\partial\partial_\mu\phi^A}\delta\phi^A\Big) = 0, \tag{14}$$

where we have used the fact that only $\mathcal{L}_m$ contains the matter field $\phi_A (A = 1, ..., N)$. Consider an infinitesimal local Lorentz transformation

$$\Lambda^a{}_b(x) = \delta^a{}_b + \alpha^a{}_b(x), \quad \alpha_{ab} = -\alpha_{ba}, \tag{15}$$

and from (9) we have

$$\delta e^\mu_a(x) = \alpha_{ab}(x)e^{\mu b}(x). \tag{16}$$

Suppose that $\mathcal{L}_m$ takes the form

$$\mathcal{L}_m = \mathcal{L}_m(e^\mu_a, \phi^A, D_\mu\phi^A), \tag{17}$$

ϕ^A's belonging to some representation of the Lorentz group with the generators I_{ab} $(a, = 0, 1, 2, 3)$, $I_{ab} = -I_{ba}$; D_μ is the covariant derivative

$$D_\mu\phi^A = \partial_\mu\phi^A - \frac{1}{2}\omega_{\mu ab}(I^{ab})^A{}_B\phi^B. \tag{18}$$

Then under the transformation (9), ϕ^A transforms as

$$\phi^A(x) \rightarrow \phi'^A(x) = [D(\alpha)]^A{}_B\phi^B(x). \tag{19}$$

$D(\alpha)$ can be linearized near the identity when α_{ab} are infinitesimal:

$$[D(\alpha)]^A{}_B = \delta^A{}_B + \frac{1}{2}(I_{ab})^A{}_B\alpha^{ab}(x). \tag{20}$$

Thus under the transformation (9), ϕ^A varies as

$$\delta\phi^A(x) = \frac{1}{2}(I_{ab})^A{}_B\phi^B(x)\alpha^{ab}(x). \tag{21}$$

Now we introduce J^μ_{ab} such that

$$eJ^\mu_{ab}\alpha^{ab} = \frac{3}{c}\Bigg[\frac{\partial\mathcal{L}_\omega}{\partial\partial_\mu e^{a\nu}}e^\nu_b\alpha^{ab} + \frac{\partial\mathcal{L}_m}{\partial\partial_\mu e^{a\nu}}e^\nu_b\alpha^{ab} + \frac{\partial\mathcal{L}_m}{\partial\partial_\mu\phi^A}\tfrac{1}{2}(I_{ab})^A{}_B\alpha^{ab}\phi^B\Bigg] \tag{22}$$

Then (14) can be written in the form

$$\partial_\mu(eJ^\mu_{ab}\alpha^{ab}) - \frac{3c^3}{16\pi G}\partial_\mu\left(\frac{\partial\Delta}{\partial\partial_\mu e^\nu_a}\delta e^\nu_a\right) = 0. \tag{23}$$

From (7) one can get

$$\frac{\partial\Delta}{\partial\partial_\lambda e^\mu_l}e^{m\mu}\alpha_{lm} = \alpha^{lm}\partial_\mu(eV^{\mu\lambda}_{lm}) \tag{24}$$

where

$$V^{\mu\lambda}_{lm} = e^\mu_l e^\lambda_m - e^\mu_m e^\lambda_l. \tag{25}$$

Substituting (24) into (23), we obtain

$$\partial_\mu(eJ^\mu_{ab}\alpha^{ab}) - \frac{3c^3}{16\pi G}\partial_\mu[\alpha^{ab}\partial_\nu(eV^{\nu\mu}_{ab})] = 0, \tag{26}$$

i. e.,

$$\partial_\mu(eJ^\mu_{ab})\alpha^{ab} + \Big[eJ^\mu_{ab} - \frac{3c^3}{16\pi G}\partial_\nu(eV^{\nu\mu}_{ab})\Big]\partial_\mu\alpha^{ab} = 0. \tag{27}$$

Since α_{ab} and $\partial_\mu\alpha^{ab}$ are mutually independent, we must have

$$\partial_\mu(eJ^\mu_{ab}) = 0 \tag{28}$$

$$J^\mu_{ab} = \frac{3c^3}{16\pi G}\nabla_\lambda V^{\lambda\mu}_{ab}, \tag{29}$$

or

$$J^{\mu}_{ab} = \frac{3c^3}{16\pi G}(\omega_a e^{\mu}_b + \omega_{ab} e^{\mu}_c - \omega_b e^{\mu}_a - \omega_{ba}{}^c e^{\mu}_c). \quad (30)$$

Since $V^{\nu\mu}_{ab}$ is an antisymmetric tensor with respect to the indices μ and ν , this means that J^{μ}_{ab} is conserved identically. As usual , we call $V^{\nu\mu}_{ab}$ superpotentials. Since the current J^{μ}_{ab} is derived from the local Lorentz invariance of the total Lagrangian, it can be interpreted as the angular momentum tensor density of the gravity-matter system. From (25) and (29) we see that the current J^{μ}_{ab} of gravity-matter system is entirely determined by the tetrad, quite similarly to the theory of the energy-momentum conservation law in general relativity [2]. The reason is that all informations about the state of motion of the whole gravity-matter system is contained in the tetrad due to the Einstein equations.

For a globally hyperbolic Riemannian manifold M, there exist Cauchy surfaces Σ_t foliating M. We choose a submanifold D of M joining any two Cauchy surfaces Σ_{t_1} and Σ_{t_2}, so the boundary ∂D of D consists of three parts Σ_{t_1}, Σ_{t_2} and A which is at spatial infinity. For an isolated system, the space-time should be asymptotically flat at spatial infinity, so the tetrad has the following asymptotic behavior [11]:

$$\lim_{r\to\infty}(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) = 0. \quad (31)$$

Since

$$eV^{\rho\sigma}_{cd} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}e^a_\mu e^b_\nu, \quad (32)$$

we have

$$\lim_{r\to\infty}\nabla_\sigma V^{\rho\sigma}_{cd} = 0. \quad (33)$$

Thus by the identity

$$\int_D \nabla_\mu J^{\mu}_{ab} e d^4x = 0 \quad (34)$$

we get the conservative angular momentum J_{ab}

$$J_{ab} = \int_{\Sigma_t} J^{\mu}_{ab} e d\Sigma_\mu \quad (35)$$

where $ed\Sigma_\mu$ is the covariant surface element of the Cauchy surface and Σ_t, $d\Sigma_\mu = (1/3!)\epsilon_{\mu\nu\alpha\beta}dx^\nu \wedge dx^\alpha \wedge dx^\beta$. The expression (35) for J_{ab} can be simplified using the Gauss theorem

$$J_{ab} = \frac{3c^3}{16\pi G}\int_{\partial\Sigma_t} eV^{i0}_{ab} ds_i \quad (36)$$

where $ds_i = (1/2!)\epsilon_{0ijk}dx^j \wedge dx^k$, $i, j, k = 1, 2, 3$ and $\partial\Sigma_t$ is the boundary of Σ_t which is at spatial infinity. We also have the following expression:

$$J_{ab} = \frac{3c^3}{32\pi G}\int_{\partial\Sigma_t} eV^{\mu\nu}_{ab} ds_{\nu\mu} \quad (37)$$

where $ds_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}dx^\alpha \wedge dx^\beta$. If we introduce

$$\tilde{J}^{ab} = \frac{3c^3}{16\pi G}\int_{\partial\Sigma_t} \epsilon^{\mu 0\alpha\beta} e^a_\alpha e^b_\beta ds_\mu, \quad (38)$$

then

$$J_{ab} = \frac{1}{2!}\epsilon_{abcd}\tilde{J}^{cd} \quad (39)$$

It is seen from (38) that the total angular momentum is entirely determined by the asymptotic behavior of the tetrad up to the order $1/r^2$ at spatial infinity.

Now let us discuss the gravitational radiation of angular momentum. Consider a subspace σ of Σ_t, choosing σ to be time-independent. The identity

$$\int_\sigma \nabla_\mu J^{\mu}_{ab} e d^3x = 0 \quad (40)$$

implies

$$\int_\sigma \partial_0(J^0_{ab}e)d^3x = -\int_\sigma \partial_i(J^i_{ab}e)d^3x = -\int_{\partial\sigma} J^i_{ab} e ds_i, \quad (41)$$

i. e.,

$$\frac{\partial}{\partial t}J_{ab}(\sigma) = -c\int_{\partial\sigma} J^i_{ab} e ds_i. \quad (42)$$

This states that the AM decrease of the system contained in σ is equal to the AM flux through $\partial\sigma$, describing the gravitational radiation of AM.

Next we show that J_{ab} is gauge-covariant. Note that any physical solution of an isolated system must be asymptotically flat, i.e., must satisfy (31). Consider another choice of the tetrad $e'^a_\mu(x)$ which also satisfies (31) and is related to e^a_μ by the local Lorentz transformation

$$e'^a_\mu = \Lambda^a{}_b(x)e^b_\mu(x)$$

Therefore at the spatial infinity $\partial\Sigma$ there exist inertial coordinate frames X^a and X'^a such that at $\partial\Sigma$ the following relations hold:

$$e^a_\mu(x) = \frac{\partial X^a(x)}{\partial x^\mu}, \quad (43)$$

$$e'^a_\mu(x) = \frac{\partial X'^a(x)}{\partial x^\mu}. \quad (44)$$

According to special relativity [4], any two inertial coordinate frames are connected by a Lorentz transformation. Hence we have

$$X'_a(x) = B_a + A_a{}^b X_b(x) \quad (45)$$

where B_a and $A_a{}^b$ are constants. On the other hand, from

$$\Lambda^a{}_b(x)e^b_\mu(x) = \frac{\partial X'^a(x)}{\partial x^\mu} \quad \text{and} \quad e^\mu_b(x) = \frac{\partial x^\mu}{\partial X^b}$$

it follows

$$\Lambda^a{}_b = \frac{\partial X'^a}{\partial X^b}, \quad (46)$$

so we have

$$\Lambda_{ab} = A_{ab} = \text{const}. \quad (47)$$

Thus it is easy to show that

$$J'_{ab} = A_a{}^c A_b{}^d J_{cd}.$$

3. Kerr space-time and many-body systems

First we would like to discuss the Kerr space-time which has the following asymptotic behavior at spatial infinity [12]:

$$ds^2 = (1-\frac{2m}{r})c^2dt^2 - (1+\frac{2m}{r})(dx^2+dy^2+dz^2) + 4mac\frac{xdydt - ydxdt}{r^3}$$

where $m = GM/c^2$, $ma = GJ/c^3$, M is the mass of the spherical source body and J is its spin angular momentum. We can get the following tetrad satisfying (31):

$$\begin{aligned}
e_{00} &= c\sqrt{1-\frac{2m}{r}}, \qquad e_{33} = \sqrt{1+\frac{2m}{r}},\\
e_{01} &= -\frac{2may}{r^3\sqrt{1-2m/r}}, \qquad e_{02} = \frac{2max}{r^3\sqrt{1-2m/r}},\\
e_{11} &= \sqrt{1+\frac{2m}{r}+\frac{4m^2a^2y^2}{r^6(1-2m/r)}},\\
e_{22} &= \sqrt{1+\frac{2m}{r}+\frac{4m^2a^2x^2}{r^6(1-2m/r)}}.
\end{aligned}$$

To calculate J_{ab}, we need to know only the behavior of $e_{a\mu}$ up to $1/r^2$ when $r\to\infty$:

$$\begin{aligned}
e_{00} &= c(1-\frac{m}{r}-\frac{m^2}{r^2}),\\
e_{01} &= -\frac{2may}{r^3}, \qquad e_{02} = \frac{2max}{r^3},\\
e_{11} &= e_{22} = e_{33} = 1+\frac{m}{r}-\frac{m^2}{r^2}.
\end{aligned} \tag{48}$$

From (48) and (38) we get

$$\begin{aligned}
\tilde{J}^{03} &= \frac{3c^3}{16\pi G}\int_{r\to\infty}\epsilon^{\mu 0\alpha\beta}e^0_\alpha e^3_\beta x_\mu r\, d\Omega\\
&= \frac{3c^3}{16\pi G}\int_{r\to\infty} 2ma\sin^3\theta d\varphi = \frac{mac^3}{G} = J,\\
\tilde{J}^{01} &= \tilde{J}^{02} = \tilde{J}^{12} = \tilde{J}^{13} = \tilde{J}^{23} = 0,
\end{aligned}$$

that is,

$$J_{12} = J, \qquad J_{01} = J_{02} = J_{03} = J_{13} = J_{23} = 0.$$

This result convinces us that J_{ab} is indeed the angular momentum of the Kerr gravity-matter system.

The discussion of gravititional radiation of AM is quite simple. Since the tetrad is time-independent,

$$\partial_\mu(eJ^\mu_{ab}) = \partial_i(eJ^i) = 0.$$

The Gauss theorem with the integral region $\Sigma_t - \sigma$ gives

$$\int_{\partial\sigma} J^i_{ab} e ds_i = \int_{\partial\Sigma_t} J^i_{ab} e ds_i \tag{49}$$

where σ contains the spinning body. To study the AM radiation, it is necessary to know only the asymptotic behavior of $J^i_{ab}e$ at spatial infinity up to $1/r^2$. From (25), (29) and (32) we have

$$eJ^\mu_{ab} = \frac{3c^3}{32\pi G}\epsilon^{\nu\mu\alpha\beta}\epsilon_{abcd}\partial_\nu(e^c_\alpha e^d_\beta) \tag{50}$$

which shows that only the behavior of $e_{a\mu}$ up to $1/r$ is relevant and Eq. (48) gives

$$\begin{aligned}
e_{00} &= c(1-m/r),\\
e_{01} &= e_{02} = 0,\\
e_{11} &= e_{22} = e_{33} = 1+m/r.
\end{aligned} \tag{51}$$

Subtituting (51) into (50), we get

$$\begin{aligned}
eJ^1_{12} &= -eJ^3_{23} = \frac{3m^2c^4}{16\pi G}\frac{y}{r^4},\\
eJ^2_{12} &= eJ^3_{13} = -\frac{3m^2c^4}{16\pi G}\frac{x}{r^4},\\
eJ^1_{13} &= eJ^2_{23} = \frac{3m^2c^4}{16\pi G}\frac{z}{r^4}.
\end{aligned}$$

Since the surface element is proportional to r^2 at spatial infinity, (49) implies

$$\frac{\partial}{\partial t}J_{ab}(\sigma) = -c\int_{\partial\sigma} J^i_{ab}e ds_i = -c\int_{\partial\Sigma_t} J^i_{ab}e ds_i = 0. \tag{52}$$

So there is no gravitational radiation of angular momentum from the Kerr system, in agreement with the results of other theories [11].

Next we will discuss a many-body system. The corresponding Einstein equations have approximate solutions [4]. Suppose that the mass center is at rest, then we have the following approximate solution to the order $1/r^2$:

$$\bar{g}^{00} = \frac{1}{c}\Big(1+\frac{4GM}{C^2r}+\frac{7G^2M^2}{c^4r^2}\Big), \qquad \bar{g}^{0i} = \frac{2Gx^jM_{ji}}{c^3r^3},$$
$$\bar{g}^{ik} = -c\delta_{ik} + \frac{G^2M^2x^ix^k}{c^3r^4}$$

where M is the total mass of the bodies, M_{ij} is their total mechanical orbital angular momentum tensor and $\bar{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$. Then $g_{\mu\nu}$ are

$$g_{00} = c(1-\frac{2GM}{c^2r}+\frac{2G^2M^2}{c^4r^2}), \qquad g_{0i} = \frac{2Gx^jM_{ji}}{c^2r^3},$$
$$g_{ik} = \delta_{ik}(-1-\frac{2GM}{c^2r}+\frac{G^2M^2}{c^4r^2}) - \frac{G^2M^2}{c^4r^4}x^ix^k. \tag{53}$$

From (53) we get the following tetrad up to $1/r^2$:

$$\begin{aligned}
e_{00} &= c(1-\frac{GM}{c^2r}+\frac{G^2M^2}{2c^4r^2}), e_{0i} = \frac{2Gx^jM^{ji}}{c^2r^3},\\
e_{ij} &= \frac{G^2M^2}{c^4r^4}x_ix_j \qquad (i\neq j),\\
e_{11} &= e_{22} = e_{33} = 1+\frac{GM}{c^2r}+\frac{G^2M^2}{2c^4r^4}.
\end{aligned} \tag{54}$$

Substituting (53) into (37), we get the total angular momentum of the the many-body system:

$$\begin{aligned}
&J_{12} = M_{12}, \qquad J_{13} = M_{13},\\
&J_{23} = M_{23}, \qquad J_{01} = J_{02} = J_{03} = 0.
\end{aligned} \tag{55}$$

Thus we see that our AM conservation law is reasonable.

4. Quadrupole radiation of angular momentum

As in the derivation of the gravitational radiation of energy, we consider the weak field at a large distance from the source bodies. The familiar expansion of the metric has the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$. To the first order of $h_{\mu\nu}$, a solution to the Einstein equations

$$\Box\phi_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{56}$$

with the coordinate condition

$$\partial_\mu\phi^\mu_\nu = 0, \tag{57}$$

where $\phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, $h = h^\mu_\mu$ has the well-known retarded potential form

$$\phi_{\mu\nu}(\mathbf{r}, t) = -\frac{4G}{c^4}\int\frac{T_{\mu\nu}(\mathbf{r}',\ t - |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'. \tag{58}$$

For large $r = |\mathbf{r}|$, we have approximately

$$\phi_{\mu\nu}(\mathbf{r}, t) = -\frac{4G}{c^4 r}\int T_{\mu\nu}(\mathbf{r}', t - \frac{r}{c})d^3\mathbf{r}'. \tag{59}$$

The spatial components ϕ_{ij} $(i, j, k, \ldots = 1, 2, 3)$ can be further expressed as

$$\phi_{ij}(\mathbf{r}, t) = -\frac{2G}{c^4 r}\frac{\partial^2}{\partial t^2}\int\rho x'_i x'_j d^3\mathbf{r}' \tag{60}$$

For later use, we now derive some useful relations for the derivatives of $\phi_{\mu\nu}$. To the first order in $1/r$, i.e., the first order in $h_{\mu\nu}$, we have

$$\partial_i\phi_{\mu\nu} = \frac{1}{c}\dot\phi_{\mu\nu}n_i, \qquad n_i = \frac{x^i}{r} \tag{61}$$

where a dots stand for a time derivative. From the coordinate condition (57) we have

$$\partial^0\phi_{0\nu} = -\partial^i\phi_{i\nu} = -\frac{1}{c}\dot\phi_{i\nu}n^i. \tag{62}$$

Eqs. (61) and (62) taken together give

$$\partial^0\phi_{00} = -\frac{1}{c}\dot\phi_{i0}n^i \tag{63}$$

Thus we have

$$\begin{aligned}
&\partial^0\phi_{00} = \frac{1}{c}\dot\phi_{ij}n^in^j, \quad \phi_{00} = \frac{1}{c}\dot\phi_{jk}n^jn^kn^i,\\
&\partial^i\phi_{oj} = -\frac{1}{c}\dot\phi_{jk}n^in^k, \quad \partial^0\phi_{0j} = -\frac{1}{c}\dot\phi_{jk}n^k,\\
&\partial^i\phi_{jk} = \frac{1}{c}\dot\phi_{jk}n^i.
\end{aligned} \tag{64}$$

All these relations hold to the first order.

The expansion for the tetrad is supposed to be

$$e_{a\mu} = \eta_{a\mu} + \frac{1}{2}f_{\mu a}, \quad e^\mu_a = \delta^\mu_a - \frac{1}{2}f_a{}^\mu. \tag{65}$$

A relation between $h_{\mu\nu}$ and $f_{\mu a}$ is easily obtained:

$$h_{\mu\nu} = \frac{1}{2}(f_{\mu\nu} + f_{\nu\mu}). \tag{66}$$

The first order of the tetrad gauge condition $\nabla_\mu\omega^\mu{}_{ab} = 0$ ensures that

$$f_{\mu\nu} = f_{\nu\mu}. \tag{67}$$

Using the relations

$$\omega_{abc} - \omega_{bac} = e^\mu_b e^\nu_a(\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}), \tag{68}$$

$$\omega_a = -\eta^{bc}e^\mu_b e^\nu_a(\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}). \tag{69}$$

It can be directly calculated that to the second order

$$\begin{aligned}
\omega_{ab}{}^\mu - \omega_{ba}{}^\mu &= \tfrac{1}{2}\big\{\partial_b h_a{}^\mu - \partial_a h_b{}^\mu\\
&- \tfrac{1}{2}\big[h^\beta{}_a(\partial_b h_\beta{}^\mu - \partial_\beta h_b{}^\mu) + h^\alpha{}_b(\partial_\alpha h_a{}^\mu - \partial_a h_\alpha{}^\mu)\\
&+ h^{\mu c}(\partial_b h_{ac} - \partial_a h_{bc})\big]\big\};
\end{aligned} \tag{70}$$

$$\begin{aligned}
\omega_a e^\mu_b &= \tfrac{1}{2}\delta^\mu_b\big\{\partial_a h - \partial^c h_{ac} - \tfrac{1}{2}\big[h^d_a(\partial_d h - \partial^c h_{dc})\\
&+ h^{dc}(\partial_a h_{dc} - \partial_d h_{ac})\big]\big\} - \tfrac{1}{4}h^\mu_b(\partial_a h - \partial_c h^c_a).
\end{aligned} \tag{71}$$

Since the radiation should be proportional to G, it is evident from (30) that only the second-order part of J^i_{ab} contributes to the radiation since it contains G^2, while the first-order part contains G, to be cancelled by G in the common constant factor. A practical evaluation may also prove this point. Hence we need only the second-order part of J^i_{ab}, which we call K^i_{ab}:

$$\begin{aligned}
K^i_{ab} = -\frac{3G^3}{64\pi G}&\Big\{\phi^c_a(\partial_b\phi^i_c - \partial_c\phi^i_b) + \tfrac{1}{2}\phi^c_a\delta^i_b\partial_c\phi\\
&+ \tfrac{1}{2}\phi\partial_a\phi^i_b - \phi^i_c\partial_b\phi^c_a + \delta^i_b\phi^d_c(\partial_a\phi^c_d - \partial_d\phi^c_a)\\
&- \tfrac{1}{2}\phi^i_b\partial_a\phi\Big\} - (a \leftrightarrow b).
\end{aligned} \tag{72}$$

Using the relations (64), we can obtain the following:

$$\begin{aligned}
K^i_{jk}n_i = -\frac{3c^2}{64\pi G}&\Big\{\phi^l_j(\dot\phi^i_l n_k - \dot\phi^i_k n_l)n_i\\
&+ \phi^0_j(-\dot\phi^i_l n^l n_k - \dot\phi^i_k)n_i + \tfrac{1}{2}\phi^l_j n_k(\dot\phi_{pq}n^pn^qn_l + \dot\phi^p_p n_l)\\
&+ \tfrac{1}{2}\phi\dot\phi^i_k n_i n_j - \phi^i_l\dot\phi^l_j n_k n_i + \phi^i_0\dot\phi_{jp}n^pn_kn_i\\
&+ n_k\big[\phi^0_0(\dot\phi_{pq}n^pn^qn_j + \dot\phi_{jl}n^l) + \phi^l_0(-\dot\phi_{lp}n^pn_j\\
&+ \dot\phi_{jp}n^pn_l) + \phi^0_l(-\dot\phi^l_p n^p n_j - \dot\phi^l_j) + \phi^q_p(\dot\phi^p_q n_j - \dot\phi^p_j n_q)\big]\\
&- \tfrac{1}{2}\phi^i_k n_i(\dot\phi_{pq}n^pn^qn_j + \dot\phi n_j)\Big\} - (j \leftrightarrow k).
\end{aligned} \tag{73}$$

Making use of the integrals

$$\frac{1}{4\pi}\int n_i n_j d\Omega = -\frac{1}{3}\eta_{ij}; \tag{74}$$

$$\frac{1}{4\pi}\int n_j n_k n_l n_m d\Omega = \frac{1}{15}(\eta_{jk}\eta_{lm} + \eta_{jl}\eta_{km} + \eta_{jm}\eta_{kl}), \tag{75}$$

we have

$$\frac{d}{dt}J_{mn} = \frac{2G}{45c^3}(\dddot{D}^l_m\ddot{D}_{ln} - \dddot{D}^l_n\ddot{D}_{lm}) \tag{76}$$

where D_{ij} is the usual quadrupole moment

$$D_{ij} = \int \rho(3x_i x_j - \eta_{ij} x^p x_p) d^3x, \tag{77}$$

or

$$\frac{d}{dt} M^l = \frac{1}{2} \epsilon^{lmn} \frac{d}{dt} J_{mn} = \frac{2G}{45c^3} \epsilon^{lmn} \dddot{D}^p_m \ddot{D}_{pn}. \tag{78}$$

This is exactly what is given in [10].

5. Discussion

To conclude this paper, we make some remarks. First. The conservative currents J^μ_{ab} apply to both open and closed systems, while the corresponding conservative angular momentum J_{ab} applies only to closed systems satisfying the boundary condition (31), which is similar to that used in Ashtekar's self-dual formalism of general relativity [13]. Since the current J^μ_{ab} depends on the tetrad, it is natural that it is not gauge covariant, and this seems inevitable for conservation laws in general relativity. However, for a closed system, the space-time at spatial infinity is, according to Eq. (31), flat, thus the conservative angular-momentum J_{ab} should be a non-scalar covariant object under Lorentz transformations (47) at spatial infinity, just as in special relativity where the angular momentum is a tensor rather than a scalar. To understand this, the key point is that in order to obtain J_{ab} one has to treat the whole system as a point at rest, so that each point of the space-time at spatial infinity belongs to the same Minkowski space-time. This means that in general relativity, for a closed system, the total angular momentum J_{ab} must be a Lorentz tensor like that in special relativity. On the other hand, the covariance of J_{ab} is very similar to that of the conservative energy-momentum in [2]. So it is quite comprehensible.

Second. One can obtain Eq. (28) immediately, starting directly from Eqs. (25) and (29). But they are based on the $SO(1,3)$ invariance of the system and the Noether theorem. So the quantity J_{ab} has a direct physical interpretation of the angular momentum. It is nothing more than a conservative quantity of the dynamical system of Einstein's gravity. If one starts simply from Eqs. (25) and (29), it is hard to interpret the corresponding J_{ab} physically. Furthermore, the conservative energy-momentum has superpotentials [2]. The existence of superpotentials for conservation laws in general relativity seems to be a very general feature (e.g., the Landau-Lifshitz definition [10], the Einstein-Tolman definition of energy-momentum [14], etc.). From the energy-momentum and AM conservation laws we came to know that the currents can also be obtained from the Noether theorem.

It is true that one can construct an arbitrary anti-symmetric tensor playing the role of a superpotential, but the resulting conservative quantities lack physical interpretation.

Third. In the metric formalism, the Lagrangian has the conventional form, Eq.(2). In the tetrad formalism, the Lagrangian can be reduced to $\mathcal{L}_\omega$ because Δ is a total divergence which does not affect the equations of motion. Some other authors also take $\mathcal{L}_\omega$ as a Lagrangian (e.g., [15]).

Fourth. The above two applications may be called simple but they are typical, since the second one contains many bodies and this is a very general case. A theoretical application of our theory is contained in our paper [16].

Last. In general, J_{ab} are geometric conserved quantities in pseudo-Riemannian geometry. From (12) and (13) we know that the currents do have physical meaning only when the tetrad satisfies the Einstein equations.

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