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AN EXPANDING SINGULAR SHELL IN A NONLINEAR SCALAR FIELD MODEL AND IN THE RELATIVISTIC THEORY OF GRAVITY

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An example of a nonlinear scalar field model is used to illustrate the main features of fields (scalar, electromagnetic, gravitational in RTG, etc.) generated by expanding spherically symmetric shells.

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1. Introduction

The problem of expanding singular shells of matter and their gravitational fields is nowadays popular in different cosmological inflationary models of general relativity (GR) (see, for example, [1]). In GR, strict equations for the expansion of a singular shell can be derived by the well-known Israel method [2]. And the gravitational field of an expanding spherically symmetric shell is taken in GR in a static form, according to the famous Birkhoff theorem.

But if one deals with another gravitational theory (or, more generally, with other fields — scalar, four-vector, etc.), an analogue of the Birkhoff theorem may be absent. So the fields of expanding spherically symmetric shells can be time-dependent, i.e., can be of a really detectable nonstatic nature.

This is just the case of the Relativistic Theory of Gravity (RTG), elaborated in [3,4], where the gravitational field of an expanding spherically symmetric shell was considered in [4,5].

Speaking generally, in RTG the observed physical quantities are constructed with the help of two metrics — the "curved" one, g_{ij}., and the "flat" one, γ_{ij}, and the gravitational field ψ_{ij} is a tensor field. Thus, for example, the four-vector ν^i in the weak-field gauge transformations, usual in GR or bimetric gravity,

$$\psi_{ij} \to \psi_{ij} + D_i\nu_j + D_j\nu_i - \gamma_{ij}D_p\nu^p,$$

has no "gauge" or "coordinate" meaning in RTG. This "nongauge" approach to gravitational phenomena in RTG differs it strongly from other bimetric theories and from GR and that is why the predictions of RTG are essentially different from those of the above theories.

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To understand better the results obtained for a shell in RTG, let us consider a simple example of a singular shell in a classical nonlinear scalar field model.

It turns out that many features of the solutions to this problem in RTG and in the scalar field model are common. Thus we can say that the nonstatic behaviour of classical fields created by such sources is a common feature of classical field theories.

2. Scalar field model

Consider a nonlinear scalar field theory in Minkowski spacetime ($\gamma_{ij} = \mathrm{diag}\,(1,-1,-1,-1)$ is the Minkowski metric with the interval ds) with the Lagrangian:

$$L = \frac{1}{8\pi}\partial^p\phi\partial_p\phi + \frac{a}{20\pi}\phi^5 + \rho\phi - \rho \tag{1}$$

where a is a coupling constant and ρ is the scalar charge density.

The field equation for ϕ reads:

$$\partial^p\partial_p\phi = a\cdot\phi^4 + 4\pi\rho. \tag{2}$$

Standard variation of (1) with respect to γ_{ij} using the relation $\delta\rho\sqrt{-\gamma}/\delta\gamma_{ij} = \frac{1}{2}\rho u^i u^j\sqrt{-\gamma}$, where $u^i = dx^i/ds$, gives the following expression for the energy-momentum tensor of the model under consideration:

$$\begin{aligned} T^{ij} &= -2\frac{\delta L\sqrt{-\gamma}}{\sqrt{-\gamma}\delta\gamma_{ij}} \\ &= \rho\Phi u^i u^j + \frac{1}{4\pi}\partial^i\phi\partial^j\phi - \gamma^{ij}\left(\frac{1}{8\pi}\partial^p\phi\partial_p\phi + \frac{a}{20\pi}\phi^5\right) \\ &= T^{ij}_m + T^{ij}_\phi \end{aligned} \tag{3}$$

where $\Phi = 1-\phi$.

Then the matter equations of motion in our model are

$$\partial_j T^{ij} = 0. \tag{4}$$

Taking into account the expression (3) and the charge conservation, $\partial_i(\rho u^i) = 0$, these laws can be rewritten in the form

$$u^j \partial_j u^i \equiv \frac{du^i}{ds} = \left(\partial^i - u^i u^j \partial_j\right) \ln \Phi. \tag{5}$$

The spatial components of Eq. (5) are

$$u^0 \frac{d\vec{v}}{ds} = -\left(\nabla + \vec{v}\partial_0\right) \ln \Phi. \tag{6}$$

One can see from (6) that the particles possessing the same sign of the scalar charge, attract each other in this model of interaction. A sign change of the interaction term $\rho\phi$ in (1) leads to a sign change in all expressions of this paper involving the scalar field source.

3. Model of an expanding singular shell

Consider an expanding spherically symmetric singular dust-like shell with total scalar charge Q. Let the shell radially collapse to the origin of the coordinate frame with an initial velocity v at spatial infinity, rebounce from the origin and expand towards spatial infinity with the final velocity v (one can adopt this asymptotic conditions due to the total system energy conservation).

It should be stressed here that the solution to the problem of an expanding shell in RTG (like all other problems in theories formulated against the Minkowski background) begins with the choice of a global coordinate frame and the appropriate initial, boundary, asymptotic, etc. conditions in this frame, and this is not a trivial operation. One can do it in electromagnetism, in RTG and so on (in every theory where one can choose a global coordinate frame in Minkowski spacetime), but in GR there is a problem: as in GR the initial choice of a coordinate system is to some extent the choice of a solution, one always faces the problem of how to take the form of the solution coinciding with the desired matter distribution, boundary and asymptotic conditions, etc. Here we see how RTG differs from GR on the level of basic concepts.

The charge density ρ of the shell, according to the conservation law, can be written as

$$\rho = \frac{q}{4\pi r^2}\delta\left(r - R(t)\right) \tag{7}$$

where $R = R(t)$ is the expanding shell radius and $q = Q/u^0$.

The shell motion obeys the strict law (4). Integrating (4) over a thin spacetime layer, including the charged shell, one can get the desired strict equation of motion:

$$\frac{d}{dt}\left(\frac{\Phi dR/dt}{\sqrt{1-(dR/dt)^2}}\right) = -\frac{4\pi R^2}{q}\left(T^{11}_{\phi,\,\mathrm{ext}} - T^{11}_{\phi,\,\mathrm{int}}\right) \tag{8}$$

where $T^{11}_{\phi,\,\mathrm{int,ext}}$ are the rr components of the energy-momentum tensor of the scalar field ϕ inside and outside the shell, taken for $r = R(t)$, in the spherical coordinates and $d\Phi/dt$ is half sum of the total outer and inner derivatives of $\Phi = 1 - \phi$.

4. Approximation scheme

One can try to find a solution to our nonlinear problem (1)–(8) in the form of an expansion in powers of a small value of the total scalar charge q (or small values of the scalar field: $\phi \ll 1$):

$$\phi = \phi^{(1)} + \phi^{(2)}, \quad R(t) = R^{(0)} + R^{(1)}$$

where the values of $\phi^{(p)}$ and $R^{(p)}$ are of the order q^p or smaller, $R^{(0)}(t)$ is the "zero" motion of the shell: $R^{(0)}(t) = |vt|, \quad v = \mathrm{const}$.

Thus in the first approximation. Eq. (2) takes the form

$$\partial^p \partial_p \phi^{(1)} = 4\pi\rho(R^{(0)}) \tag{9}$$

where the density $\rho(R^{(0)})$ is determined from the relation (7) with the "zero" shell motion $R^{(0)}(t) = |vt|$.

Integrating (9) in terms of retarded potentials, one gets:

$$\phi^{(1)} = \frac{1}{2r}\int\limits_0^\infty dr' \int\limits_{t-|r+r'|}^{t-|r-r'|} dt' \quad r' 4\pi\rho(t', r'). \tag{10}$$

Integration of (10) yields the following fields:

inside the shell:

$$\phi^{(1)} = \frac{q}{2rv}\ln\frac{t+r}{t-r}; \tag{11a}$$

outside the shell:

$$\phi^{(1)} = \frac{q}{2rv}\ln\frac{1+v}{1-v}. \tag{11b}$$

These solutions are written for $t - r > 0$, for $t - r < 0$ one must change the sign of the velocity: $v \to -v$.

The solutions (11) tend to zero at spatial infinity, are bounded on the line $r = 0$ in the (t, r) plane (except the singular point $t = 0 = r$, where the shell collapses to the origin; the singular lines $t = \pm r$ in (11a) lie outside the area of validity of the solution (11a)), are continuously differentiable in the whole spacetime (except $r = R(t)$, the shell itself, where the solutions are continuous and the jump of the first derivatives conforms to the chosen form of surface density (7)) and contain no wave term at spatial infinity.

Thus due to the theorems of mathematical physics the solutions (11) are unique under the chosen boundary, asymptotic and initial conditions.

For $v \to 0$ the solution (11b) has the conventional static limit $\phi \to q/r$ as the inside-shell area degenerates into the zero point $r = 0$.

The values of v, t and r in (11) are restricted by the approximation $\phi^{(1)} \ll 1$: outside the shell r must not be close to zero and v must not be close to its unit value; inside the shell for $r = 0$ the value of t must not be close to zero (these are the ordinary restrictions for a point-like source in classical field theories).

The second approximation

. A substitution of the fields (11) to the shell equation of motion (8) yields

$$\frac{d}{dt}\left(\frac{dR/dt}{\sqrt{1-(dR/dt)^2}}\right) = -\frac{qk}{2v^2t^2(1-v^2)^{3/2}} \tag{12}$$

where

$$k = (1+v^2)\left(\ln\frac{1+v}{1-v} - 1\right).$$

. Integration of (12) with the asymptotic conditions $R(t \to \pm\infty) \to |vt|$ gives the following law of shell motion:

$$R(t) = |vt| + \frac{qk}{2v^2}\ln|t| + \text{const} \tag{13}$$

where, due to the approximation method, $R^{(0)} \gg R^{(1)}$, or

$$v \gg qk/tv^2.$$

Thus one can see that our method restricts the 4-dimensional validity domain of the solution (11) to

$$t \gg qk/v^3. \tag{14}$$

Now let us find the field $\phi^{(2)}$. This field is contributed by both the $R^{(1)}$ component of shell motion in $\rho(R^{(1)})$, i.e., $\phi_1^{(2)}$, and the nonlinear terms in (2), i.e., $\phi_2^{(2)}$:

$$\partial^p\partial_p\phi_1^{(2)} = 4\pi\rho(R^{(1)}), \tag{15}$$

$$\partial^p\partial_p\phi_2^{(2)} = a(\phi^{(1)})^4. \tag{16}$$

Intergation of (15) and (16) in terms of the retarded potentials (10) yields (for $t-r>0$; for $t-r<0$ one should replace $v \to -v$):

inside the shell:

$$\phi_1^{(2)} = \frac{q}{2rv^2}\left[\left(\text{const} + \frac{qk}{2v^2(1-v)}\right)\left(\frac{1}{t+r} - \frac{1}{t-r}\right) + \frac{qk}{2v^2}\left(\frac{1}{t+r}\ln\frac{t+r}{1+v} - \frac{1}{t-r}\ln\frac{t-r}{1+v}\right)\right], \tag{16a}$$

$$\phi_2^{(2)} = \frac{2aq^4}{r(t-r)v^4}\left[I_1\left(\frac{t+r}{t-r}\right) + \frac{2r}{t+r}\left(I_2\left(\frac{1+v}{1-v}\right) - I_2\left(\frac{t+r}{t-r}\right)\right)\right]$$
$$+\frac{aq^4}{64v^4}\ln^4\left(\frac{1+v}{1-v}\right)\frac{(1+v)^2}{v^2}\left(\frac{1}{t-r} - \frac{1}{t+r}\right); \tag{16b}$$

outside the shell:

$$\phi_1^{(2)} = \frac{q^2k}{4rv^4(t-r)}\left[\ln\left(\frac{1+v}{1-v}\right) - \frac{v}{1-v^2}\right]; \tag{16c}$$

$$\phi_2^{(2)} = \frac{2aq^4}{r(t-r)v^4}I_1\left(\frac{1+v}{1-v}\right) + \frac{aq^4}{64v^4}\ln^4\left(\frac{1+v}{1-v}\right)\left(-\frac{1}{r} + \frac{2}{v(t-r)}\right), \tag{16d}$$

where the integrals

$$I_1(x) = \int_0^{\ln x} dt\ t^4\sinh^{-2}t, \quad I_2(x) = \int_0^{\ln x} dt\ t^4\sinh^{-3}t$$

can be rewritten in terms of polylogarithms.

The solutions (11) and (16), taken together, are continuous at all spacetime points, including the boundary (13) ($r = R(t)$), are regular at $r=0,\ t=r$, and at spatial infinity tend to zero with no wave terms. So they are unique.

One can see that the out-of-shell solution in the second approximation (16c,d), contrary to the first approximation (11b), does possess a time dependence.

As our approximation method implies $\phi^{(1)} \gg \phi^{(2)}$, the solutions (11) and (16a,c) lead to the restriction

$$t - r \gg qk/v^3; \tag{17}$$

(a restriction that follows from (11) and (16b,d), is

$$t - r \gg q^3a/v^4 \tag{17'}$$

and its "force" depends on the relation between the values of q and a).

Thus our solutions (11) and (16) to the above problem of an expanding singular shell in the spacetime domain specified by the inequalities (17,17'), are well-behaving.

The appearance of a singularity on the null line $|t| = r$ (in the (t,r) - plane) is not a surprise: the singular line at the origin $r=0$ of the static solution for a shell at rest transforms, for an expanding shell, to the singularity at the null line $|t| = r$ due to the null-cone singularity of the Green function of Eq. (2). So this line must be beyond the validity domain of our approximation method.

The nonstatic nature of the field ϕ both inside and outside the shell can be detected by a test observer subject to a force from the scalar field of the shell. In the first approximation from (6) it follows that

$$(u^0)^2\frac{d\vec{v}}{dt} = (\nabla + \vec{v}\partial_0)\,\phi \tag{18}$$

where ϕ outside and inside the shell is taken from (11).

If the test body (observer) is at rest, $v=0$, then its acceleration is

inside the shell:

$$-\frac{q\vec{r}}{2vr^3}\left(\ln\frac{t+r}{t-r} - \frac{2tr}{t^2-r^2}\right), \tag{19a}$$

outside the shell:

$$-\frac{q\vec{r}}{2vr^3}\ln\frac{1+v}{1-v}. \qquad (19b)$$

Thus one can easily detect the nonstatic nature of the inside-shell field (19a). It is also interesting to note that the sign of the inner acceleration (19a) is always positive for $q > 0$ (for $v = 0$ it vanishes) and the absolute value of the outer acceleration (19b) is always greater than its static value $q\vec{r}/r^3$.

The motion of the singular shell can be defined as well by jump conditions, resembling the jump conditions in classical electrodynamics or those for singular shells in GR:

$$[n^p\partial_p\phi] = -4\pi\sigma$$

where $[F]$ denotes the jump of the function F across the shell, n^p is the unit external four-vector, perpendicular to the hypersurface of the expanding shell, σ is the surface scalar charge density (this jump condition can be derived by simply intergating Eq. (1) over a thin layer containing the hypersurface of the expanding shell).

The substitution of (11),(16a,c) into this jump condition gives the result $\sigma = Q/(4\pi R^2)$, where R obeys the relation (13), while the substitution of (16b,d) gives further corrections to the equation of motion of the order q^4a.

In the linear case, $a \equiv 0$, our problem has an exact solution which can be written in the following general form:

$$\phi_{\text{ext,int}} = \frac{Q}{2r}\int\limits_{t_0}^{t_{\text{ext}},t_{\text{int}}}\frac{dt}{R(t)}\sqrt{1-v^2}$$

where the limits of the intergal are found from the following relations: $t_{\text{ext}} = t - r + R(t_{\text{ext}})$, $t_{\text{int}} = t + r - R(t_{\text{int}})$, $t_0 = t - r - R(t_0)$, and the shell equation of motion is

$$\left(1-\frac{Q}{2R}\right)\int\limits_B^t dx\frac{\sqrt{1-v(x)^2}}{R(x)}\frac{\ddot{R}}{1-v^2}$$
$$= -\frac{Q}{2R^2}\int\limits_B^t dx\frac{\sqrt{1-v(x)^2}}{R(x)} + \frac{Q(1-v)}{2R}\left[\sqrt{\frac{1-v}{1+v}}\frac{1}{R}\right]_B,$$

where $\ddot{R} = d^2R(t)/dt^2$, $B = t - R(t) - R(B)$. The time dependence of this solution is explicit.

One can see that the nonstatic nature of the shell field ϕ (11), (16) is not something induced by an approximation method but is the real feature of an exact solution.

5. Comparison with the gravitational field of an expanding shell in RTG

Comparing the above solutions of our scalar model with the solutions for the gravitational field of an expanding spherically symmetric shell in RTG [4,5], one can easily see that all features of the scalar field mentioned in our paper, are precisely the same for the gravitational field [4,5].

It is convenient to cite here this gravitational field in the first approximation (in the second approximation the form of the field is too cumbersome to be cited here (see [4,5]), nevertheless it should be mentioned that the gravitational field behaviour in the second approximation near the null line is just the same as that of the solution (16), accordingly the validity domain for the solution of [4,5] is defined by inequalities similar to (17),(17')).

The field equations in RTG read:

$$\begin{aligned}\sqrt{g/\gamma}g^{ij} &= \gamma^{ij} - \psi^{ij},\\ R_{ij} - g_{ij}R/2 &= 8\pi T_{ij},\\ D_j\psi^{ij} &= 0,\end{aligned}$$

where g_{ij} is the Riemannian metric; γ_{ij} is the Minkowski metric; D_j is a covariant derivative in terms of γ_{ij}.

In the first approximation the gravitational field of an expanding singular shell in the spherical coordinates $\gamma_{ij} = (1, -1, -r^2, -r^2\sin^2\theta)$ is

outside the shell:

$$\psi_{tt} = -\frac{2mu^0}{vr}\ln\frac{1+v}{1-v}, \qquad (20a)$$

$$\psi_{tr} = -\frac{2mu^0vt}{(vr)^2}\left(2v - \ln\frac{1+v}{1-v}\right), \qquad (20b)$$

$$\psi_{rr} = \frac{mu^0}{(vr)^3}\left\{\left[(vt)^2(v^2-3) - (vr)^2(v^2-1)\right]\ln\frac{1+v}{1-v}\right.$$
$$\left. + 2v\left[3(vt)^2 - (vr)^2\right]\right\}; \qquad (20c)$$

inside the shell:

$$\psi_{tt} = -\frac{2mu^0}{vr}\ln\frac{t+r}{t-r}, \qquad (20d)$$

$$\psi_{tr} = -\frac{2mu^0}{(vr)^2}\left(2vr - vt\ln\frac{t+r}{t-r}\right), \qquad (20e)$$

$$\psi_{rr} = \frac{mu^0}{(vr)^3}\left\{\left[(vt)^2(v^2-3) - (vr)^2(v^2-1)\right]\ln\frac{t+r}{t-r}\right.$$
$$\left. - 2(vr)(vt)(v^2-3)\right\}, \qquad (20f)$$

where $t - r > 0$, v is the velocity of a shell of mass m, $u^0 = 1/\sqrt{1-v^2}$; for $t - r < 0$ one must replace $v \to -v$.

It should be mentioned that outside the shell one can find the translation vectors ν_i allowing one to rewrite the out-of-shell solution in terms of its static form $\psi_{ij}(v = 0)$ [4,5] (this is an analogue of the Birkhoff theorem in RTG in the approximation under study):

$$\psi_{ij} = \psi_{ij}(v=0) + D_i\nu_j + D_j\nu_i - \gamma_{ij}D_p\nu^p.$$

Thus the curvature tensor R_{ijkl} in Cartesian coordinates outside the shell has its common static form. For the inside field the solutions (20e-f) can be also put in the form $\psi_{ij} = D_i\mu_j + D_j\mu_i - \gamma_{ij}D_p\mu^p$, and all the components of R_{ijkl} are zero for (20e-f).

Nevertheless, the covariant expression for a force acting on a test body with the four-velocity u^i, $F^i = -\Delta^i_{mn}u^m u^n$ (Δ^i_{mn} are covariant Christoffel symbols, i.e. quantities formed from the standard Christoffel symbols by replacing the partial derivatives ∂ by the covariant ones D with respect to γ; this definition is common to gravitational theories against flat background, see e.g. [6]) and is time-dependent, so the observed quantities ($F^iF_i,\ F^iu_i, etc.$) in RTG give the opportunity to detect the nonstatic nature of the gravitational field of a shell [4,5].

The inner acceleration of a test body at rest is always negative (for $v = 0$ it vanishes, cf. (19a)):

$$\frac{m(3-v^2)\vec{r}}{2vr^3}\left(\ln\frac{t+r}{t-r} - \frac{2tr}{t^2-r^2}\right).$$

Speaking generally, one can say, that, since in RTG the observed physical quantities are constructed with the help of two metrics — the "curved" one, g_{ij}, and the "flat" one, γ_{ij}, and the gravitational field is a tensor field, the mentioned nonstatic behaviour of the gravitational field of a shell is always detectable. Thus the above four-vector ν^i has not a "gauge" or "coordinate" meaning in RTG. This "nongauge" approach to gravitational phenomena in RTG differs it strongly from other bimetric theories.

One can see as well that the change of mu^0 to $-4Q$ in the tt and tr components of the above field ψ_{ij} (20) yields the ϕ and A_r components of the electromagnetic potentials (in the Lorentz gauge) of an expanding charged shell with the corresponding electromagnetic fields $\vec{E} = 0,\ \vec{H} = 0$ inside the shell, $E_r = Q/r^2,\ \vec{H} = 0$ outside the shell in accordance with the Gauss theorem of classical electrodynamics. And the change of mu^0 to $-4q$ in the tt component of the above field ψ_{ij} leads to the scalar-field solution (11).

Moreover, one can state that the nonstatic nature of a physical field with a source like an expanding spherically symmetric shell is a common feature of a classical field theory.

In particular, one can consider such a nonstatic nature of the gravitational field of a shell as a theoretical and experimental test of the Relativistic Theory of Gravity in the weak-field approximation.

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