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NONEQUILIBRIUM STATES OF A SCALAR QUANTUM FIELD IN THE UNSTEADY UNIVERSE

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Keldysh's covariant technique has been constructed to describe nonequilibrium fluctuations of a self-interacting scalar field in curved space-time. The Dyson theorem is proved for the Green functions. A general expression is obtained for a mass operator in the self-consistent field approximation. A kinetic equation is derived for the distribution function for scalar field quasiparticles in the collision approximation. The quantum scalar field energy-momentum tensor is shown to be calculable with the aid of perturbation theory in curved space-time. The approach being developed allows one to solve the problem of regularizing the energy-momentum tensor for nonequilibrium states.

1. Introduction

Quantum cosmology was previously considered within the framework of the adiabatic theory allowing for only vacuum short-wave fluctuations of the matter field. But the actual Universe is nonstationary, general equations of state of the matter field are nonequilibrium, and the matter tensor should be determined by averaging over an arbitrary nonequilibrium state. To describe such processes, Keldysh's diagram technique [1] extended to the case of curved space should be used instead of Feynman's.

2. Covariant Liouville Equation

A covariant description of a self-interacting quantum scalar field is given by the Tomonaga-Schwinger equation [2]

$$\frac{\delta \mid \psi[t(x)] \rangle}{\delta t(x)} = \mathcal{H}_I(x) \mid \psi[t(x)] \rangle, \tag{1}$$

where $t(x)$ is a spacelike surface, the functional $\psi(x)$ is a field wave function, $\mathcal{H}_I$ is the scalar field self-interaction Hamiltonian density. For a derivative-free constraint we have $\mathcal{H}_I = -\mathcal{L}_I$, where $\mathcal{L}_I = -\lambda\varphi^4/4$ is the coupling Lagrangian. The field operators $\varphi(x)$ satisfy the commutation relations

$$[\varphi(x),\ \varphi(x')] = i\Delta(x, x') \tag{2}$$

and the Klein-Fock equations [3]

$$[\Box - m^2 + \xi R(x)]\varphi(x) = 0, \quad \Box := g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{3}$$

obtained by varying the free Lagrangian

$$\mathcal{L}_o = \tfrac{1}{2}\sqrt{-g(x)}[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - (m^2 + \xi R(x)\varphi^2(x))]. \tag{4}$$

The total Lagrangian is $\mathcal{L} = \mathcal{L}_o + \mathcal{L}_I$. In the Tomonaga-Schwinger representation, the system evolution is carried out by the wave function rather than the scalar field operators. A noncovariance of the description may be related here only to the arbitrariness of the representation of an evolution parameter t numbering the layers $t_o < t_1 < t_2 < \dots$. Contrary to the conventional delusion, in general relativity (GR) there is a primary time shift along normals to the levels of the harmonic function $t(x)$. Then the time-space-volume relation is normalized, and the positiveness of gravity manifests itself in a positive focusing of these volumes that may, in principle, play the role of an internal time. For $R_{00} \geq 0$, $T_{00} \geq |T_{0i}|$ (where $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the matter tensor) the absence of volume focusing (when the proper time coincides with the harmonic one but does not contract with respect to it) involves almost triviality of the Riemannian space-time curvature [3].

In our opinion, the transition from metric gravitational potentials to tetrad ones is a more consistent approach to the covariant description of quantum gravity. In the Tomonaga-Schwinger equation (1) the operator $\delta/\delta t(x)$ should designate a Lie derivative $e_0^\mu\partial_\mu$ along the 4-velocity of the observer e_0. In more precise terms, we take an arbitrary (e.g., extremal or corresponding to equidense matter, etc.) spacelike hypersurface and move equidistantly along e_0^μ. This equidistant displacement arises from a local displace-

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ment (bubble derivatives).

The transition from a coordinate base dx^μ, $\partial_\nu = \partial/\partial x^\nu$ of GR to a tetrad one $e^a_\mu dx^\mu$, $e^\nu_a \partial_\nu$ $(a, b$ are Lorentzian indices) is an attempt to conserve in GR what we had in special relativity (SR): orthogonality of the coordinate surfaces $dx^\mu = 0$ and normalization of the base $\|dx^\mu\| = 1$ (in SR, in an inertial frame of reference x^μ are not only the parameters numbering the world points, but invariant intervals of time and length). But in GR (contrary to SR) an orthonormalized frame of reference is local (this locality is equivalent to gravitation).

It may be argued that the introduction of an additional invariant structure (tetrads, or a (1+1+1+1) 4-metric splitting) is a translation of the problem of coordinate conditions in GR (local coordinates are taken as a basis of a tangential bundle of the space-time) into the language of tetrads. Do privileged tetrads exist in GR? The gravitational field itself is a tetrad field in this approach. It determines a proper tetrad of the Einstein tensor (the timelike Ricci canonical vector accompanies the sources of the gravitational field). If the space-time is Ricci-flat (vacuum), then the Weyl canonical Petrov tetrad may be considered to be privileged. Its timelike principal Riemannian vector accompanies the free part of the gravitational field described by the Weyl conformal curvature tensor. In gravitational fields of algebraically special types the canonical tetrad is determined ambiguously, but it does not matter. If we are interested in gauge rather than conditions, then the tetrad field with a constant dilation (best of all, with a zero one: $\nabla^\mu e_{a\mu} = 0$) may play the role of a privileged gauge. In this case the Hilbert gravitational Lagrangian density (the scalar curvature $R(x)$) differs from the quadratic Lagrangian by a constant giving the cosmological term (equal to zero for a dilation-free tetrad). Thus, the SR gauge conditions (inertial frames of reference) may be partly preserved in GR (dilation-free tetrads; canonical tetrads; semiharmonic tetrads for which $g_{00} + \det g_{ij} = g_{0i} = 0$). The ambiguity of choice of a privileged tetrad does not mean such an arbitrariness meant by Einstein when relating GR to the general principle of relativity. Einstein is right in that each tetrad giving a specified space-time curvature, is admissible. The transition to another observer should be considered to be an actual deformation of the tetrad field in order that the choice of the observer make an actual sense. The change of the tetrad potentials due to the change of the argument is reformulated as a change of forms themselves of the function-potentials with the argument being unchanged. The curvature may be therewith conserved, but the gravitational energy (Hamiltonian) and momentum may be unconserved: the conservation laws should be formulated only for conserving structures (fixed by the gauge or accompanying conditions). In this case the conserving integral quantities may be nonlocal invariants.

Based on the Tomonaga-Schwinger equation, one can construct the Liouville covariant equation for the density matrix of a coupled scalar field in curved space

$$\rho[t(x)] := \overline{\mid \psi[t(x)] >< \psi[t(x)] \mid}$$
$$i\delta\rho[t(x)]/\delta t(x) = \mathcal{H}_I(x)\rho[t(x)] - \rho[t(x)]\mathcal{H}_I(x). \quad (5)$$

Here the overbar means averaging over an arbitrary nonequilibrium statistical ensemble. The solution of equation (5) is written in the form

$$\rho[t(x)] = S[t, t_0]\rho[t_0]S^+[t, t_0], \quad (6)$$

where the evolution operator S satisfies the equation

$$i\delta S[t, t_0]/\delta t(x) = \mathcal{H}_I(x)S[t],$$
$$S[t_0, t_0] := 1, \quad (7)$$

$t_0(x)$ is an initial surface. The solution of equation (7) is of the form

$$S[t, t_0] = T \exp[-i \int_{t_0}^{t} \mathcal{H}_I(x')d^4x], \quad (8)$$

where T is the chronological ordering operator allowing for the succession order of layers t_i.

3. Correlation functions of a scalar field

An arbitrary nonequilibrium state of the field is represented by a sequence of distribution functions

$$f_1(1;1) \equiv f_1(x_1, x'_1) = \langle N\varphi(x'_1)\varphi(x_1)\rangle_0,$$
$$f_{12}(1,2;1,2) \equiv f_{12}(x_1, x_2, x'_1, x'_2)$$
$$= \langle N\varphi(x'_1)\varphi(x'_2)\varphi(x_2)\varphi(x_1)\rangle_0 \quad (9)$$

where N is the normal ordering operator and the angular brackets mean an ensemble with the initial density matrix

$$< A_0 > := \mathrm{Sp}\, A\rho\,[t_0]. \quad (10)$$

The distribution functions (9) are closely related to the field correlation functions:

$$f_{12}(1,2;1,2) = f_1(1,1)f_2(2,2) + f_1(2,1)f_2(1,2)$$
$$+ g'_{12}(1,2;1,2) + g'_{21}(2,1;1,2),$$
$$f_{123} = \gamma_{123}(f_1f_2f_3 + f_1g'_{23} + f_2g'_{13} + f_3g'_{21} + g'_{123}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\gamma_{123} := (1 + p_{12})(1 + p_{13} + p_{23}), \ \ldots \quad (11)$$

Here p_{12} is the permutation operator acting on the first pair of function arguments.

A possibility of constructing a diagram technique for nonequilibrium states is provided by the theorem on attenuation of initial correlations [4].

Theorem. If the spectrum of a system is continuous and the correlation functions $g'_{1\dots n}$ are integrable in the momentum representation, then

$$\lim_{t\to-\infty} g'_{1\dots n} = 0. \tag{12}$$

The integrability condition for the correlation functions means the absence of a long-range order in the system, i.e. the absence of crystallization or Bose condensation of the φ field. For simplicity assume that there is a flat domain at $t_0 \to -\infty$. The condition $t_0 \to -\infty$ means that the times $t - t_0$ considerably exceed the relaxation time of short-wave fluctuations, but not that of a transition to the initial state of the Universe. Therewith the entropy is determined in terms of the distribution function for quasiparticles, and the gravitational field is a purely dynamic object in the classical approximation and does not contribute to the entropy.

To prove the theorem on attenuation of initial correlations, we take into account that the correlation functions in the momentum representation in the in-region are solutions to the free Klein-Fock equation. According to the Riemann-Lebesgue theorem, the integrals of such functions with arbitrary continuous multipliers tend to zero. The attenuation of initial functions means a phase mixing in the system. In a finite system (the Universe with a finite spatial volume V) the initial correlations can be undamped and can determine an initial level of noise whose amplitude does not exceed $V^{-1/2}$ for a stable evolution. In an unstable case the initial level of noise may increase, resulting in a turbulent state of the system. In the present paper we assume that the initial correlations attenuate. The disappearance of initial correlations means that for $t \to \infty$ the interacting field density matrix $\rho[t_0]$ coincides with the one $\rho_0[t_0]$ diagonal in the representation of quantum numbers for free field operators $\varphi(x)$.

From the above theorem follows the Wick theorem for free field operators:

$$\begin{aligned}\langle N\varphi(x_1)...\varphi(x_n)\rangle_0 &= \mathrm{Sp}\,\varphi(x_1)...\varphi(x_n)\rho_0(t_0)\\ &= \sum_{i_1\dots i_n} \Pi_{i,j}\langle N\varphi(x_i)\varphi(x_j)\rangle.\end{aligned} \tag{13}$$

The summation is here conducted over products of various pairwise contractions. The chronological products of any Heisenberg operators $TA(t)B(t')$, being functionals of $\varphi(x)$ averaged over an arbitrary nonequilibrium state, may be written in the form:

$$\begin{aligned}&\langle TA(t)B(t')\rangle\\ &= \mathrm{Sp}\,\rho_0[t_0]S^+[t_m,t_0]\,TA_0(t)B_0(t')S[t_m,t_0]\\ &\equiv \langle S^+[t_m,t_0]\,TA_0(t)B_0(t')S[t_m,t_0]\rangle_0.\end{aligned} \tag{14}$$

Here A_0, B_0 are the functionals for free field $\varphi_0(x)$ and t_m corresponds to the level hypersurface for $\max(t,t')$.

Eq. (14) is conveniently written in the form

$$\langle TA(t)B(t')\rangle = \langle T_cA_0(t)B_0(t')S_c[t_m,t_0]\rangle \tag{15}$$

where T_c is chronological ordering along the Keldysh contour going from the initial surface $t_0(x)$ in the in-region to the surface $t_m(x)$ and back. Denote the lower branch going from the past to the future by the minus sign, the back one by the plus sign; S_c is the evolution operator along the Keldysh contour. The use of the Keldysh contour permits usual averages to be written in the form of chronological products

$$\langle A(t)B(t')\rangle = \langle T_cA(t_+)B(t'_-)S_c[t_m,t_0]\rangle, \tag{16}$$

if the time t is considered to be on the "plus" branch and the time t' on the "minus" branch of the contour.

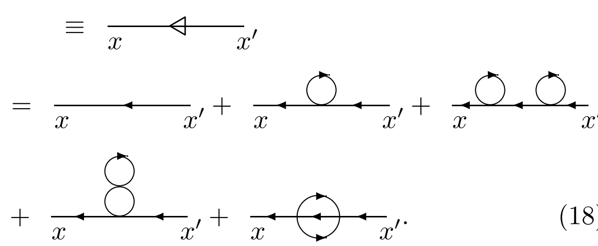
4. Keldysh's covariant diagram technique in a curved space

Keldysh's diagram technique is based on introducing the matrix Green functions

$$\begin{aligned}iG(x,x') &= \langle T_c\varphi(x)\varphi(x')\rangle\\ &= \begin{pmatrix} iG^{--} & iG^{-+}\\ iG^{+-} & iG^{++}\end{pmatrix} = iG^{SS'}(x,x')\end{aligned} \tag{17}$$

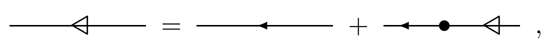
where $S,S'{=}\pm$, $iG^{--}(x,x') = \langle T\varphi(x)\varphi(x')\rangle$ is the causal Green function, $iG^{-+}(x,x') = \langle\varphi(x')\varphi(x)\rangle$ and $iG^{+-}(x,x') = \langle\varphi(x)\varphi(x')\rangle$ are the Whightman functions averaged over a nonequilibrium statistic ensemble, $iG^{++}(x,x') = \langle\tilde{T}\varphi(x)\varphi(x')\rangle$ is the anticausal Green function. Using Eq. 15 and expanding the evolution operator in a series, we obtain the diagram expansion of the Green function (17):

$$iG(x,x') = \langle T_c\varphi_0(x)\varphi_0(x')S_c[t_m,t_0]\rangle_0$$



(18)

Here a continious line denotes the free field matrix Green function $iG(x,x')$, a point denotes the vertex $-\lambda\sqrt{-g}$ ($g = \det g_{\mu\nu}$). At each vertex an integration over the space-time is performed from t_0 to t_m and back along the Keldysh contour. There are no disconnected graphs. Connected graphs are grouped into reducible and irreducible ones, the latter cannot be dissected to pieces by cutting a line. Summation of the diagram (18) results in the matrix integral Dyson equation



(19)

$$\bullet := -i\Sigma(x,x') = -i\Sigma^{SS'}(x,x'). \tag{20}$$

An analytic form of (19) reads:

$$G(x,x') = G_0(x,x') + \int\limits_c G_0(x,y)\Sigma(y,z)G(z,x')d^4yd^4z \qquad (21)$$

where $\int\limits_c$ is taken over the Keldysh contour.

In the sign S-representation we have

$$G^{SS'}(x,x') = G_0^{SS'}(x,x') + \sum_{S_1,S_2}\int\limits_c G_0^{SS_1}(x,y)\Sigma^{S_1S_2}(y,z)G^{S_2S'}(z,x')d^4yd^4z. \qquad (22)$$

5. Relation of Green functions to the Einstein equations

The scalar field energy-momentum tensor $< T_{\mu\nu} >$, averaged over an arbitrary nonequilibrium state, is related to space-time geometry by the Einstein equation with the gravitational constant γ:

$$G^E_{\mu\nu} \;=\; R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\gamma < T_{\mu\nu} >$$
$$< T_{\mu\nu} > \;=\; d_{\mu\nu}G^{-+}(x,x')\,|_{x=x'}\;. \qquad (23)$$

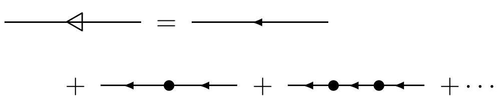
Here $d_{\mu\nu}$ is a differential operator acting on x,x' (henceforth $x \to x'$). At $\gamma = 0$ we take a flat space-time $G_0 = G_0^{(0)}$, $g_{\mu\nu} = \eta_{\mu\nu}$. Due to averaging over nonvacuum states, $< T_{\mu\nu} >_0^{(0)} \neq 0$, which implies $G^E_{\mu\nu} \neq 0$. As a result, we have an iteration:

$$G^E_{\mu\nu} \neq 0 \Rightarrow g^{(1)}_{\mu\nu} \neq \eta_{\mu\nu} \Rightarrow G_0^{(1)} \Rightarrow g^{(2)}_{\mu\nu} \Rightarrow G_0^{(2)} \Rightarrow ... \qquad (24)$$

In a flat world the tensor $< T_{\mu\nu} >_0^{(0)}$ is regularized, therefore the regularization problem does not arise in this approach.

6. Secular divergences of the Keldysh diagram technique

Nonequilibrium states of the quantum scalar field in the Keldysh technique are described by the matrix Green function (GF) (17). This GF may be expanded in a series of the perturbation theory in coupling (18). All insertions in (18) may be gathered in a mass operator (20). As a result, for the GF we obtain the diagram series



(25)

An element of the type contains the integral

$$\sum_{S_1S_2}\int\limits_{t_0}^{t_m}\int\limits_{t_0}^{t_m} G_0^{SS_1}(t-t_1)\Sigma^{S_1S_2}(t_1-t_2)G_0^{S_2S'}(t_2-t')dt_1dt_2 \qquad (26)$$

where $t_m = \max(t,t')$; the matrix elements $G_0^{SS'}(t-t')$ contain products $n_p\theta(t-t')$ and $(1-n_p)\theta(t-t')$ where n_p are occupation numbers at the initial instant t_0, the function $\theta(t)$ equals zero for $t \leq 0$ and unity for $t > 0$. The integral in t_1 can be shown to diverge as $(t_m - t_0)$ in the terms containing a product of the mass operator Σ by the occupation number of the form $n_p\Sigma n_p$ for $t_0 \to -\infty$. While repeating the unit (26)

, there arise divergences proportional to $(t_m - t_0)^2$, etc.

We have shown that these divergences can be summed and the result is the substitution of the initial distribution function of quasiparticles $n_p(t_0)$ by the exact one, $n_p(t_m)$, at the current instant t_m. As a result, all secular divergences of the diagram may be dropped, and in each remaining Keldysh diagram the function $n_p(t_0)$ should be substituted by $n_p(t_m)$. The GF satisfies the Dyson equation (19) containing an unknown distribution function of quasiparticles n_p. To find this function, it is necessary to obtain a closed kinetic equation.

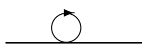
7. Kinetic equation for a distribution function

The equation for the distribution function $n_p(t)$ is obtained as an additional condition imposed on the solutions to the Dyson equation (19).

This condition is of the form

$$\partial n_p(t)/\partial t = [\Sigma, G]_-^{S=-,\,S'=+}, \qquad (27)$$

where $[\Sigma, G]$ is the commutator of the matrices Σ and G. As a result of the renormalization, Σ and G are functionals of the exact distribution function of quasiparticles $n_p(t)$. Hence, substituting them into (27), we obtain a closed kinetic equation for $n_p(t)$. In the self-consistent field approximation we have

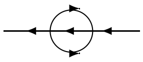
$\Sigma^{(1)} =$  $\equiv U.$ (28)

The kinetic equation (concisely written) takes the form

$$\partial n_p(t)/\partial t = [U, n_p]. \qquad (29)$$

The particle entropy is shown to be conserved in the self-consistent field approximation (28). Thus, the unsteadiness of the Universe leads to an adiabatic rotation of the quasiparticle eigenfunction base, but does not lead to a change in the quasiparticle nonequilibrium state population, which corresponds to the inverse process.

In the collision approximation

$\Sigma^{(2)} =$  . (30)

The kinetic equation takes the form

$$\partial n_p(t)/\partial t = [\Sigma^{(2)}, n_p] \quad (31)$$

and proves to be irreversible in time. It contains a quasiparticle collision integral which leads to increasing entropy. As a result, the nonstationarity of the gravitational field, as regards quasiparticle collisions, leads not only to a base function change, but to a restribution of quasiparticles in states, i.e. to actual transitions from some states to others, which implies quasiparticle scattering.

8. Relation of the energy-momentum tensor to the quasiparticle distribution function

Solving the kinetic equations allows one to calculate the energy-momentum tensor in an arbitrary nonequilibrium state characterized by the quasiparticle distribution function n_p. The calculation of the energy-momentum tensor, following the perturbation theory and the Chapman-Enskog hydrodynamic approximation, indicates the proportionality of this tensor and the distribution function:

$$< T_{\mu\nu} > \sim n_p(t). \quad (32)$$

9. Conclusion

The results of the present paper are summarized as follows:

1. Keldysh's covariant diagram technique has been constructed to describe nonequilibrium fluctuations of a coupling scalar field in curved space-time.

2. A theorem of attenuation of initial correlations has been proved.

3. The Dyson theorem is derived for the Green function of a coupling scalar field.

4. The general expression is obtained for a mass operator in the self-consistent field approximation.

5. Based on the Dyson equation for the Green-Keldysh function, the kinetic equation has been obtained for the distribution function of quasiparticles of a scalar field in the collision approximation in curved space-time.

6. The distribution functions are renormalized in the Dyson equation. The diagram technique divergences are shown to be summable and included in the renormalized quasiparticle distribution function. The kinetic equations obtained therewith play a role of concordance equations for the Dyson distribution function.

7. The quantum scalar field energy-momentum tensor averaged over nonequilibrium states is shown to be calculable following the perturbation theory in a curved space: this space distinguishes the nonequilibrium situation from those occured while averaging over the vacuum state where the perturbation theory is unfit, and one has to use the self-consistent field approximation. The approach being developed permits one to solve the problem of regularizing the energy-momentum tensor nonequilibrium states.

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